Sway buckling of multistory frames

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Flambage d'ensemble de cadres à étage multiples

Seitliches Ausknicken von Stockwerkrahmen

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SUMMARY

In this contribution the sway buckling of a plane structure, composed of beams and columns, will be studied.
As the number of stories is assumed to be great, the stiffening resulting from the beams can be distributed continuously over the height of the columns. In this way it has become possible to concretize the sway buckling in one differential equation. The buckling load may then be directly read in a diagram as a function of a dimensionless coefficient K', which indicates the relative stiffness of the beams with regard to the columns.

RÉSUMÉ

Cette contribution étudie le flambage d'ensemble d'une ossature plane composée de poutres et de colonnes.
Le nombre d'étages étant grand, l'effet de raidissement des poutres peut être réparti uniformément sur la hauteur des colonnes. De cette façon, il a été possible de représenter le flambage d'ensemble au moyen d'une équation différentielle unique. La charge critique peut alors être directement tirée d'un diagramme, en fonction d'un coefficient adimensionnel K' qui indique la rigidité relative des poutres par rapport aux colonnes.

ZUSAMMENFASSUNG

In diesem Beitrag wird das seitliche Ausknicken eines aus Riegeln und Stützen bestehenden ebenen Stockwerkrahmens untersucht.
Da die Zahl der Geschosse als gross vorausgesetzt wird, kann die versteifende Wirkung der Riegel kontinuierlich über die Stützenhöhe verteilt werden. Dies ermöglicht die Beschreibung des seitlichen Ausknickens in einer einzigen Differentialgleichnung. Die Knicklast kann demzufolge in Abhängigkeit eines dimensionslosen Koeffizienten K', welcher die Steifigkeitsverhältnisse von Riegeln und Stützen enthält, direkt einem Diagramm entnommen werden.

1. INTRODUCTION - HYPOTHESES

Besides the local buckling of a particular element of a structure, the buckling of the structure as a whole may occur in the case of high-rise buildings. In this contribution we first consider the stability of plane frames, the columns of which each have a constant moment of inertia (I_i) from the foundation up to the roof. The moment of inertia of one column, however, needs not to be equal to that of another.

We further assume that the moment of inertia (I) of the beams is constant on all floors and that the distance ℓ of two successive floors is equal. The number of stories is assumed to be high (see further). The material behaves elastically up to the critical load. The floor load is the same on all floors. Afterwards the influence of an additional heavy roof load will be examined.

Moreover, it is assumed that the elastic deformations are only the result of bending moments : the influence of the transverse force and of the normal force is left out of consideration.

We consider the frame in sway buckling.

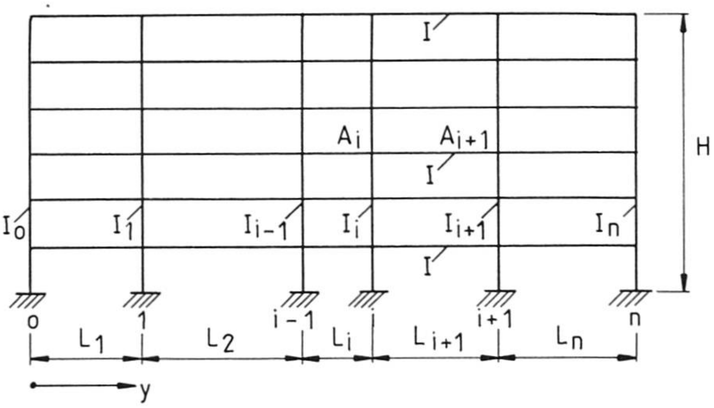


Fig. 1

2. FUNDAMENTAL DIFFERENTIAL EQUATION

As the number of floors is great and the beams, at the deformation of the whole are assumed to remain of constant length, it may be accepted that the deflection curves of all columns are equal.

It may then be approximately assumed that the end points A_i and A_{i+1} of any given element of a beam will have the same rotation when the structure deflects in its plane. This causes a point of inflection in the middle of each part beam (between two successive columns).

