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Rational Analysis of Shear in Reinforced Concrete Beams

Analyse d'effort tranchant dans les poutres en béton armé

Plastische Berechnung der Schubtragfähigkeit von Stahlbetonbalken

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SUMMARY
The shear strength of beams is analysed by the truss analogy with variable strut inclination. The web crushing criterion is derived as a solution satisfying equilibrium. If the materials are assumed to be perfectly plastic, the web crushing criterion is also an upper bound,corresponding to a failure mechanism with vertical deformations only. The exact plastic solution is also given for beams without shear reinforcement. The solutions are compared with experimental evidence and with the design rules of building codes, particularly the CEB Model Code.

RÉSUMÉ
La résistance ultime des poutres à l'effort tranchant est étudiée à l'aide de l'analogie du treillis avec inclinaison variable des bielles. Un critère d'écrasement de l'âme est dérivé comme solution satisfaisant l'équilibre. Si on suppose un comportement parfaitement plastique des matériaux, ce critère est aussi une borne supérieure, correspondant à un mécanisme d'écoulement aux déformations verticales seulement. La solution exacte est aussi donnée pour des poutres sans armature d'âme. Les solutions sont comparées avec des résultats d'essais et avec différentes normes, particulièrement avec le Code Modèle du CEB.

ZUSAMMENFASSUNG
Die Schubtragfähigkeit von Balken wird mit der Fachwerkanalogie mit variabler Neigung der Druckstreben berechnet. Ein Stegbruchkriterium wird als Gleichgewichtslösung hergeleitet. Für ideal plastisches Materialverhalten entspricht dem Stegbruchkriterium ein Bruchmechanismus mit vertikalen Verschiebungen allein und führt zu einer oberen Grenze der Tragfähigkeit. Die genaue Lösung wird auch für Balken ohne Schubbewehrung angegeben. Die Lösungen werden mit Versuchsergebnissen und mit Bemessungsvorschriften verglichen, insbesondere mit der CEB Mustervorschrift.

1. INTRODUCTION

Since extensive use of reinforced concrete as a building material started in the last century, virtually all problems concerning the bending of reinforced concrete have been solved. In contrast, the design of reinforced concrete beams with respect to shear rests on a shaky theoretical basis. Consequently, most codes of practice are very conservative in their requirements to shear reinforcement.

In the present paper, it is intended to review the considerations lying behind the building codes and to discuss a more realistic calculation of the shear strength of reinforced concrete beams.

The purpose of a beam is to transfer a load from its point of application to the support. This transfer causes diagonal tension cracks in the concrete, and unless the load is close to the support (compared with the beam depth) this means that the load will rest on the longitudinal reinforcement. If no countermeasures are taken, the reinforcing bars will be torn out of the concrete, and we get the type of failure shown on Fig.1.

A diagonal crack runs from the load to the reinforcement and then splits the beam along the reinforcing bars. This diagonal tension failure should be avoided for two reasons. Firstly, it may occur at a load which is considerably lower than the flexural capacity of the beam. Secondly, it is a sudden failure which may cause disastrous collapse.

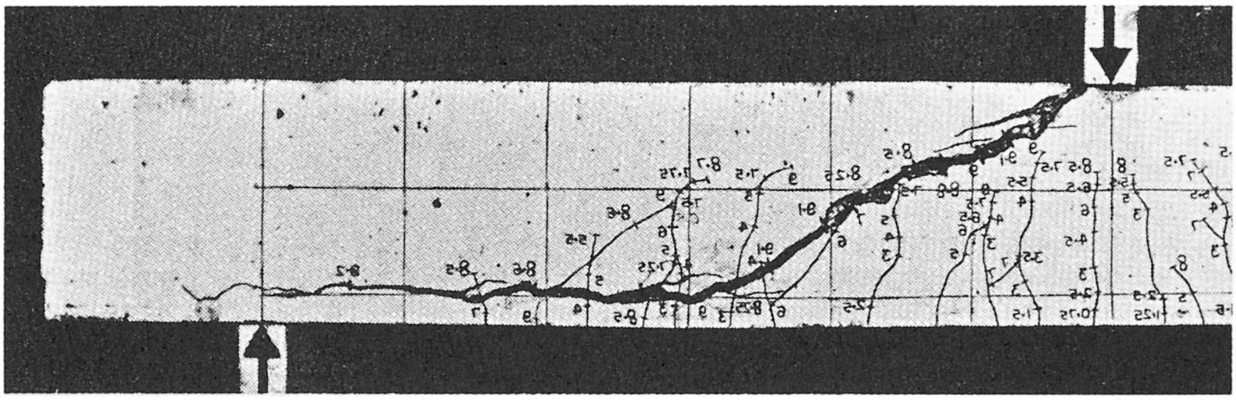

Fig. 1 Diagonal tension failure of beam without web reinforcement (reproduced from TAYLOR [32])

Diagonal tension failure may be prevented if the longitudinal bars are supported by an additional web reinforcement. Usually this shear reinforcement consists of closed stirrups encasing the longitudinal reinforcement and bent around the top bars or otherwise anchored in the compression zone.

Since the turn of the century, the action of the web reinforcement has been studied by hundreds of shear tests and dozens of theoretical investigations. Most of the latter are based upon the truss analogy, introduced by RITTER in 1899 and developed by MÖRSCH (cf. the historical study by HOGNESTAD [14]). A precise formulation of the truss analogy is given in Section 2.

A very important parameter of the analogy is the strut inclination, and the majority of present day building codes are based upon the truss model with 45° struts. It has been known for a long time (see e.g. CHAMBAUD [4], 1957) that this inclination is not the one observed at shear failure of beams with web reinforcement.

That it is not valid in the elastic state either, was shown in 1964 by KUPFER [15], who applied the principle of minimum complementary energy. In 1976 GROB & THÜRLIMANN [10] suggested a truss model with variable strut inclination, introducing limits for the truss angle based upon kinematic considerations.

