Analysis of cylindrical shell systems

Analyse des systèmes de voiles cylindriques

Berechnung von zylindrischen Schalensystemen

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SUMMARY

A method for analysis of cylindrical shell systems is described. The shells are simply supported at the curved ends and can be connected in an arbitrary manner in the transverse direction. A trigonometric expansion in the longitudinal direction is combined with a matrix displacement solution for each series term in the transverse direction. The efficiency of the approach rests on a transformation of shell element matrices to nodal (beam) parameters.

RÉSUMÉ

Une méthode pour l'analyse des systèmes de voiles cylindriques est présentée. Les voiles sont appuyées simplement aux bords incurvés et sont attachés d'une manière arbitraire en direction transversale. Une série trigonométrique en direction longitudinale est combinée à une solution de déplacement sous forme matricielle en direction transversale. La réussite de la méthode se base sur une transformation des matrices des voiles aux paramètres nodaux.

ZUSAMMENFASSUNG

Eine Methode für die Berechnung von Systemen von Zylinderschalen wird beschrieben. Die Schalen sind an den Schalenenden frei drehbar gelagert und in der Querrichtung beliebig verbunden. Eine trigonometrische Entwicklung in Längsrichtung wird mit einer Verschiebungsformulierung in Querrichtung kombiniert. Die Effektivität der Methode basiert auf einer Transformation der Schalenelementmatrizen zu Knotenpunkt-(balken-)-parametern.

1. INTRODUCTION

In the fifities and sixties the finite element methods grew up parallel with the computers, and conquered gradually almost the whole market of complex structural analyses. Illustrating examples of shell structures now analyzed by the finite element method are the cellular rafts of the CONDEEP platforms (Fig. 1). Such runs comprising 6000 elements, 100000 unknowns and with a total computing time of 16 hours CPU on a UNIVAC 1108 have been described by Lindvik [1].

Even if the finite element method in principle is applicable for nearly all structural computations, a warning should be raised against the present tendency of considering it as the only method for complex problems. In particular, faster methods that allow a description of the main features of the structural action can be very useful for parametric studies to establish principal dimensions. A final finite element analysis can be run if deemed necessary to account for structural performance not considered by the more rapid but perhaps less accurate methods.

The purpose of the present paper is to demonstrate how rapid and efficient analyses of systems of cylindrical shells may be established by a modern interpretation of traditional cylindrical shell theory.

2. LINEAR THEORY OF CYLINDRICAL SHELLS

The traditional theory of cylindrical shells is based on a set of linear first order differential equations in the two surface coordinates. The equations express equilibrium of stress resultants, constitutive laws relating stresses to strains, and kinematic relations between strains and displacements. Traditionally, these equations were reduced to higher order differential equations. By ignoring secondary terms rather simple equations could be obtained, the most neat and systematic of which are due to Donnell [2]. The equations could be integrated by expansion into single trigonometric series of the Levy type. The solution of concern here uses trigonometric functions in the direction of the generator, imposing the conditions of simple support at the shell ends.

This traditional shell theory is utilized also in the present paper, the only difference being that the equations in the circumference coordinate for individual shell panels are integrated by the transfer matrix method. This method uses the set of first order equations directly, and secondary terms can easily be included. Thus the accuracy of the shell theory used is on the same level as the so-called improved Donnell theory [3]. The advantage of this accuracy is that the limiting case of pure arch action is described correctly. Details of the approach are omitted here, they may be found in the report [4].

3. FORMULATION OF SYSTEM CONNECTIVITY

Cylindrical shell panels seldom or never are used as independent structures, but connected to edge beams, plates, walls or other shell panels. The connection forces depend on the displacements, and thus an external redundancy exists in addition to the internal redundancy in each panel expressed by the differential equations. Traditionally, the redundant quantities were found by the force method. A serious drawback of this method is its complexity if the system is not of the very simplest kind.