Fig. 2

The bending moment in the beam against the columns i and i+1 is $\mathfrak{M}^{i}_{i+1} = \frac{6\,EI}{L_{i+1}}\varphi$

Such moments form a stiffening of the columns.

Distributing these beam moments over the height (ℓ) of a story, which is admissible because of the great number of stories one obtains

$$m_i = \frac{6\,EI}{\ell}\cdot\varphi\left(\frac{1}{L_i}+\frac{1}{L_{i+1}}\right) = \frac{6\,EI}{\ell}\frac{dy}{dx}\left(\frac{1}{L_i}+\frac{1}{L_{i+1}}\right) \qquad (1)$$

Let p be the total load of the entire building devided by the total height H; i.e. the load per unit length (measured vertically). Each column (i) bears a certain part (p_i) of this load, which is assumed to be constant over the height.

There always must be $\sum_{o}^{n} p_i = p$.

Let us separate a small element dx from the column i.

After deformation (deflection of the structure) the element AB is deformed to A'B'.
The internal forces M_i, D_i and N_i at the point A' have increased at the point B' to $M_i + dM_i$, $D_i + dD_i$ and $N_i + dN_i$.
From the vertical equilibrium follows :

$$p_i \, dx + dN_i = 0,$$

or $$p_i = - \frac{dN_i}{dx} .$$

Expressing the equilibrium of moments round the point A' and neglecting terms of smaller order of magnitude, one obtains :

$$- \frac{dM_i}{dx} + D_i + N_i \frac{dy}{dx} - m_i = 0 \qquad (2)$$

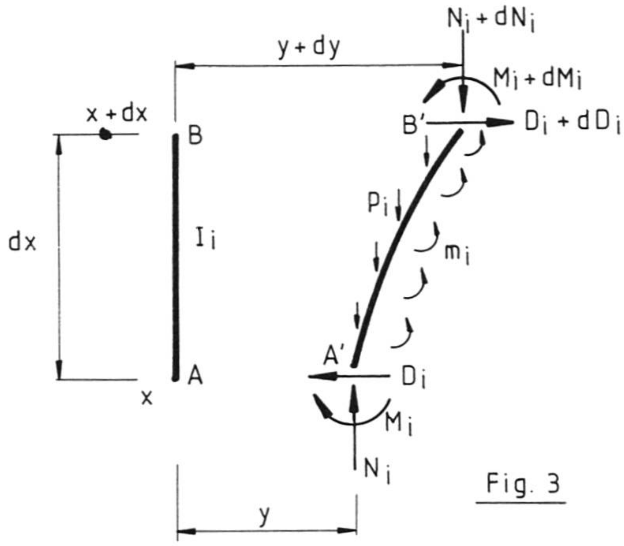


Fig. 3

The normal force N_i is given by :

$$N_i = p_i \,(H - x) + P_i \quad ; \qquad (3)$$

P_i represents an additional heavy load on the roof acting on the colomn i.
The bending moment may be expressed as a function of the deflection

$$M_i = - EI_i \frac{d^2y}{dx^2} \quad \text{and} \quad \frac{dM_i}{dx} = - EI_i \frac{d^3y}{dx^3} \qquad (4)$$

Introducing (1), (3) and (4) in (2), one obtains :

$$EI_i \frac{d^3y}{dx^3} + \left[p_i \,(H-x) + P_i - \frac{6\, EI}{\ell} \left(\frac{1}{L_i} + \frac{1}{L_{i+1}} \right) \right] \frac{dy}{dx} + D_i = 0 \qquad (5)$$

If this differential equation is written for each column and these equations are added up, taking into account the supposition that the deflection curve is also equal for all columns up to the third derivative (see suppositions), one obtains:

$$E \sum_o^n I_i \frac{d^3y}{dx^3} + \left[\sum_o^n p_i \,(H-x) + \sum_o^n P_i - \frac{12\, EI}{\ell} \sum_1^n \frac{1}{L_i} \right] \frac{dy}{dx} + \sum_o^n D_i = 0 \qquad (6)$$