A realistic strut inclination at ultimate load may be determined by plastic analysis. In 1964, NIELSEN [20] considered reinforced concrete members in a state of plane stress and derived formulas for the stresses in reinforcement and concrete. The same expressions were used in the paper by GROB & THÜRLIMANN [10]. The formulas are valid when both longitudinal and web reinforcement are yielding, and the strut inclination is determined by the relative strengths of the two types of reinforcement. Applying the theory to beams in shear, NIELSEN [21] in 1967 determined the strut inclination which corresponds to minimum volume of total reinforcement. In a subsequent discussion, NIELSEN [22] gave the strut angle when the shear resistence is determined by the compressive strength of the concrete. A similar equation had been proposed a decade earlier by CHAMBAUD [5]. In Section 3, the optimal strut inclination and the corresponding shear capacity are deduced from simple engineering concepts.

The ultimate load may also be determined by considering the mechanism of beams failing in shear. Most attempts in this direction have been based upon shear compression failure, where the beam end is rotating about a hinge in the compression zone (cf. the review in reference [23]). In 1975, NIELSEN & BRÆSTRUP [23] considered a pure shearing mechanism, without any rotation of the beam end. It was found that the corresponding ultimate load coincided with NIELSEN's lower bound corresponding to web concrete failure. This formula for the shear strength is termed the web crushing criterion. The failure mechanism and the upper bound solution are briefly reviewed in Section 4.

In Section 5, the web crushing criterion is compared with available test results, and the agreement is found to be reasonable. Furthermore, the theory gives a rational explanation of the phenomena observed at shear failure, which is applicable not only to beam shear, but also to shear in walls, corbels and joints, punching shear of slabs, etc. (cf. NIELSEN et al. [24], BRÆSTRUP et al. [3]).

The theory of plasticity may also be used to predict the shear strength of beams without web reinforcement. These results are summarized in Section 6.

Finally, in Section 7, the theoretical and experimental results are compared with the design rules of the Danish building code and the Model Code proposed by CEB (Comité Euro-International du Béton).

2. THE TRUSS MODEL

A simple way of visualizing the effect of the web reinforcement is by regarding the beam as a plane truss. The longitudinal bars and the stirrups (vertical or inclined) constitute the tension members. The compression members are formed by the concrete in the top chord and the web. The web width is termed b and the inclination of the stirrups is α. We introduce the geometrical ratio of shear reinforcement as:

$$\rho = \frac{A_s}{bc \sin\alpha} ,$$

where A_s is the cross-sectional steel area per stirrup and c is the stirrup spacing along the beam axis.

The truss analogy is given a precise formulation through the assumptions:

(a) The reinforcing bars are unable to resist lateral forces. The steel stress in the stirrups is σ_a. The compression zone and the longitudinal reinforcement act as stringers with a compressive force C and a tensile force T, respectively.

(b) The action of the stirrups is described by an equivalent stirrups stress $\rho\sigma_a$ per unit area perpendicular to the stirrup direction.

(c) The concrete of the web is in a state of uniaxial compression, the compressive stress σ_b being inclined at the angle θ to the beam axis.

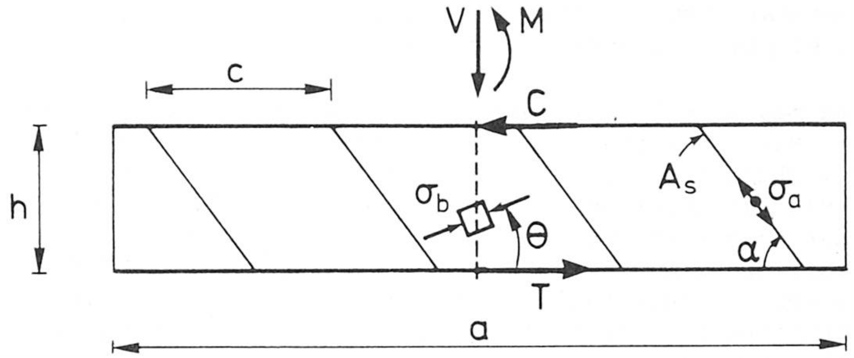

Fig. 2 Truss model of reinforced concrete beam

Assumption (a) expresses that we neglect dowel action of the reinforcement and shear in the compression zone. The meaning of assumption (b) is that the spacing of the stirrups (longitudinally and transversely) is required to be sufficiently small to permit a description of their action as continuously distributed over a section perpendicular to the stirrups. Assumption (c) implies that the individual struts of the truss model are replaced by a diagonal compression field.

The mathematical model, taken to represent the beam, is shown on Fig.2. The beam depth h is defined as the distance between the compression and the tension stringer. For simplicity, we consider a part of a beam, the shear span a, with constant shear force.

A section of the beam is subjected to the shear force V and the bending moment M. Using the truss model, we find the equilibrium equations:

$$V = \sigma_b\, bh\cos\theta\, \sin\theta + \rho\sigma_a bh\cos\alpha\, \sin\alpha \tag{1}$$

$$M = hT - \frac{1}{2}\sigma_b\, bh^2\cos^2\theta + \frac{1}{2}\rho\sigma_a\, bh^2\, \cos^2\alpha \tag{2}$$

The condition that the stress be zero in a horizontal section leads to the relation:

$$\sigma_b\, \sin^2\theta = \rho\sigma_a\, \sin^2\alpha \tag{3}$$

Inserting (3) into (1) and (2), we find:

$$V = \rho\sigma_a bh \sin^2\alpha(\cot\theta + \cot\alpha) \tag{4}$$

and

$$M = hT - \frac{1}{2}\rho\sigma_a bh^2 \sin^2\alpha(\cot^2\theta - \cot^2\alpha)$$

or

$$M = h[T - \frac{1}{2}V(\cot\theta - \cot\alpha)] \tag{5}$$

3. EQUILIBRIUM ANALYSIS

From the equilibrium equations, the load-carrying capacity of the beam may be derived if we introduce the material strength parameters. Thus we add the assumptions:

(d) The yield strength of the tensile stringer is $T = T_y$. The yield stress of the stirrups is $\sigma_a = f_y$.