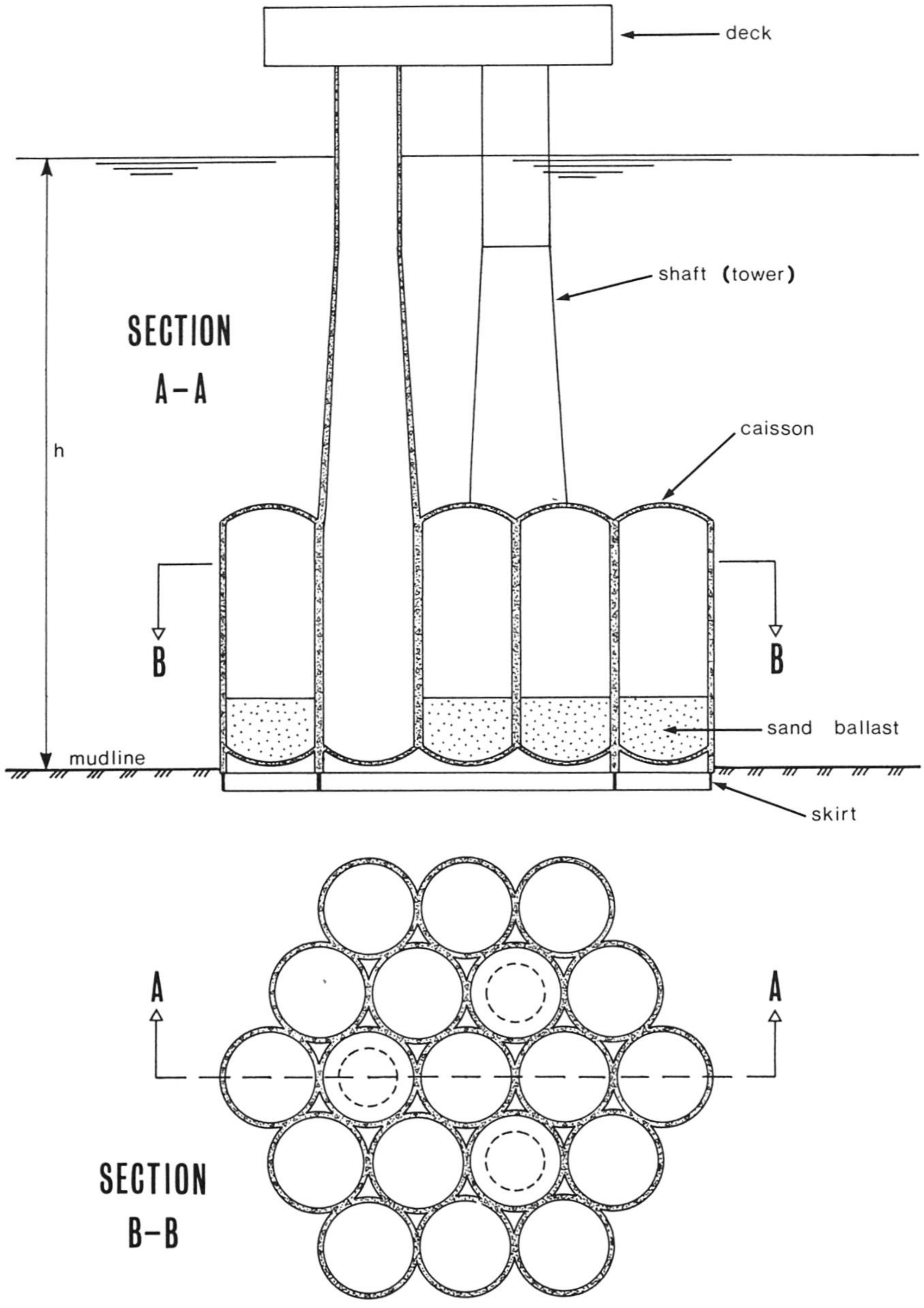

FIGURE 1 CONDEEP PLATFORM

The transfer matrix method offers a simple and systematic way of formulating the system connectivity if the topology is of the line type. It has thus been used to some extent in shell programs [5]. However, the method in its proper form fails as soon as a point is met on the transfer path, where two or more shells branch off in different directions.

Experience from frame analysis has demonstrated that the matrix displacement method is unsurpassed as far as a simple description of a general topology is concerned. Thus the choice for the present method was

- a trigonometric expansion in the longitudinal direction
- a matrix displacement formulation of the solution for each individual term in the transverse direction

4. SHELL ELEMENT MATRICES

The system analysis being carried through for each series term, the shell element matrices refer to a single series term. The longitudinal distribution of each term is either according to a sine or a cosine function. This is illustrated in Fig. 2, showing the distribution of the shear force and the hoop force for the first series term. The distribution of higher terms is analogous, the only difference being that the shell length is divided by an integer n, the number of the series term.

The notations used for stress resultants and displacements in the shell theory are shown in Figs. 3 and 4. The quantities entering the element matrices are denoted by the vector symbols

$$V^T = [u_x \; u_s \; w \; \theta_s] \tag{1}$$

$$N^T = [N_{sx} \; N_s \; R_s \; M_s] \tag{2}$$

The longitudinal distribution of the quantities is

$$u_s,\ w,\ \theta_s,\ N_s,\ R_s,\ M_s \ : \ \sin\lambda x$$

$$u_x,\ N_{sx} \ : \ \cos\lambda x$$

where $\lambda = n\pi x/\ell$

The transfer matrix solution delivers a result in the form

$$\begin{bmatrix} V_2 \\ N_2 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ N_1 \end{bmatrix} + \begin{bmatrix} V_p \\ N_p \end{bmatrix} \tag{3}$$

In Eq. (3) and the following equations related to the system analysis, the vector symbols refer to the amplitude values of the quantities in question, the trigonometric longitudinal distribution being tacitly understood. Subscripts 1 and 2 refer to the edges 1 and 2, respectively.