Putting $\sum_o^n I_i = J$; $\sum_o^n p_i = p$; $\sum_o^n P_i = P$; $\sum_1^n \frac{1}{L_i} = \frac{1}{L}$; $a = \frac{pH}{EJ} + \frac{P}{EJ} - \frac{12 I}{\ell JL}$; $b = \frac{p}{EJ}$

and considering that in the absence of any horizontal load $\sum_o^n D_i = 0$, equation (6) will pass into :

$$EJ \frac{d^3y}{dx^3} + \left[p \,(H-x) + P - \frac{12\, EI}{\ell\, L} \right] \frac{dy}{dx} = 0 \qquad (7)$$

This intrinsic differential equation describes the problem of sway buckling.

3. GENERAL SOLUTION

In order to convert the differential equation into an easily solvable one, we pass on to a new variable z :

$$z = A\,(h - x)^{3/2} \qquad (8)$$

with A, a constant still to be determined, and h, a fictive height of the structure, function of the relative load proportion P/p and the effect of the transverse members :

$$h = H + \frac{P}{p} - \frac{12\, EI}{\ell\, Lp} \qquad (9)$$

The differential equation may be written :

$$- EJ \frac{27}{8} A^2 \left[\frac{d^3y}{dz^3} z + \frac{d^2y}{dz^2} - \frac{dy}{dz}\frac{1}{9z} \right] - \frac{3}{2} p z \frac{dy}{dz} = 0 \qquad (10)$$

Putting

$$A = \frac{2}{3} \sqrt{\frac{p}{EJ}} \qquad (11)$$

and, if $z \neq 0$, then one obtains

$$\frac{d^3y}{dz^3} + \frac{d^2y}{dz^2}\frac{1}{z} + \frac{dy}{dz}\left(1 - \frac{1}{9z^2}\right) = 0 \qquad (12)$$

Finally, putting $\frac{dy}{dz} = u$ (13)

then (12) is converted into the well-known differential equation

$$\frac{d^2u}{dz^2} + \frac{du}{dz}\frac{1}{z} + u\left(1 - \frac{1}{9z^2}\right) = 0 \qquad (14)$$

The Bessel functions of the order 1/3 and -1/3 form a set of independent functions, which satisfy the differential equation.
The general solution of (14) can be written as :

$$u = C_1 J_{-1/3}(z) + C_2 J_{1/3}(z) \qquad (15)$$

with C_1 and C_2 two integration constants to be determined from the boundary conditions.
A third integration constant follows from the relation : $\frac{dy}{dz} = u$.
The boundary conditions at the foundation (x = 0) usually are : $y = 0$ and $\frac{dy}{dx} = 0$ (16)

As to the third boundary condition two possibilities may be examined.
In the case of a plane structure, as indicated in fig. 4, one may put

for $x = H$: $M = 0$ or $\frac{d^2y}{dx^2} = 0$.

In the case of a water tower, for instance, where the water reservoir has a very great stiffness, one should rather put

for $x = H$: $\frac{dy}{dx} = 0$

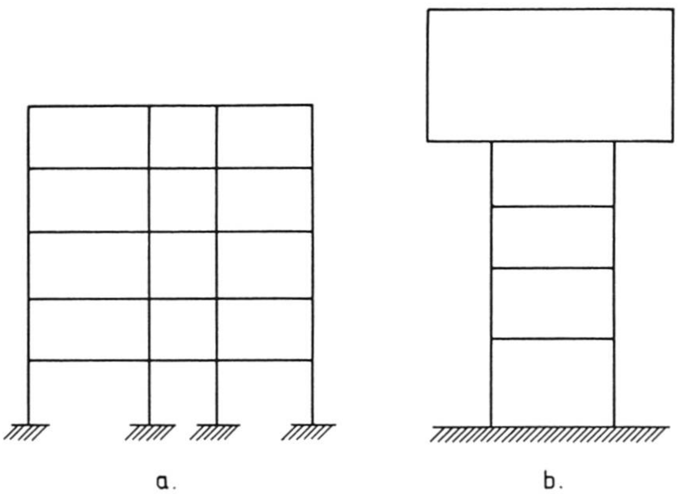


Fig. 4

4. TYPICAL SOLUTIONS

The general solution is now applied in some typical cases, which are of frequent occurrence in practice.
In all these applications it is assumed that the columns are fixed into the foundation.