(e) The crushing strength of the web concrete is $\sigma_b = \nu f_c$ where f_c is the cylinder strength and ν is a web effectiveness factor.

The beam is assumed not to be overreinforced in flexure, therefore the strength of the compression stringer is immaterial.

The effectiveness factor ν is introduced to account for the limited ductility of the concrete.

With a fixed strut inclination θ, the shear strength is given by equation (4) with $\sigma_a = f_y$:

$$V = bh\rho f_y(\cot\theta + \cot\alpha)\sin^2\alpha \tag{7}$$

Equation (7) is valid as long as the concrete strength of the web is not exceeded. By equation (3), this requires

$$\rho\sigma_a \leq \frac{\sin^2\theta}{\sin^2\alpha}\nu f_c$$

Inserting into equation (7), we find the strength limit imposed by the web concrete:

$$V \leq bh\nu f_c(\cot\theta + \cot\alpha)\sin^2\theta \tag{8}$$

There is no reason to believe, however, that the strut inclination should remain constant. A generally accepted principle of mechanics states that the internal forces of a structure accomodate themselves to carry the maximum load. In the theory of plasticity, this principle is formalized as the lower bound theorem. From equation (7), we note that the flatter the concrete compression, the higher the shear force. Thus, if the ductility of the beam is sufficient, the web stresses will be redistributed in such a way that the strut inclination decreases with increasing load. This effect is indeed observed during beam tests (cf. reference [2]). However, equation (3) imposes a lower limit on the strut inclination, i.e. an upper limit on the shear resistance. Eliminating θ between equation (3) and (4), we get:

$$V = bh\sqrt{\rho\sigma_a \sin^2\alpha(\sigma_b - \rho\sigma_a \sin^2\alpha)} + bh\rho\sigma_a \cos\alpha \sin\alpha \qquad (9)$$

By equation (9), V is an increasing function of σ_b, hence the maximum shear load is obtained at crushing of the concrete, $\sigma_b = \nu f_c$. Also V is an increasing function of $\rho\sigma_a$, as long as

$$\rho\sigma_a \leq \frac{1}{2}\frac{1 + \cos\alpha}{\sin^2\alpha}\nu f_c = \rho_1\sigma_a \qquad (10)$$

hence the maximum shear load is obtained with yielding of the stirrups, $\rho\sigma_a = \rho f_y$. Inserting into equation (9), we find the shear resistance as a function of the material strength parameters:

$$V = bh\sqrt{\rho f_y \sin^2\alpha(\nu f_c - \rho f_y \sin^2\alpha)} + bh\rho f_y \cos\alpha \sin\alpha \qquad (11a)$$

valid for $\rho f_y \leq \rho_1\sigma_a$

For $\rho f_y > \rho_1\sigma_a$, the maximum shear load is obtained with $\rho\sigma_a = \rho_1\sigma_a$, i.e. the stirrups do not yield at failure of the concrete. By equation (9), the shear strength is then:

$$V = \frac{1}{2}bh\nu f_c \cot\frac{\alpha}{2} \qquad (11b)$$

valid for $\rho f_y \leq \rho_1\sigma_a$

Equations (11) constitute the web crushing criterion. It gives the maximum shear force that can be carried by a particular concrete section. With a given shear reinforcement strength ρf_y, the optimal strut inclination is the one corresponding to failure of the web concrete. This value, $\theta = \theta_F$, is found from equation (3) with $\sigma_b = \nu f_c$ and $\rho\sigma_a = \rho f_y$ for $\rho f_y \leq \rho_1\sigma_a$ and $\rho\sigma_a = \rho_1\sigma_a$ for $\rho f_y \geq \rho_1\sigma_a$.

Thus we get:

$$\cot\theta_F = \sqrt{\frac{\nu f_c}{\rho f_y \sin^2\alpha} - 1} \qquad (12a)$$

valid for $\rho f_y \leq \rho_1\sigma_a$

and

$$\cot\theta_F = \tan\frac{\alpha}{2} \qquad (12b)$$

valid for $\rho f_y \geq \rho_1\sigma_a$

If the beam is to achieve the maximum shear resistance given by the web crushing criterion, then it is a necessary condition that the tension stringer be sufficiently strong. By equation (5), this requires:

$$T_y \geq \frac{M}{h} + \frac{1}{2}V(\cot\theta - \cot\alpha) \qquad (13)$$

Thus the tension stringer must be designed for a force which is greater than the pure bending term M/h. In particular, we note that a stringer force must be anchored at a simple support, where M = 0. If the longitudinal reinforcement is curtailed or insufficiently anchored, then equation (13) may impose a lower limit on the admissible strut inclination, and hence an upper limit on the shear strength. Equation (13) does not apply at the maximum moment, because the diagonal compression field, used in deriving equation (5), is not valid (except possibly for indirect loading). At point loads and supports, the stress distribution must be modified, cf. NIELSEN [21] or NIELSEN & BRÆSTRUP [23].

For weak shear reinforcement ($\rho f_y \ll 1$), the strut inclination given by equation (12a) becomes very flat, and the diagonal compression field degenerates to a single strut running from the load to the support. The same happens with deep beams. Also in this situation, a lower bound solution can be derived. A particularly simple case is formed by beams without shear reinforcement, considered in Section 6.

4. FAILURE MECHANISM

The upper bound method of the theory of plasticity may be used to determine an estimate of the ultimate load. However, in order to carry out a rigorous upper bound analysis, we must assume plastic properties of the materials. Thus we introduce the additional assumptions:

f) The stringers and the stirrups are rigid, perfectly plastic. The yield strengths are given by assumption d).

g) The web concrete is rigid, perfectly plastic with the square yield condition for plane stress and the associated flow rule. The tensile strength is zero and the compressive strength is νf_c.