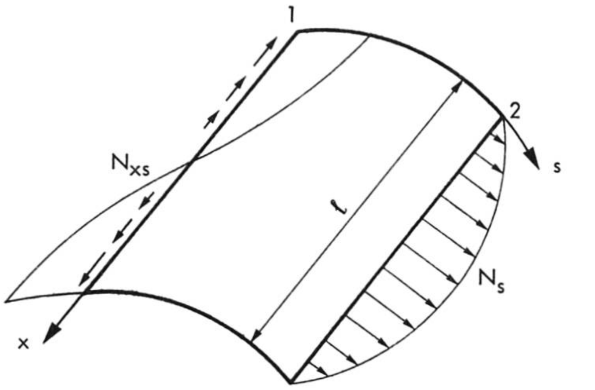

FIGURE 2 LONGITUDINAL DISTRIBUTION OF HOOP AND SHEAR FORCES FOR THE FIRST SERIES TERM

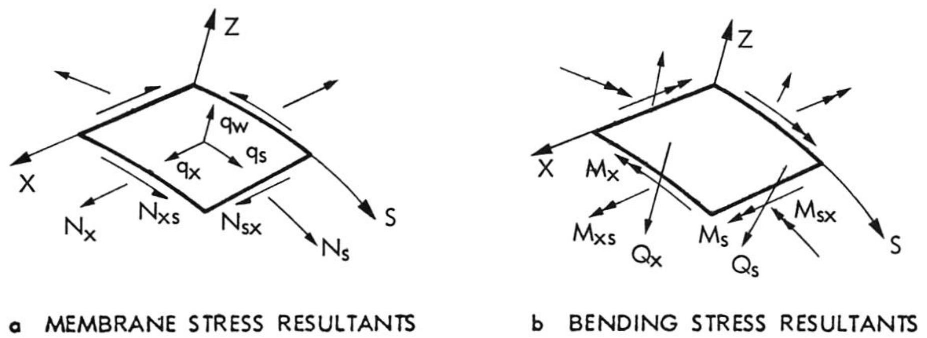

FIGURE 3 STRESS RESULTANTS IN SHELL THEORY-NOTATIONS AND SIGN CONVENTIONS

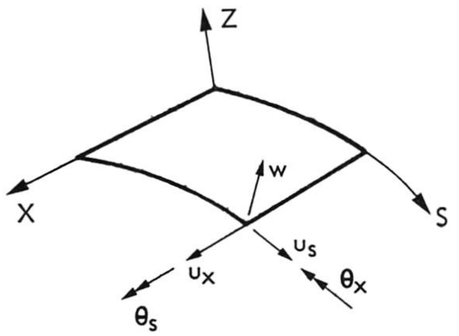

FIGURE 4 DISPLACEMENT COMPONENTS IN SHELL THEORY-NOTATIONS AND SIGN CONVENTIONS

In the matrix displacement method a relation is needed in the form

$$\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = k \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + N_0 \qquad (4)$$

It is seen that Eq. (4) is easily obtained from Eq. (3) by part-inversion.

For the best possible numerical accuracy, the coordinate s should be kept within certain limits in the computation of transfer matrices. For this purpose each shell element in the actual program is automatically divided into a number of shell segments. Stiffness and load matrices are computed for each segment according to the procedure described above, and the segments are assembled to a shell element by a standard stiffness method.

Plate elements are included as a limiting case of a shell element. Details of the computation of G are given in [4].

5. BEAM ELEMENT MATRICES

The shell elements may be connected to beam elements along an arbitrary generator. The state of displacement of the beam is described by four displacement quantities as given for the shell in Eq. 1. These displacements and the corresponding stress resultants are expanded in trigonometric series in the same manner as the shell quantities. A stiffness relation is established for each term by using conventional bending theory and St. Venant's theory of torsion. The point of gravity is assumed to coincide with the shear centre. If the beam is of an open thinwalled type, the beam must be subdivided into plate elements or possibly beams or shells, in such a manner that each element satisfies the assumptions with sufficient accuracy.

The stiffness matrix of a beam element is a 4 by 4 matrix, and the displacement components and stress resultants entering this matrix are referred to the centre of gravity k, see Figs. 5 and 6. S_x is a shear force applied at the centre line. It has a cosine distribution as shown for the corresponding shell quantity in Fig. 2, and is positive when it tends to elongate the beam. Those interested in the detailed appearance of stiffness and load matrices should study Ref. [4].

6. TRANSFORMATION OF SHELL ELEMENT MATRICES

As the unknowns of the system the 4 displacement components of each beam are chosen. Thus the beam centres are defined as the nodes. Locally, the y- and z-axes are with advantage taken in the direction of the principal axes of the beam. The method as such allows that these axes are kept unchanged in the stiffness assembly and final solution, but it was found convenient in the program to use a common global x-y-z frame of reference. Thus all beam stiffnesses are transformed to this system.

The four displacement components of the centre line of the beam define uniquely the corresponding displacement components of an arbitrary generator of the beam when the assumptions of technical bending and St. Venant torsion are used. Thus the centre of gravity, the node, may be considered as master, and all other points in the cross section as slaves, that follow the master, see Figs. 5 and 6. The stress and displacement components of a node k and a slave point k_i are

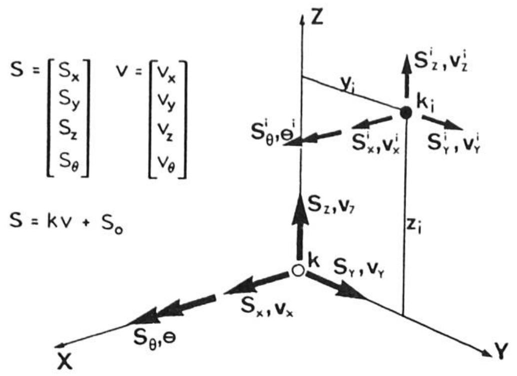

FIGURE 5 FORCES AND DISPLACEMENTS AT NODAL POINT (k) AND SLAVE POINT (k_i)