4.1. Structure, for which the equations $\left(\frac{d^2y}{dx^2}\right)_{x=H} = 0$ en $P = 0$ hold

A structure of that kind is thus only subjected to a vertical load p, distributed all over the height. The roof floor has no exceptional stiffness so that, in addition to the boundary conditions at the foundation, may be put as third boundary condition :

$$\left(\frac{d^2y}{dx^2}\right)_{x=H} = 0$$

With these specific data and the general solution as a basis, the boundary conditions may be expressed as follows :

1. In $x = 0$: $\frac{dy}{dx} = 0$ or $\left[- u \sqrt{\frac{p}{EI}} (h - x)^{1/2}\right]_{x=0} = 0$

As h, p and EI are constant, this condition is reduced to : $u_{x=o} = 0$,

or, u being a function of z, $u(z_o) = 0$ (17)

in which z_o represents the value of z for $x = 0$: $z_o = \frac{2}{3} \sqrt{\frac{p}{EI}}\, h^{3/2}$ (18)

Putting $\frac{pH^3}{EJ} = K$ and $\frac{12\, IH^2}{J\ell L} = K'$ into (18) (19)

then $z_o = \frac{2}{3} \sqrt{K} \left(1 - \frac{K'}{K}\right)^{3/2}$. (20)

The condition (17) may now be written generally :

$$C_1 J_{-1/3} (z_o) + C_2 J_{1/3} (z_o) = 0 \qquad (21)$$

2. In $x = 0$: $y = 0$

For $x = 0$ always holds $z_o \neq 0$. The function values $J_{-1/3}(z_o)$ and $J_{1/3}(z_o)$ at the point z_o are real and finite.

From $\frac{dy}{dx} = u$ follows

$$y = \int u\, dx + C_3 = \int \left[C_1 J_{-1/3}(z) + C_3 J_{1/3}(z)\right] dx + C_3$$

In the limite case $z = z_o$ the integral disappears. This means that the deflection at the origin only becomes zero if $C_3 = 0$.

3. For $x = H$, or $z = z_1$ must be $\frac{d^2y}{dx^2} = 0$, or $\left[3\, z^{2/3} \frac{du}{dz} + z^{-1/3}\, u\right]_{x=H} = 0.$

The value z_1 may be calculated as a function of K and K' :

$$z_1 = \frac{2}{3} \sqrt{\frac{p}{EJ}} \left(H - \frac{c}{p} - H\right)^{3/2} = \frac{2}{3} \sqrt{K} \left(- \frac{K'}{K}\right)^{2/3}.$$

As z_1 is always different from zero, the boundary condition becomes

$$3\, z_1 \left(\frac{du}{dz}\right)_{z=z_1} + u(z_1) = 0.$$

or $3\, z_1 \left[C_1 \frac{dJ_{-1/3}(z_1)}{dz} + C_2 \frac{dJ_{1/3}(z_1)}{dz}\right] + C_1 J_{-1/3}(z_1) + C_2 J_{1/3}(z_1) = 0$ (22)

Expressing the derivatives of Bessel functions in the basic functions themselves, one finds

$$\left[- C_1 J_{2/3} (z_1) + C_2 J_{-2/3} (z_1)\right] 3\, z_1 = 0 .$$

As $z_1 \neq 0$: $\quad - C_1 J_{2/3} (z_1) + C_2 J_{-2/3} (z_1) = 0.$ (23)

In order that there should be another than the trivial solution for the system formed by the equations (21) and (23)

$$\begin{cases} C_1 J_{-1/3} (z_o) + C_2 J_{1/3} (z_o) = 0 \\ -C_1 J_{2/3} (z_1) + C_2 J_{-2/3} (z_1) = 0 \end{cases}$$

the determinant must be zero : $J_{-1/3} (z_o) J_{-2/3} (z_1) + J_{1/3} (z_o) J_{2/3} (z_1) = 0$ (24)