These assumptions mean that the elastic deformations are neglected in the analysis. The yield locus for concrete in plane stress is shown on Fig. 3. It is identical to the modified Coulomb failure criterion with a zero tensile cut-off. σ_1 and σ_2 are the principal stresses and the concrete is unable to resist stress combinations outside the square locus. The associated flow rule means that when the stress point is on the yield locus, then the ratio between the possible strain rates ε_1 and ε_2 is such that the vector $(\varepsilon_1, \varepsilon_2)$ is an outwards directed normal to the locus at the stress point. At the corners, the vector $(\varepsilon_1, \varepsilon_2)$ is situated between the adjacent normals.

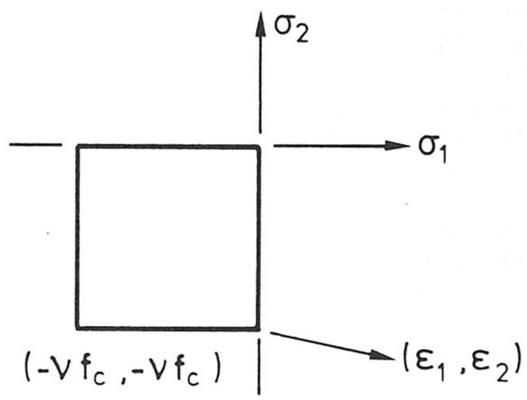

Fig. 3 Yield locus for concrete in plane stress

A possible shear failure mechanism is shown on Fig. 4. The deformations are taking place in yield lines at the inclination β, forming a parallelogram-shaped deformation zone. For comparison, Fig. 5 shows a photograph of a test beam after failure. Note the absence of any rotation of the beam end, and the tensile cracks in the flange near the support which indicate that a yield zone has been formed.

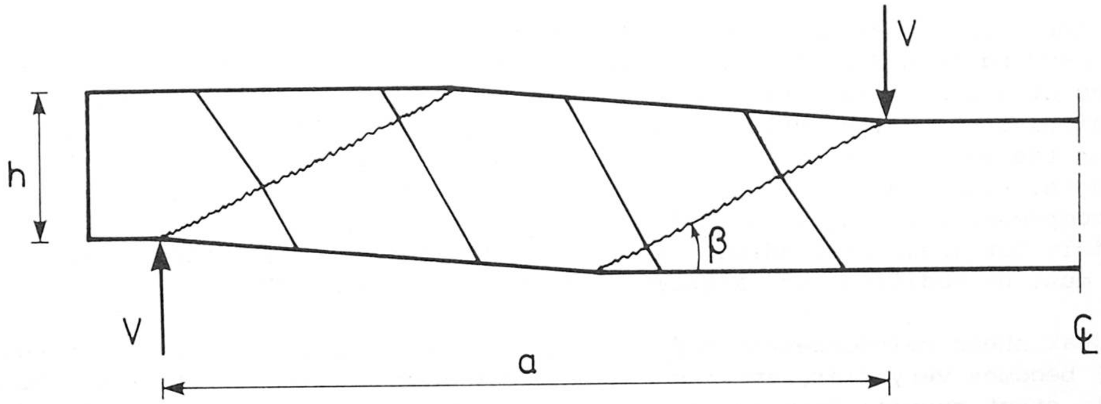

Fig. 4 Shear failure mechanism for reinforced concrete beam

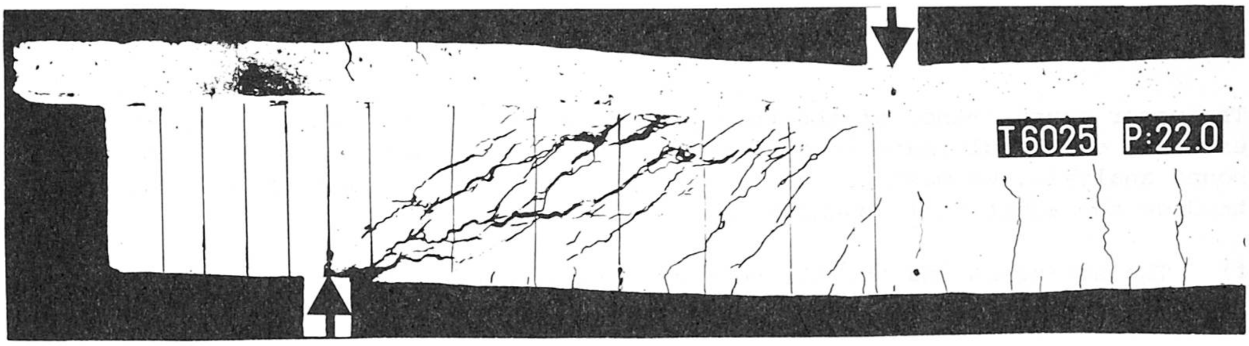

Fig. 5 Shear failure of beam with web reinforcement (BRÆSTRUP et al [2])

Using assumptions f) and g), the rate of internal work dissipated in the failure mechanism is calculated. An upper bound for the ultimate shear force is found by equating the rate of internal work to the rate of external work done by the load. The lowest upper bound is determined by minimizing with respect to the yield line inclination β. As shown by NIELSEN & BRÆSTRUP [23], the result is identical with the web crushing criterion, equations (11). Since this solution is also a lower bound, it is in fact the complete solution corresponding to the assumptions made.

In the failure mechanism giving the lowest upper bound, the inclination $\beta = \beta_F$ of the yield lines is:

$$\beta_F = 2\theta_F \tag{14}$$

where θ_F is the strut inclination given by equations (12), corresponding to failure of the web concrete. The fact that strut inclination is different from yield line inclination, means that shear stresses are transferred in the yield lines (possibly by aggregate interlock). The situation is similar to a compressed concrete cylinder failing along an inclined plane.