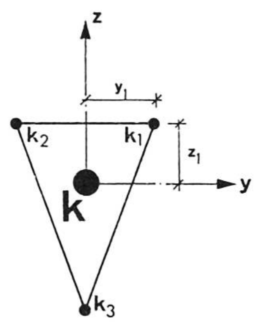

FIGURE 6 BEAM ELEMENT WITH SLAVE POINTS $(k_1,\ k_2$ AND $k_3)$

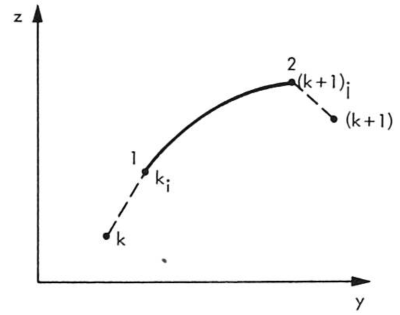

FIGURE 7 ECCENTRICALLY CONNECTED SHELL ELEMENT

shown in Fig. 5. The transformation of displacements may be written

$$v_{k_i} = T_{k_i} v_k \quad (5)$$

Provided that the longitudinal distributions of displacements are

$$v_x \cos\lambda x \quad , \quad v_y \sin\lambda x \quad , \quad v_z \sin\lambda x \quad , \quad \theta \sin\lambda x$$

the transformation written out in full is

$$\begin{bmatrix} v_x \\ v_y \\ v_z \\ \theta \end{bmatrix}_{k_i} = \begin{bmatrix} 1 & -y_i\lambda & -z_i\lambda & 0 \\ 0 & 1 & 0 & -z_i \\ 0 & 0 & 1 & y_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \\ \theta \end{bmatrix}_k \quad (6)$$

The particular important aspect of this relation is the coupling between the displacements v_y and v_z of the node, and the displacement v_x of the slave. This is easily understood when the displacement component v_x is interpreted as a description of longitudinal strain. The other transformations needed are obtained by transposing and inverting Eq. (6).

The shell elements are generally connected to slave points. In the stiffness and load assembly, however, all quantities must be referenced to the nodes. Thus, the shell elements are considered as being eccentrically connected to the nodes. Fig. 7 shows a shell element eccentrically connected to two nodes k and (k+1). Suppose that a stiffness relation of the form (4) has been found for the shell element, and that this relation has been transformed to the global y-z system already. Denoting now element quantities related to the local numbering 1-2 by index L and quantities related to the nodes k and (k+1) by index G, the following relations apply

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_L = \begin{bmatrix} T_{k_i} & 0 \\ 0 & T_{(k+1)_j} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_G \quad (7)$$

$$\begin{bmatrix} S_1 \\ S_2 \end{bmatrix}_G = \begin{bmatrix} T^T_{k_i} & 0 \\ 0 & T^T_{(k+1)_j} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}_L \quad (8)$$

from which a global relation

$$S_G = k_G v_G + S_{oG} \quad (9)$$

is found. The approach of introducing only the beam centres as nodes and transforming shell element matrices to these node parameters is one of the decisive features of the method. It limits the number of unknowns to a minimum, allows a simple and systematic programming, and takes care of all linear relationships between the displacements of different points in a beam *ab initio*.

7. LOAD REPRESENTATION

As the analysis is based on a Fourier expansion longitudinally and a numerical integration circumferentially, all loads can in principle be handled. In the program, however, the longitudinal distribution has been restricted to a piecewise linear variation, that in the program is obtained by superposition of linear cases extending over part of the structure as shown in Fig. 8. This distribution applies for load components in as well x-,y- and z-directions, with the additional restriction that the resultant in the x-direction must be zero.

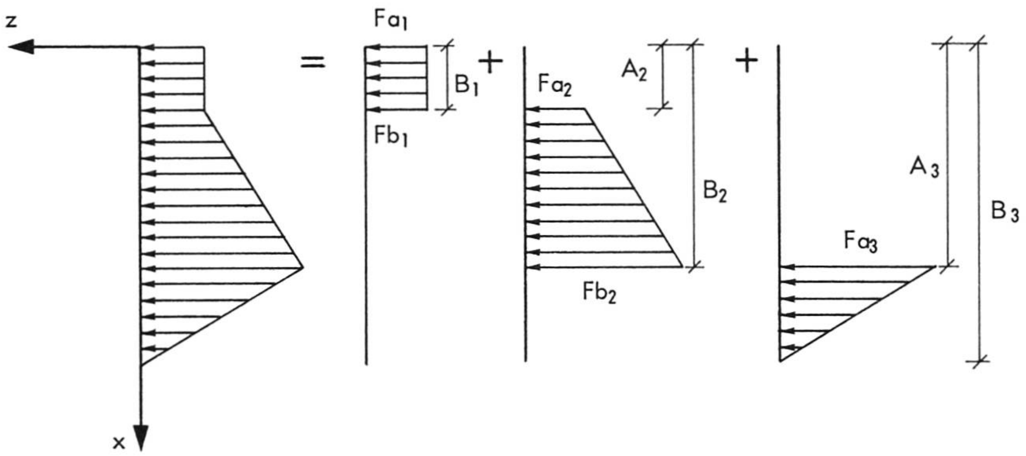

FIGURE 8 SUPERPOSITION OF LOAD DISTRIBUTIONS

The load on the shell elements is supposed to be constant in the circumferential direction within each segment. A rapid variation in the s-direction must therefore be reflected by narrow segments. All beam loads are line loads, including line moments about the x-axis.

