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Autor: Ghali, A. / Tadros, G.S.

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On Finite Strip Analysis of Continuous Plates

Analyse par bandes finies de plaques continues

Über die Finite-Streifen-Berechnung von Platten

A. GHALI
M. ASCE, Professor of Civil Engineering,

G. S. TADROS
Post-doctoral Fellow,

The University of Calgary, Calgary, Alberta, Canada

Introduction

Semi-analytical finite element procedures have been used to reduce drastically the number of equations and solution cost for two-dimensional and three-dimensional situations. In the application of these procedures to plates in bending, long elements or strips are used (Fig. 1a and b) and the transverse deflection of a strip is expressed in the form

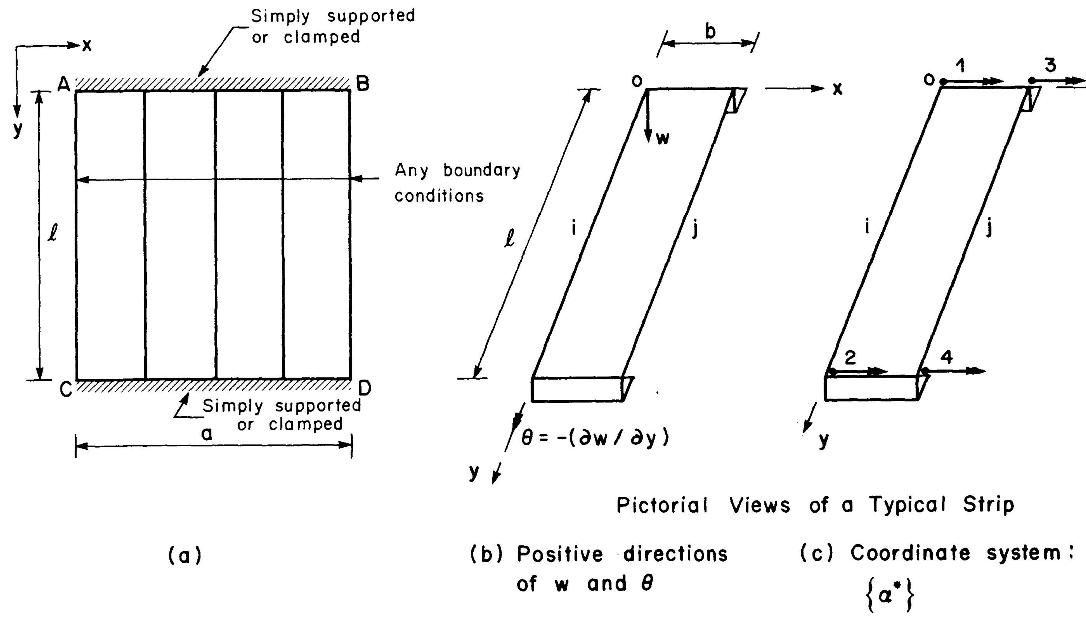

Fig. 1. Finite Strip Idealization.

$$w^* = \sum_{m=1}^{r} w_m^* = \sum_{m=1}^{r} f(x)\, Y_m, \tag{1}$$

where $f(x)$ is a polynomial of x only and Y_m are basic functions satisfying the end conditions in the y direction. When a third degree polynomial is used for $f(x)$, the unknown displacement parameters are two for each nodal line and for each term of the series: w_{im} and θ_{im}, where $\theta = -(\partial w/\partial x)$. This is the finite strip method, first developed by CHEUNG, for the analysis of simply-supported slabs [1] and later by CHEUNG and others for other structures idealized into strips subjected to in-plane as well as to bending forces. For the simply-supported slab, the basic function is a trigonometric series:

$$Y_m = \sin \frac{m \pi y}{l}. \tag{2}$$

Applied loads must also be resolved into series similar to the displacement function, and a load vector $\{F^*\}$ is then related to the nodal parameters. For the strip in Fig. 1b, this relation takes the form:

$$[S^*]\{D^*\} = \{F^*\}, \tag{3}$$

where $[S^*]$ is a square matrix of order $4r$ representing the stiffness of the strip, and the displacement vector

$$\{D^*\} = \{\{D^*\}_1, \{D^*\}_2, \ldots, \{D^*\}_r\}, \tag{4}$$

where

$$\{D^*\}_m = \{w_i, \theta_i, w_j, \theta_j\}_m. \tag{5}$$

The stiffness matrix of the strip can be partitioned into $r \times r$ submatrices $[S^*]_{mn}$ corresponding to each term of the series. Due to the orthogonality of the chosen function Y_m and its derivatives, it can be shown that the submatrix $[S^*]_{mn} = [0]$ when $m \neq n$. Thus, Eq. (3) will actually uncouple, and for each term of the series we can write for one strip

$$[S^*]_{mm}\{D^*\}_m = \{F^*\}_m. \tag{6}$$

For each term of the series, a stiffness matrix of the plate $[S]_m$ of order $s \times s$ (where s is twice the number of nodal lines) is assembled from the stiffness matrices $[S^*]_{mm}$ of the individual strips. Adding the forces on each nodal line from the two adjacent strips, we obtain a load vector $\{F\}_m$. Solution of the equation

$$[S]_m\{D\}_m = \{F\}_m \tag{7}$$

gives the s displacement parameters for the mth term of the series. Thus, the uncoupling makes it necessary to solve r sets of s equations instead of one set of $r \times s$ equations. The half-band width of the matrix is reduced by a factor r; thus, the reduction in computer time due to uncoupling can be quite considerable.

CHEUNG [2] used the finite strip method with the basic function Y, other than trigonometric, to analyze plates with two other end conditions: simply-supported-clamped and clamped-clamped. Unfortunately, with these functions the uncoupling described above can not occur.

The object of the present paper is to use the trigonometric basic function of Eq. (2) for all the three boundary conditions described above. The effect of the fixing moment at the clamped edge is accounted for by superposition. Thus, in all cases benefit is obtained from the uncoupling of the equations.

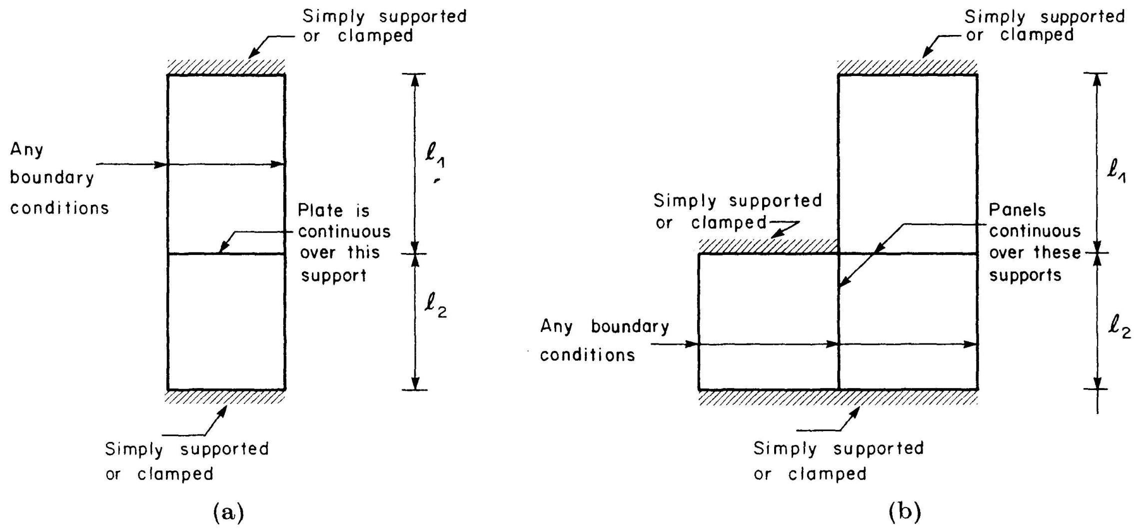

Fig. 2. Continuous plates in Bending.