For deep beams and for beams with weak shear reinforcement, the yield line inclination given by equation (14) is too flat to be geometrically possible, and the deformation zone of Fig. 4 degenerates into a single yield line running from the load to the support. It is a simple matter to calculate the upper bound in this case (cf. NIELSEN & BRÆSTRUP [23]), but for slender beams the (safe) approximation of the web crushing criterion remains adequate. An exception is formed by beams without shear reinforcement, treated in Section 6.

Beams with weak longitudinal reinforcement will usually fail in flexure. However, deep beams and beams with few stirrups may get a shear failure involving yielding of the main reinforcing bars. A general treatment of this case is outside the scope of the present paper (cf. NIELSEN et al. [24], BRÆSTRUP et al. [3]), but in Section 6 on beams without stirrups, the influence of longitudinal reinforcement is taken into account.

5. APPLICATION OF THE THEORY

The formulas for the shear resistance of reinforced concrete beams are visualized on Fig. 6 in the case of vertical stirrups ($\alpha = 90^{\circ}$). The non-dimensional shear strength V/bhf_c is shown as a function of the mechanical degree of shear reinforcement $\rho f_y/f_c$. Equation (7) corresponds to a straight line with the inclination $\cot\theta$. The web crushing criterion, equations (11), is represented by a quarter-circle with diameter ν and centre at $(\nu/2,\ 0)$, plus the horizontal tangent.

Suppose we have chosen a fixed strut inclination θ. The shear strength as a function of the stirrup reinforcement is then given by equation (7), until it reaches the limit determined by equation (8) and represented by the circle on Fig. 6. Then the shear capacity can be increased no further, unless greater dimensions or stronger concrete are prescribed.

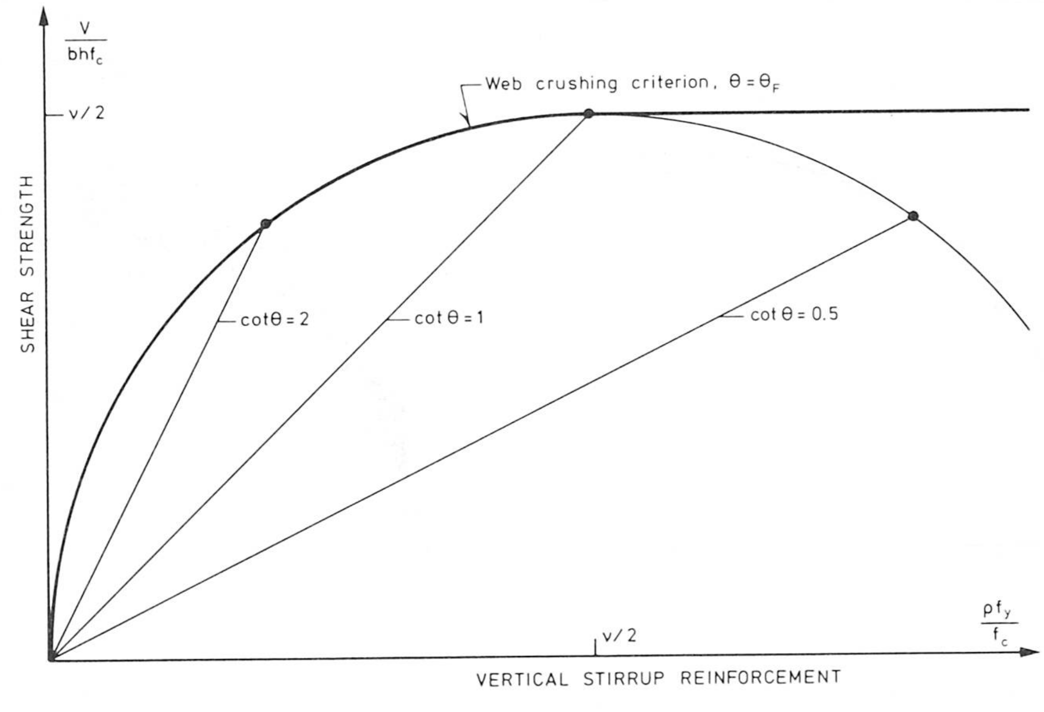

Fig. 6 Shear resistance of reinforced concrete beams (vertical stirrups)

It is more reasonable to assume that the strut inclination varies with the shear load. The most ecomomical inclination is $\theta = \theta_F$, corresponding to the web crushing criterion. It is determined by equation (12a), where the necessary stirrup reinforcement ρf_y is found from equation (11a), inserting the applied shear force V. This shear load must be inferior to the upper limit given by equation (11b).

For week shear reinforcement, the web crushing inclination θ_F is very small. Therefore the design may be unfeasible, due to the increase in tensile stringer force, as given by equation (13). Also, the stress distribution at failure will be very different from the one at service load, leading to unacceptable requirements to concrete ductility. For these reasons, it is advisable to impose a minimum strut inclination $\theta = \theta_{min} < 45^o$. This means that equation (7) with $\theta = \theta_{min}$ determines the shear strength until the limit set by equation (8). Then the shear strength is given by the web crushing criterion, equation (11a), with $\theta = \theta_F$ until the limit given by equation (11b). From that point the shear strength cannot be increased by adding more stirrup reinforcement.

In order to use the web crushing criterion for design, it is necessary to assess the values of the shear depth h and the effectiveness factor ν by correlation with experimental evidence. In doing so, it should be borne in mind that equations (11) represent the absolute maximum of the ultimate load. Thus it must be assured that the tensile stringer is sufficiently strong, and that the reinforcement is properly detailed, so as to exclude secondary failure causes.