This is the intrinsic relation, which defines the buckling load p_{cr}.
As both z_o and z_1 are functions of K and K', the relation (24) represents a function of K and K' : $\quad F (K,K') = 0$ (25)

For a given structure the coefficient of relative rigidity is constant,

$$K' = \frac{12 \; IH^2}{J\ell L} ,$$

so that in the function F only K appears as a variable. The smallest value of K (K_{cr}) determines the critical load p (p_{cr}), corresponding to K'.
The critical buckling load is thus given by $\quad p_{cr} = \frac{EJ}{H^3} \cdot K_{cr}$ (26)

in which K_{cr} is a function of K.

The zero points of the function F have been determined numerically; the diagram K/K' has been plotted by means of the computer (see fig. 5 and 6-curve K_1)
The value K' = 0 corresponds with a free-standing shaft (buckling under dead load).
For that case the known value K_{cr} = 7,83 is found.
If P_{cr} represents the critical, total load (H p_{cr}), a critical stress (σ_{cr}) can then be defined by

$$\sigma_{cr} = \frac{7{,}83 \; E}{(\frac{A}{J}) \; H^2}$$

with A the section and $\sqrt{\frac{A}{J}}$ H the slenderness of the column.

From the zero point (K' = 0) the function rises continuously with the stiffness of the members.
When the stiffness of the members becomes infinitely large, the columns buckle locally between the foundation and the first floor.
The critical buckling load of the column i is then the Euler buckling load :

$$(P_{cr})_i = (p_{cr})_i \; H = \frac{\pi^2 I_i E}{\ell^2} .$$

Summing up over all columns, one obtains : $\quad p_{cr} = \frac{\pi^2 \; E \; J}{H \; \ell^2} .$

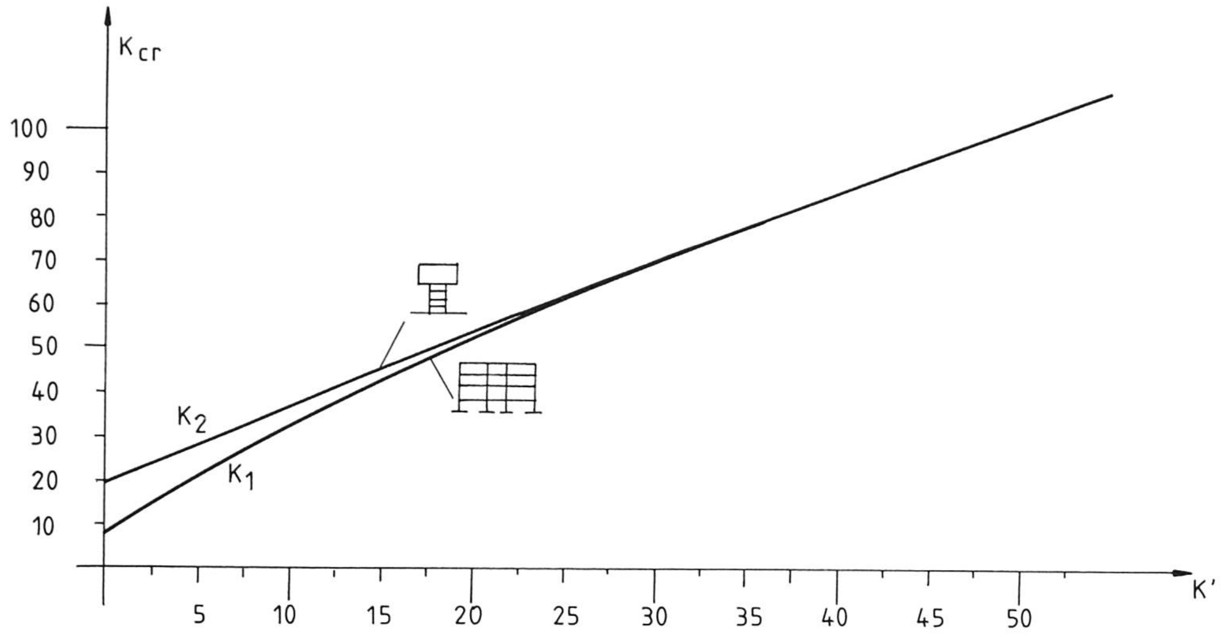
As a limit value of K_{cr} may be put

$$K_{cr} = \frac{p_{cr} \; H^3}{EJ} = \frac{\pi^2 \cdot E \; J \; n^2 \; H^3}{H^3 \; E \; J} = \pi^2 n^2 ; \quad (27)$$

n represents the number of stories (H = $n\ell$)