8. NUMERICAL EXAMPLE

As a numerical example consider the analysis of a cellular platform raft of the type shown in Fig. 1. The load is supposed to be radial external pressure with a slight variation over the height as shown in Fig. 9.

Due to the symmetry it is sufficient to analyze 1/12 of the total section, see Figs. 10 and 11.

The program assumes a beam element to be associated with every nodal point. In node 9, where only two shell elements meet, the beam must be assigned zero stiffness. This node has been introduced to reduce the number of element types.

A more refined model was used in the numerical results included. In this model fictitious shell elements were introduced between the shell and beam elements to simulate the flexibility of the connection between shell and beam elements.

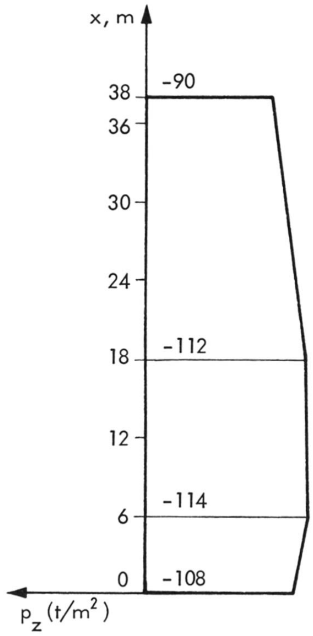

FIGURE 9 DISTRIBUTION OF RADIAL LOAD ON CELL WALLS

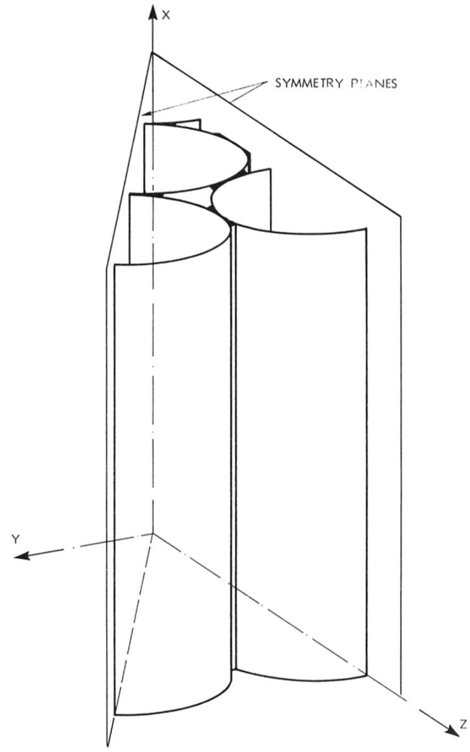

FIGURE 10 EXAMPLE STRUCTURE

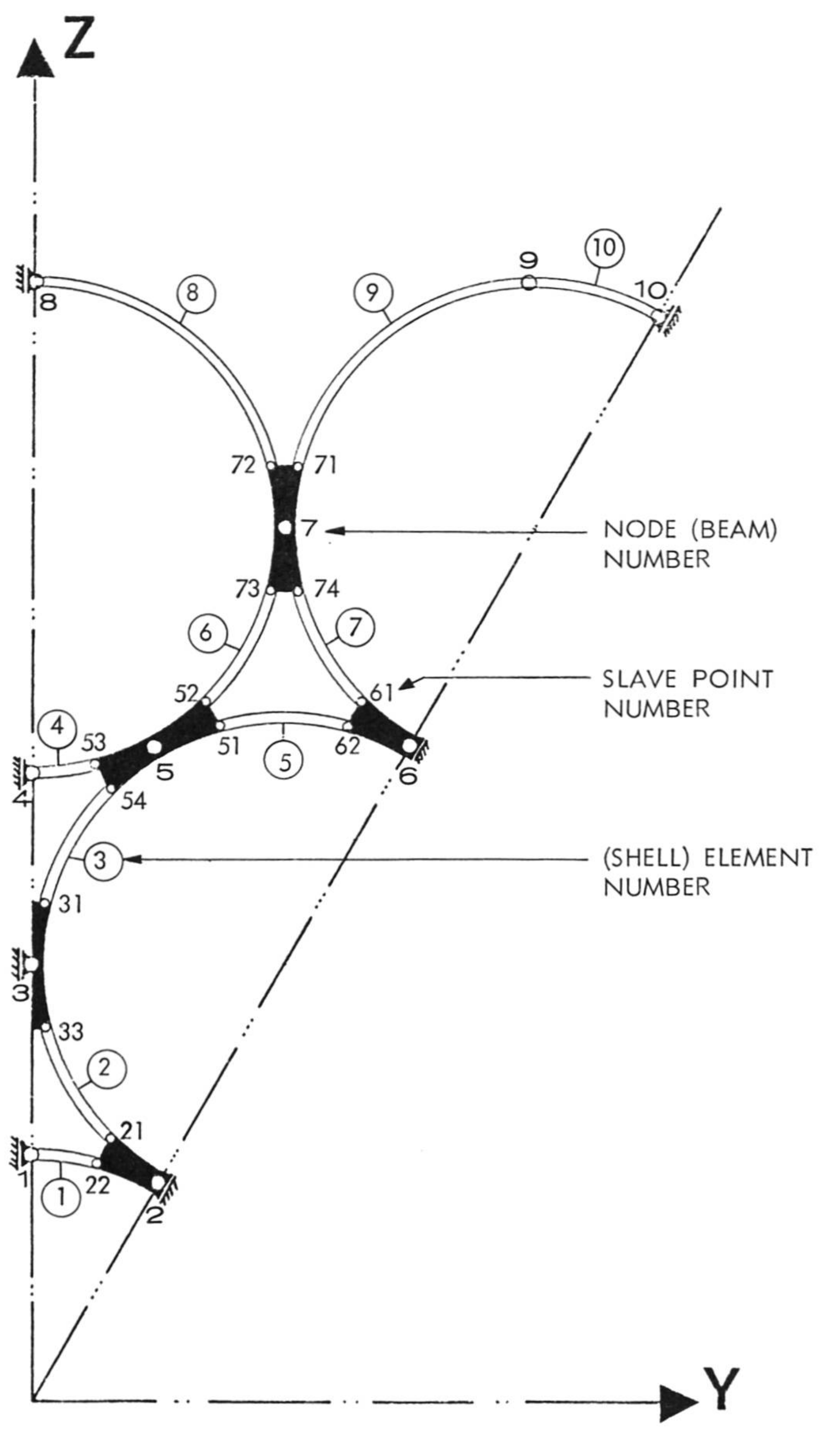

FIGURE 11 MODELLING OF THE EXAMPLE STRUCTURE IN FIG. 12

Characteristic data for the two models are given in Table 1. The example shows that the number of degrees of freedom is kept small, even for a very complex structure. This is also reflected in the computing time, that was 30 s CPU-time for an analysis with 15 Fourier terms for the refined model.

TABLE 1 CHARACTERISTIC DATA OF RAFT ANALYSIS

MODEL	FIG. 11	REFINED
SHELL SEGMENT TYPES	1	2
SHELL ELEMENT TYPES	3	4
SHELL ELEMENTS	10	24
BEAM ELEMENT TYPES	3	3
BEAM ELEMENTS (NODES)	10	24
DEGREES OF FREEDOM	26	88

Typical results from the analysis are shown in Figs. 12, 13 and 14, showing direct forces, moments and shear forces in a ring section at midheight, 19 m above the bottom. The more simple model of Fig. 11 showed an increase of peak moments by about 5 per cent, whereas the differences in hoop forces were negligible.