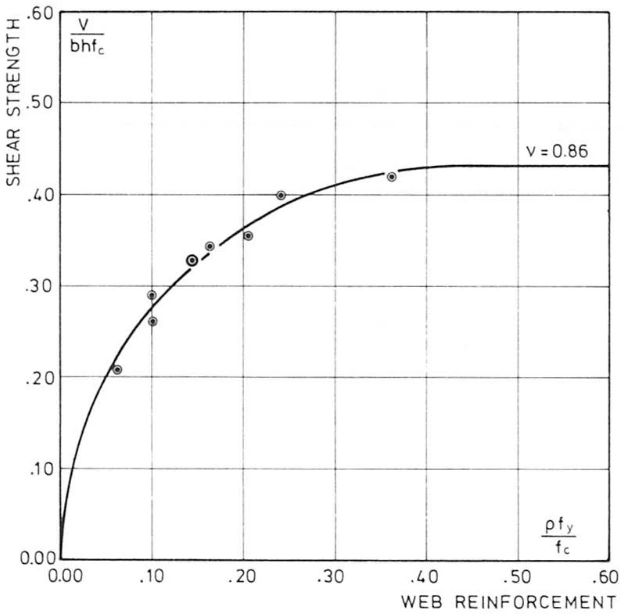

Fig. 7 Web crushing criterion compared with test results (LEONHARDT & WALTHER [17])

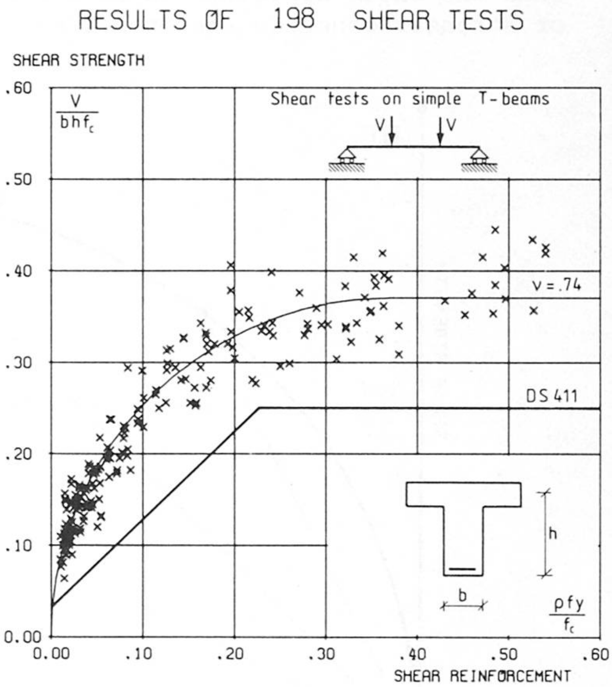

Fig. 8 Shear test results compared with web crushing criterion and Danish Code

On Fig. 7 are plotted some test results reported by LEONHARDT & WALTHER [17]. The series comprises 18 beams with vertical stirrups. In two of these, the main reinforcement was curtailed, and in three it was bent up. One beam had additional shear reinforcement in the form of horizontal bars. Of the remaining 12 beams, three have been omitted from the plot because they were reported to have failed in flexure. The non-dimensional shear strength is plotted against the shear reinforcement degree. As shear depth is used the internal moment lever arm z, calculated as the distance between the centroid of the main reinforcement and the centre of the compression flange. For comparison is shown the web crushing criterion with $\nu = 0.86$, which is the value giving closest fit by orthogonal regression. The coefficient of variation is 1.0%.

Fig. 8 shows the results of 198 shear tests on simply supported T-beams with vertical stirrups. 72 tests have been carried out at the Structural Research Laboratory (references [2] and [1]), while the remainder are reported in the literature (references [5],[11]-[13],[16]-[19],[25]-[28],[30],[31],[33]-[35]). In cases where the cylinder strength is not given, f_c is taken as 80% of the cube strength. Beams that are reported to have failed by flexure, bond failure, or flange shear have been omitted. The plot also excludes beams with bent-up bars, curtailed reinforcement, or tensile flange, as well as beams with no or very few stirrups ($\rho f_y \leq 0.01\ f_c$), insufficient longitudinal reinforcement ($T_y \leq 0.3\ bhf_c$), or short shear span ($a \leq 2.4d$).

A detailed documentation on the plot is available from the authors, who would also appreciate information about test series not included.

We would expect the web effectiveness ratio ν to depend upon various factors, principally the concrete ductility and the lay-out of the reinforcement. For the tests on Fig. 8, plotted as on Fig. 7, the best fit is optained with $\nu = 0.74$, the coefficient of variation being 3%. Thus for reasonably designed beams, the effectiveness factor appears to be fairly constant. There is a trend, however, of a decreasing web effectiveness with increasing concrete strength, although it is not as pronounced as for beams without web reinforcement, considered in the section below.

6. BEAMS WITHOUT SHEAR REINFORCEMENT

As mentioned in Section 1, beams without stirrups fail in shear by diagonal tension, without any apparent truss action (Fig. 1). At first glance, it would therefore not seem reasonable to apply the theory of plasticity, since there is no web reinforcement to assist with the necessary redistribution of stresses. It is very simple, however, to construct a statically admissible stress field consisting of a single strut between the load and the support (Fig. 9a). The shaded regions are in biaxial compression at the effective concrete strength νf_c, and it is assumed that the support conditions are such that the tensile stringer force T can be transferred to the concrete (sufficient anchorage). The corresponding highest lower bound is found to be (NIELSEN et al. [24]):

$$V = \frac{1}{2} bh\nu f_c \left(\sqrt{\left(\frac{a}{h}\right)^2 + \frac{4\Phi(\nu-\Phi)}{\nu^2}} - \frac{a}{h} \right) \qquad (15a)$$

valid for $\Phi \leq \frac{1}{2}\nu$

and

$$V = \frac{1}{2} bh\nu f_c \left(\sqrt{\left(\frac{a}{h}\right)^2 + 1} - \frac{a}{h} \right) \qquad (15b)$$