Any of the plates of Fig. 2 continuous over a number of panels can be analyzed by the proposed method. Each panel is first analyzed with its two edges parallel to the x axis simply supported, then the compatibility of rotation at these edges is satisfied by the application of couples and the final result is obtained by superposition.

CHEUNG et al. [3] analyzed continuous bridges similar to the plate in Fig. 2a by solving a simply supported plate of span equals $(l_1 + l_2)$, and superposing the effect of concentrated transverse forces to bring the deflections at the intermediate support back to zero (force method of analysis). This approach may result in ill-conditioned equations when the plate is continuous over a number of spans and a large number of strips is used.

Procedure

The finite strip method for analysis of plates in bending is given in the references mentioned above and in more detail with simplified example in Ref. [4]. For this reason, the derivation of some of the equations which is available in these references is deleted from the following presentation.

The deflection of a strip (Eq. (1)) can be expressed in terms of the nodal parameters for its sides i and j. A general term of the series is

$$w_m^* = [L^*]_m \{D^*\}_m, \tag{8}$$

where

$$[L^*]_m = \left[\left(1-\frac{3x^2}{b^2}+\frac{2x^3}{b^3}\right) \middle| \left(-x+\frac{2x^2}{b}-\frac{x^3}{b}\right) \middle| \left(\frac{3x^2}{b^2}-\frac{2x^3}{b^3}\right) \middle| \left(\frac{x^2}{b}-\frac{x^3}{b^2}\right)\right] Y_m. \tag{9}$$

Define a system of four coordinates for each strip: The rotation $(\partial w/\partial y)$ at the ends $y=0$ and $y=l$ of its nodal lines (Fig. 1c).

$$\{\alpha^*\}_m = \left\{\left(\frac{\partial w_m^*}{\partial y}\right)_{x=0,\,y=0}, \left(\frac{\partial w_m^*}{\partial y}\right)_{x=0,\,y=l}, \left(\frac{\partial w_m^*}{\partial y}\right)_{x=b,\,y=0}, \left(\frac{\partial w_m^*}{\partial y}\right)_{x=b,\,y=l}\right\}. \tag{10}$$

Substitution of Eq. (8) into Eq. (10) gives

$$\{\alpha^*\}_m = [C^*]_m \{D^*\}_m, \tag{11}$$

where $[C^*]_m$ is the strip displacement transformation matrix for the mth term

$$[C^*]_m = \begin{bmatrix} [H]_m & [0] \\ [0] & [H]_m \end{bmatrix} \tag{12}$$

and

$$[H]_m = \frac{m\pi}{l}\begin{bmatrix} 1 & 0 \\ \cos m\pi & 0 \end{bmatrix}. \tag{13}$$

The edge bending moment normal to the short sides of the strip is assumed to vary linearly between nodal lines. Define a vector $\{Q^*\}$ of the four values of the moment (per unit length) at the ends of the nodal lines. The edge bending moments can be expressed in a form of trigonometric series of a transverse load of intensity

$$q = \sum q_m Y_m, \tag{14}$$

where

$$q_m = \frac{2m\pi}{l^2}\left[\frac{b-x}{b} \middle| \frac{b-x}{b}\cos m\pi \middle| \frac{x}{b} \middle| \frac{x}{b}\cos m\pi\right]\{Q^*\}. \tag{15}$$

The load vector (see Eq. (20.108) of Ref. [4]).

$$\{F^*\}_m = \int_0^l\int_0^b [L]_m^T q_m Y_m \, dx\, dy = [G^*]_m \{Q^*\}. \tag{16}$$

Substitution of Eqs. (9) and (15) in Eq. (16) gives the force transformation matrix

$$[G^*]_m = \frac{m\pi}{l}\begin{bmatrix} \frac{7b}{20} & \frac{7b}{20}\cos m\pi & \frac{3b}{20} & \frac{3b}{20}\cos m\pi \\ -\frac{b^2}{20} & -\frac{b^2}{20}\cos m\pi & -\frac{b^2}{30} & -\frac{b^2}{30}\cos m\pi \\ \frac{3b}{20} & \frac{3b}{20}\cos m\pi & \frac{7b}{20} & \frac{7b}{20}\cos m\pi \\ \frac{b^2}{30} & \frac{b^2}{30}\cos m\pi & \frac{b^2}{20} & \frac{b^2}{20}\cos m\pi \end{bmatrix}. \tag{17}$$