A typical feature of the results is the pronounced deviation from the membrane action. The narrow shell elements 2-7 (see Fig. 11) behave like clamped arches with large negative and positive bending moments. The wider shell elements 8 and 9 have a moment distribution similar to that which is known for short shell roofs. The rigid connection between the cells prevents a development of the full membrane hoop force, except in the mid-portion of the wide shells.

Figs. 12, 13 and 14 also include results from a finite element analysis accomplished by CDC Data Centers in Stockholm and Oslo, on behalf of A.S. Höyer-Ellefsen, Oslo. The results are included here by courtecy of A.S. Höyer-Ellefsen. For this analysis the MSC/NASTRAN program system was used, adapting quadrilateral shell elements, beam and spring elements and rigid connections. The model included the complete structure, that was of the type shown in Fig. 1.

As a whole, the results obtained by the two completely different models agree remarkably well. Some discrepencies appear in the narrow shell panels in the interior part of the raft (elements 2, 3 and 4, Fig. 11). Fig. 12 shows that larger hoop forces are developed in these shells according to the finite element model that according to the shell model. The difference can be explained by a compression of the upper and lower dome systems (see Fig. 1) in the finite element model. This effect is not accounted for in the shell analysis, assuming plane end sections that are rigid in their plane. A correction for this effect is, however, possible, if desired.

An advantage of the shell model is that it describes the gradients of moments and shear forces in the boundary areas properly.