valid for $\Phi \geq \frac{1}{2}\nu$

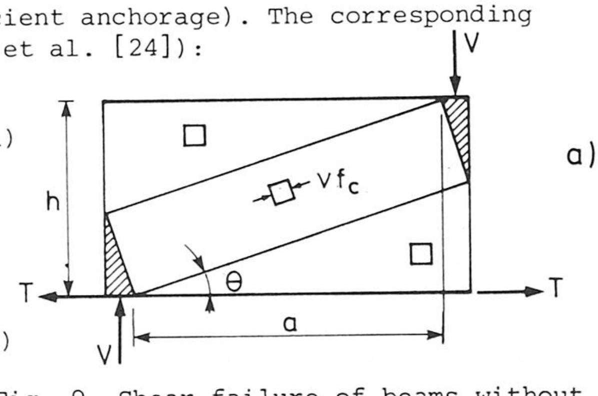

Fig. 9 Shear failure of beams without web reinforcement
a) Stress distribution

Here Φ is the degree of longitudinal reinforcement, defined as

$$\Phi = \frac{T_y}{bh\,f_c}$$

Equation (15b) was given by NIELSEN & BRAESTRUP [23].

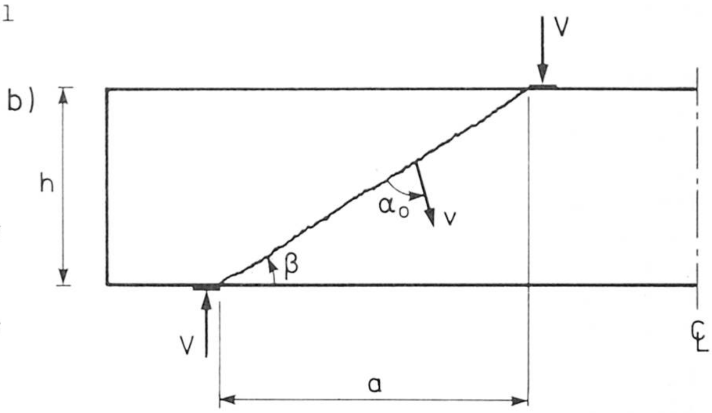

Fig. 9 Shear failure of beams without web reinforcement

b) Failure mechanism

An upper bound solution is found by assuming a mechanism consisting of a yield line running from the load to the support (Fig. 9b). The relative displacement rate v is inclined at the angle α_o to the yield line, and it is not necessarily perpendicular to the beam axis, as in the mechanism of Fig. 4. Thus the tensile stringer contributes to the rate of internal work. The lowest upper bound is determined from the work equation, minimizing with respect to the angle α_o. As shown in reference [24], the result is identical with equations (15), which constitute the exact plastic solution. Note that the shear span is measured between the edges of the load and support platens (cf. Fig.9).

The assumed failure mechanism is obviously not the one observed in reality (cf. Fig 1). This fact does not affect the validity of the solution, however, since the failure mechanism of a rigid-plastic body is not uniquely determined.

Equations (15) give the shear strength as a function of the shear span ratio and the strength of the longitudinal reinforcement, the only empirical parameter being the effectiveness factor ν. To investigate the applicability of the solution, a series of tests was carried out at the Structural Research Laboratory (ROIKJAER [29]). The beams were prestressed, but theoretically this should not affect the ultimate load. The test results are plotted on Fig. 10. The shear span ratio a/h was varied, the reinforcement degree being constant, $\Phi = 0.21$. The experimental points fit nicely to the theoretical curve, equation (15a), corresponding to an effectiveness factor $\nu = 0.46$.

It is remarkable that the predicted shear strength depends on the compressive strength f_c and not the tensile strength f_t (in fact, we have assumed $f_t = 0$, cf. Section 4). On the other hand, experience shows that for beams without stirrups, the effective strength νf_c varies with the concrete quality in much the same way as does the tensile strength. Examination of a great number of test results suggests that νf_c is proportional to $\sqrt{f_c}$. The reason for this is that the effectiveness factor is a measure of the concrete ductility, which decreases with increasing strength level. The rather low value of ν found above is explained by the fact that the concrete of the beams was very strong ($f_c \approx 55$ MP_a). The dependence upon concrete strength is the most important, but it appears that the effectiveness factor is influenced by a number of other circumstances as well. These matters are discussed in detail in reference [24].