For the assembled structure two vectors are derived: $\{\alpha\}_m$ and $\{Q\}$, each of order s. An element of $\{Q\}$ corresponding to one end of a nodal line is obtained by adding Q^* for the strips at its left and its right. In a similar way a transformation matrix $[G]_m$ of the structure (of order $s\times s$) can be assembled from the transformation matrices $[G^*]_m$ of the individual strips. Thus, for the assembled structure Eq. (16) becomes

$$\{F\}_m = [G]_m\{Q\}. \tag{18}$$

The rotation vector of the assembled structure is related to the nodal displacement parameter by the equation

$$\{\alpha\}_m = [C]_m\{D\}_m, \tag{19}$$

where

$$[C]_m = \begin{bmatrix} [H]_m & & \\ & [H]_m & \\ \text{submatrices} & & \cdots \\ \text{not shown are null} & & [H]_m \end{bmatrix}. \tag{20}$$

Combining Eq. (7) with Eqs. (19) and (18)

$$\{\alpha\}_m = [C]_m[S]_m^{-1}\{F\}_m \tag{21}$$

and

$$\{\alpha\}_m = [f]_m\{Q\}, \tag{22}$$

where

$$[f]_m = [C]_m[S]_m^{-1}[G]_m. \tag{23}$$

$[f]_m$ represents the contribution of the mth term to the "flexibility matrix relating $\{\alpha\}_m$ and $\{Q\}$. The "flexibility" matrix relating $\{\alpha\}$ and $\{Q\}$ is

$$[f] = \sum_{m=1}^{r}[C]_m[S]_m^{-1}[G]_m. \tag{24}$$

For an example of application of the above equations consider a one panel plate with the edges parallel to the x axis clamped. The bending moment normal to the clamped edges is of intensities $\{Q\}$ at the nodal lines ends. For compatibility the rotations due to $\{Q\}$ on a simply-supported slab $(=[f]\{Q\})$ must be equal and opposite to the rotations $\{\alpha\} = \sum\{\alpha\}_m$ due to actual loading; thus, combining this condition with Eq. (21) gives

$$\{Q\} = -[f]^{-1}\sum_{m=1}^{r}\{\alpha\}_m = -[f]^{-1}\sum_{m=1}^{r}[C]_m[S]_m^{-1}\{F\}_m. \tag{25}$$

It is to be noted that the above equation satisfies the compatibility condition $(\partial w/\partial y)=0$ only at the ends of the nodal lines. The error resulting from this approximation is negligible and decreases as the width of the strips is decreased. (When $b\to 0$, the compatibility condition is satisfied at all points.)

The clamped plate can now be analyzed as a simply-supported one, subjected to the combined load $\{F\}_m$ and $\{Q\}$. The contribution of the mth term to the displacement parameters (see Eqs. (7) and (18)) for the clamped plate

$$\{\bar{D}\}_m = [S]_m^{-1}\{F\}_m + \sum_{n=1}^{p} [S]_n^{-1}[G]_n\{Q\}. \tag{26}$$

Choosing $p=r$, Eq. (26) becomes

$$\{\bar{D}\}_m = [S]_m^{-1}\{\bar{F}\}_m, \tag{27}$$

where

$$\{\bar{F}\}_m = \{F\}_m + [G]_m\{Q\}. \tag{28}$$

Thus, the analysis of a clamped plate is reduced to that of a simply-supported plate with the load vector $\{F\}_m$ replaced by $\{\bar{F}\}_m$.