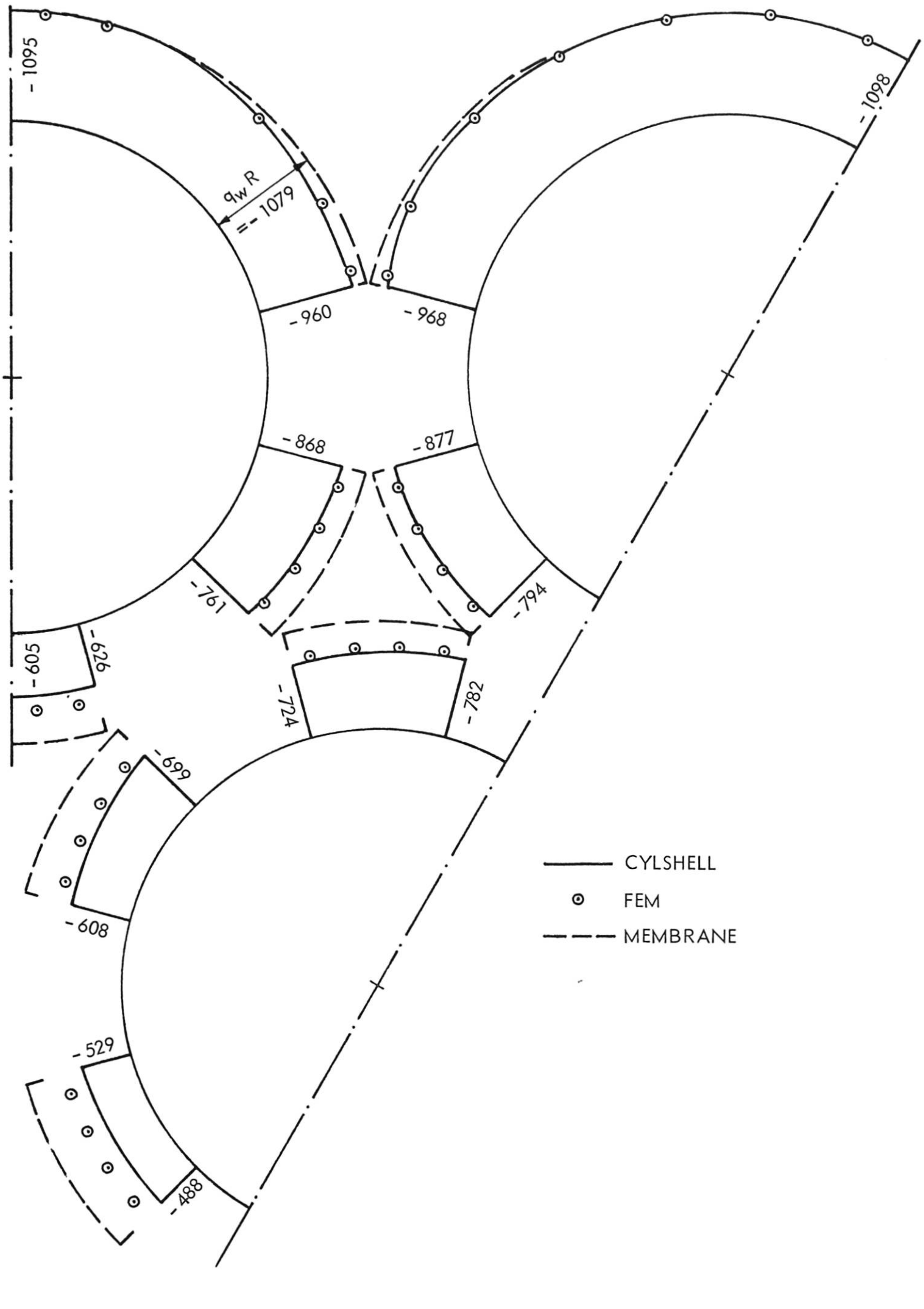

FIGURE 12 HOOP FORCES N_s (Mp/m) 19m ABOVE THE BOTTOM

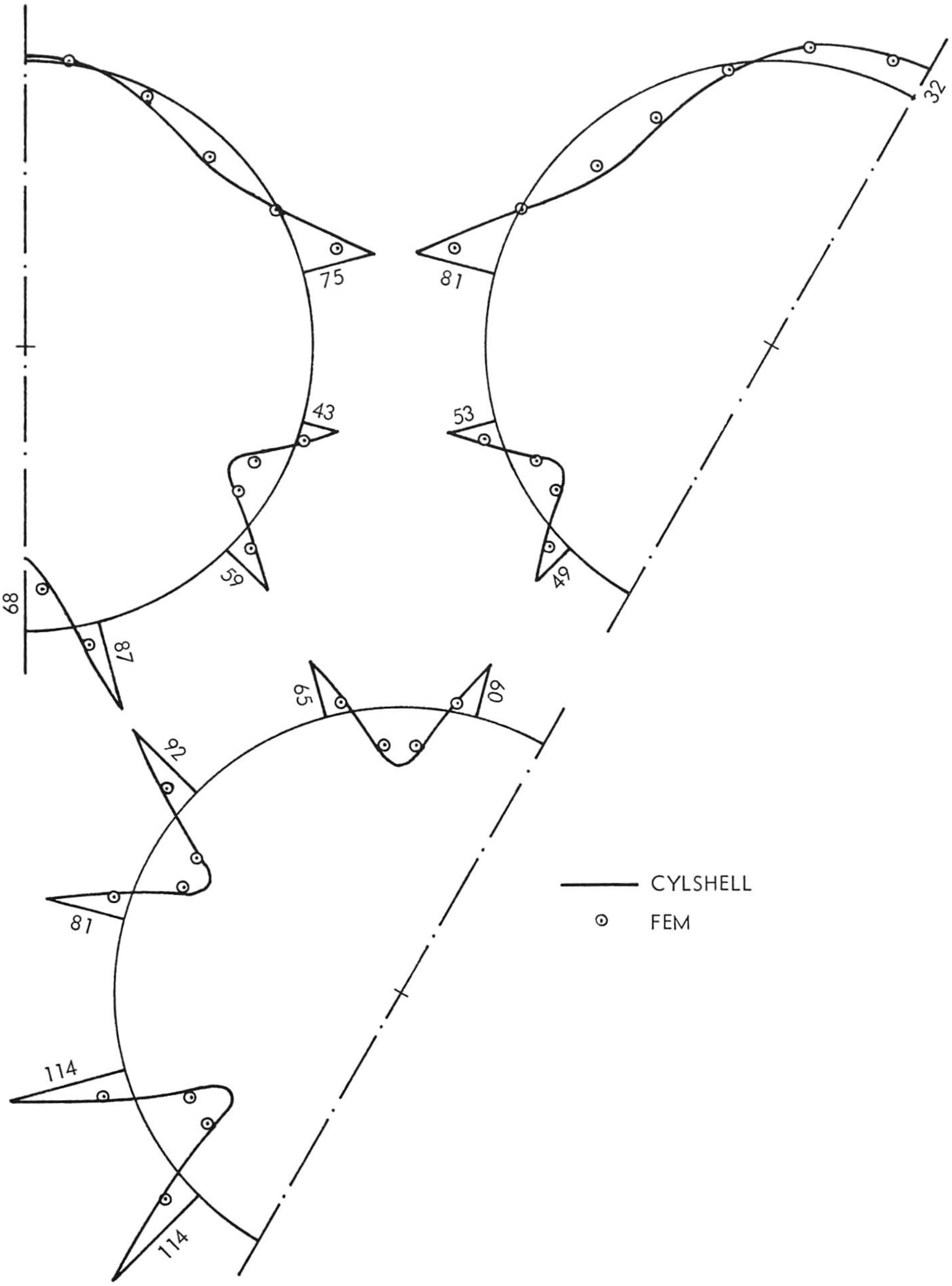

FIGURE 13 HOOP MOMENTS M_s (Mp) 19m ABOVE THE BOTTOM

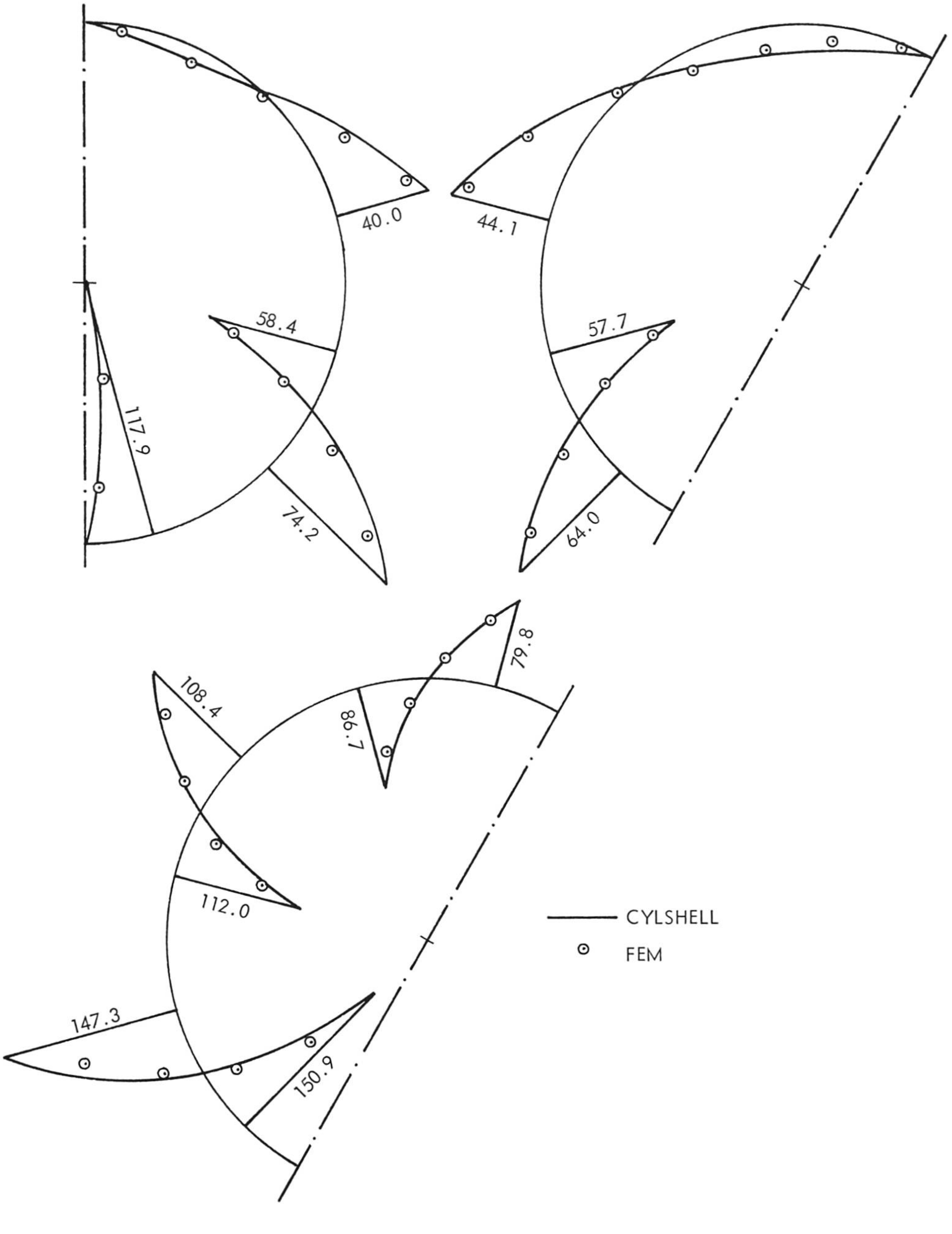

FIGURE 14 SHEAR FORCES Q_s (Mp/m) 19m ABOVE THE BOTTOM

9. POST-PROCESSING

The results of an analysis of this type should be presented graphically to facilitate the interpretation of the results. For this purpose a post-processor has been written [6], presenting the different quantities computed as contour maps.

10. REFERENCES

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