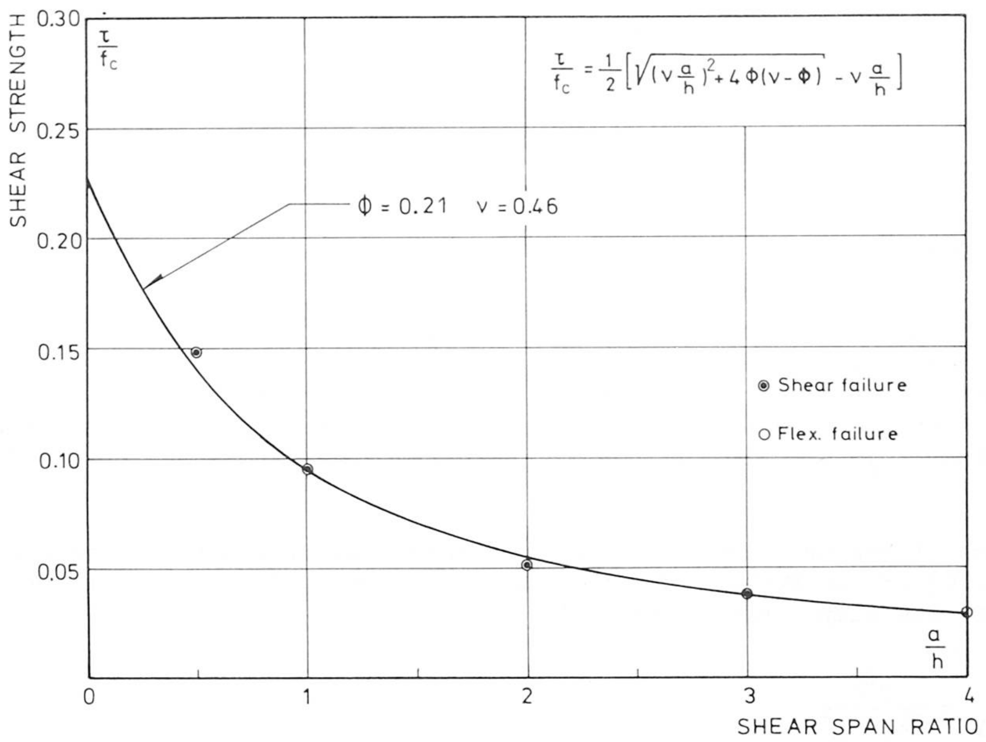

Fig. 10 Shear tests on beams without web reinforcement (ROIKJAER [29])

7. COMPARISON WITH BUILDING CODES

Proposals for the use of the web crushing criterion for the design of stirrup reinforcement are given in reference [24] and shall not be repeated here. Instead, we shall compare the theoretical formulas, derived in the preceeding sections, with the design rules of building codes.

Fig. 8 shows the shear capacity as calculated by the Danish Code of Practice, DS 411 [8],[9]. The code requires the use of the internal moment lever arm z as shear depth, and a strut inclination of $\theta = 45^o$. Thus the shear strength is given by equation (7) with $h = z$ and $\theta = 45^o$. As upper limit on the shear load is imposed the value $V = 0.25\ f_c\ bz$. Comparing with equation (11b), we see that this corresponds to an effectiveness factor $\nu = 0.50$. For 45^o stirrups, the upper limit is $V = 0.45\ f_c\ bz$, corresponding to an effectiveness factor $\nu = 0.37$. In addition, the code allows a 'shear contribution from the concrete' of $V = 0.5\ f_t\ bz$, f_t being the uniaxial tensile strength. (In Fig. 8, the actual strength parameters have been used. Of course, design values are to be inserted when the code is applied). This additional term is devoid of any theoretical justification when shear reinforcement is present. It seems mainly to be included to compensate for the unfavourable choice of strut inclination.

Nevertheless, it is obvious from Fig. 8 that the code is very conservative, and that even with an effectiveness parameter as low as $\nu = 0.50$, the use of the web crushing criterion would lead to a substantial saving of stirrups for small and moderate degrees of shear reinforcement.

The "Comité Euro-International du Béton" (CEB) recently completed a Model Code [6] [7], which introduces a "More Accurate Method" for the design of shear reinforcement, using variable strut inclination. In clause 11.2.4.2 of the Model Code (equation [11.19]), we find equation (7) with a shear depth $h = 0.9d$, d being the effective depth of the beam. As a lower limit for the strut inclination, the Model Code originally proposed $\cot\theta_{min} = 2.0$, as suggested by GROB & THÜRLIMANN [10]. Recently [7], this was changed to the more conservative bound $\cot\theta_{min} = 5/3$. The shear strength limit imposed by the web concrete is given by equation [11.17] of the Model Code, which corresponds to equation (8) with $\nu = 0.60$ and $h = d$. However, the applications of this equation is restricted by the requirement

$$V < 0.45\ f_c\ bd\ \sin 2\theta$$

The 'concrete term' which is given as $V = 0.6\ f_t\ bd$ for very small shear loads, is very reasonably phased out when any significant shear reinforcement is necessary. However, a rational estimate of the shear strength of beams without shear reinforcement should include the effects of the shear span ratio a/h and the longitudinal reinforcement degree Φ, as is the case with equations (15), given above.

The design of the main reinforcement requires a special note. According to Equation [11.20] of the Model Code, the tensile stringer force is increased (with respect to the simple moment term) by the amount:

$$\Delta\ T_y = \frac{V^2 c}{2A_s\ f_y d} \tag{16}$$

using the notation of the present paper. Assuming $\sigma_a = f_y$ and $h = d$, the applied shear force is given by equation (4):

$$V = \rho f_y\ bd\ \sin^2\alpha(\cot\theta + \cot\alpha)$$

$$= \frac{A_s}{c} f_y\ d\ \sin\alpha(\cot\theta + \cot\alpha)$$

Inserting into equation (16), we find

$$\Delta T_y = \frac{1}{2} V \sin\alpha(\cot\theta + \cot\alpha)$$

On the other hand, equation (13) requires

$$\Delta T_y = \frac{1}{2} V\ (\cot\theta - \cot\alpha)$$

Thus we note that equation (16) is correct in the case of vertical stirrups ($\alpha = 90^o$), but generally not when the stirrups are inclined.

8. CONCLUSION

In the preceding sections, we have shown that a rational analysis of the shear strength of reinforced concrete beams with stirrups may be based upon the truss analogy with variable strut inclination. The assumption of perfectly plastic properties of the materials leads to a solution, the web crushing criterion, which is both an upper and a lower bound. The web crushing criterion is found to agree reasonably well with experimental evidence, provided we introduce an empirical web effectiveness factor.

The same assumptions have been applied to beams without web reinforcement. The shear strength is determined as a function of shear span ratio and main reinforcement strength, which shows excellent agreement with exploratory test results. A reliable prediction of the shear resistence of beams without stirrups could lead to important savings in reinforcement and concrete.

An important step towards the application of the web crushing criterion in practical design is taken by the CEB Model Code. It should be noted, however, that the formula for the increase of main reinforcement due to shear is incorrect in the case of inclined stirrups. The Model Code almost abolishes the socalled addition principle, i.e. the inclusion of a shear stress term proportional to the tensile concrete strength. Still wanting is a formula for the shear strength of beams with little or no stirrups, taking account of the shear span ratio and the amount of longitudinal reinforcement.

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