Examples

To check the method of analysis for the effect of edge moments derived above, consider isotropic plates simply supported on four sides and subjected to uniform moment M_0 per unit length normal to the two sides AB and CD (see Fig. 1a and Table 1). Due to symmetry, half the plate is analyzed with $a/2$ divided into five strips. Table 2 gives the results of plates as in the first example but subjected to uniform load and the edges AB and CD clamped instead of simply supported. Poisson's ratio in both examples equals 0.3.

A study of convergence is made in Table 3 for one of the plats in Table 2 with $l/a=1.0$. Odd terms only contribute to the results. A somewhat large number of terms is required if high accuracy is desired. However, for the values considered the contribution of any two consecutive odd terms are of opposite signs, and the average of the answers with two consecutive odd terms is close to the exact solution. The results in Tables 1 and 2 are averages of solutions using 7 and 9 terms. For further study of convergence, see Ref. 6.

Table 1. Analysis of Plates Simply Supported on Four Sides and Subjected to Moments M_0 on Edges AB and CD (Fig. 1a)

l/a	Central Deflection	Central M_x	Central M_y	Source
2.0	1.746	0.156	–0.005	F. S.
	1.740	0.153	–0.010	Exact [5]
1.5	2.800	0.267	0.052	F. S.
	2.800	0.264	0.046	Exact
1.0	3.685	0.397	0.262	F. S.
	3.680	0.394	0.256	Exact
0.75	6.199	0.425	0.482	F. S.
	6.200	0.424	0.476	Exact
0.50	9.647	0.385	0.775	F. S.
	9.640	0.387	0.770	Exact
Multiplier	$M_0\,\xi^2/(100N)$ *)	M_0	M_0	

*) ξ is the smallest value of l and a.

Table 2. Analysis of Plates Clamped on Sides AB and CD, Simply Supported on the Two Other Sides Subjected to Uniform Load of Intensity q (Fig. 1a)

l/a	Central Deflection	Central M_x	Central M_y	M_y at Middle of AB	Source
0.5	0.00253 0.00260	0.0135 0.0142	0.0410 0.0420	–0.0852 –0.0842	F. S. Exact [5]
1.0	0.00186 0.00192	0.0239 0.0244	0.0325 0.0332	–0.0718 –0.0697	F. S. Exact
2.0	0.00834 0.00844	0.0862 0.0869	0.0475 0.0474	–0.1269 –0.1191	F. S. Exact
3.0	0.001162 0.001168	0.1144 0.1144	0.0421 0.0419	–0.1370 –0.1246	F. S. Exact
Multiplier	$q\,\xi^4/N$ *)	$q\,\xi^2$	$q\,\xi^2$	$q\,\xi^2$	

*) ξ is the smallest value of l and a.

Table 3. Study of Convergence. Uniformly Loaded Square Plate Clamped on Sides AB and CD, Simply Supported on the Two Other Sides (Fig. 1a)

Number of Terms in Series	Central Deflection		Central M_x		Average of Two Consecutive Solutions	
	Sum of Terms	Contribution of Last Term	Sum of Terms	Contribution of Last Term	Central Deflection	Central M_x
1	16.713	16.713	1.837	1.837		
					17.898	2.264
3	19.083	3.770	2.691	0.854		
					18.756	2.423
5	18.428	–0.655	2.155	–0.536		
					18.554	2.348
7	18.681	0.253	2.541	0.386		
					18.620	2.391
9	18.559	–0.122	2.240	–0.301		
					18.593	2.364
11	18.627	0.068	2.487	0.247		
					18.606	2.382
13	18.585	–0.042	2.277	–0.210		
					18.599	2.368
15	18.612	0.027	2.459	0.182		
					18.603	2.379
17	18.594	–0.018	2.299	–0.160		
					18.600	2.371
19	18.607	0.013	2.442	0.143		
					18.602	2.377
21	18.597	–0.010	2.312	–0.130		
					...	...
..	...	...	...	...		
					...	...
39	18.602	0.001	2.408	0.070		
					18.602	2.375
41	18.601	–0.001	2.342	–0.066		
Multiplier	$10^{-4}\,q\,l^4/N$		$10^{-2}\,q\,l^2$		$10^{-4}\,q\,l^4/N$	$10^{-2}\,q\,l^2$

Conclusion

The solution of plates simply supported on two opposite sides when analyzed by the finite strip requires solution of a small number of equations because of the orthogonal properties of the trigonometric basic functions used. If other functions are used to satisfy boundary conditions other than simply supported, the orthogonality does not occur and both the number of simultaneous equations to be solved and their half-band width are much increased. The method presented allows the use of trigonometric functions for all cases of clamped or continuous plates. The analysis is done for simply-supported panels and the compatibility of rotation at the clamped or continuous edges is achieved by the application of unknown "connecting" moments determined by the force method. The accuracy of the proposed method is demonstrated by examples.

Notation

$[C]$ = transformation matrix.
$\{D\}$ = displacement parameters.
$\{F\}$ = load vector.
$[f]$ = flexibility matrix.
$\{Q\}$ = intensity of end moment normal to the short sides of the strip. The positive directions of the end moments are the same as the directions of $\{\alpha\}$.
$[S]$ = stiffness matrix.
$\{\alpha\}$ = nodal line end rotation.
a = width of the plate.
b = width of the strip.
l = length of the strip.
N = flexural rigidity.
r = number of terms.
s = twice the number of the nodal lines.
Y_m = basic function.
w = transverse deflection.

Note: A star used as superscript refers to the strip while the absence of a star means that the symbol refers to the assemblage of strips.

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Summary

The finite strip solution for rectangular plates in bending is a semi-analytic procedure which reduces drastically the number of equations involved compared with the finite element method. This is particularly true in the case of a plate having two of its opposite edges simply supported, and the deflection variation perpendicular to these edges is expressed as a sum of basic trigonometric series. With other edge conditions, other basic functions must be used, which lack the orthogonality of the trigonometric functions and result in a large increase in the number of simultaneous equations and their half-band width. An alternative finite strip procedure is presented, in which using superposition, the trigonometric basic functions can be maintained for all edge conditions including clamped or continuous.

Résumé

La résolution par la méthode des bandes finies du problème des plaques rectangulaires soumises à la flexion est un procédé semi-analytique qui réduit considérablement le nombre des équations nécessaires comparé à la méthode des éléments finis. Ceci est particulièrement valable dans le cas de plaques dont deux côtés opposés sont simplement appuyés, et dont la variation de la flèche perpendiculairement à ces côtés est exprimée sous forme de somme de séries trigonométriques. Pour d'autres conditions aux limites, on devra utiliser d'autres fonctions de base, qui nuisent à l'orthogonalité des fonctions trigonométriques et impliquent une augmentation du nombre d'équations simultanées et de la largeur de leurs demi-bandes. On présente une méthode basée sur les

bandes finies alternatives pour lesquelles, par superposition, on peut conserver les fonctions trigonométriques de base pour toutes les conditions aux limites, y compris les encastrements ou les appuis continus.

Zusammenfassung

Die Finite-Streifen-Lösung für rechteckige Platten unter Biegung ist eine halbanalytische Methode, die die Anzahl Gleichungen, verglichen mit der Methode der finiten Elemente, wesentlich reduziert. Dies trifft besonders für den Fall einer Platte mit zwei entgegengesetzten frei aufliegenden Rändern zu, wo die Variation der Durchbiegung senkrecht zu diesen Rändern durch eine Summe fundamentaler trigonometrischer Reihen dargestellt werden kann. Andere Randbedingungen erfordern andere Grundfunktionen, die die Orthogonalität der trigonometrischen Funktionen nicht besitzen, und aus denen eine grosse Zunahme der Anzahl simultaner Gleichungen und deren Bandbreiten resultiert.

Alternativ wird eine andere mögliche Finite-Streifen-Methode beschrieben, bei der, durch Gebrauch von Superposition, die fundamentalen trigonometrischen Funktionen für alle Randbedingungen einschliesslich Einspannung und Auskragung beibehalten werden können.