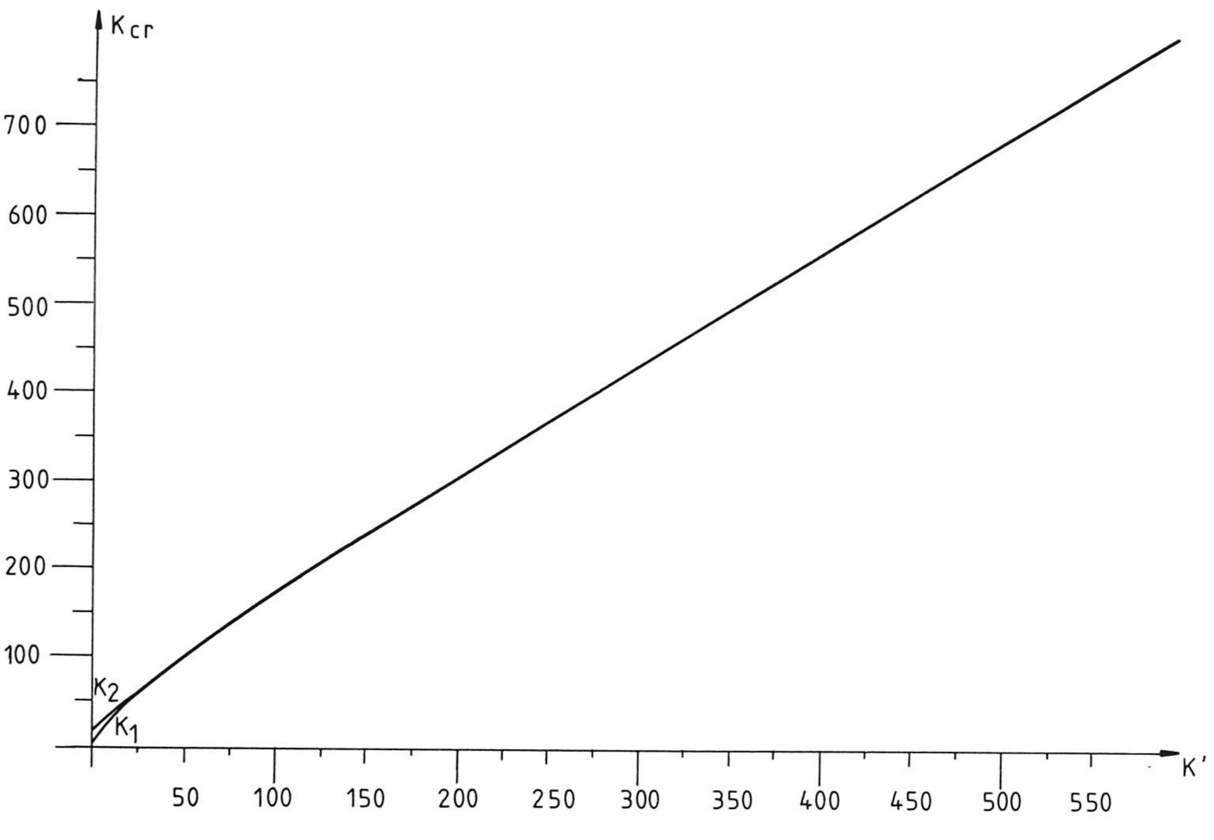
At this limit value the structure buckles locally.
The number of floors at which the danger of global and local buckling is equal, is given by :

$$n = \frac{\sqrt{K_{cr}}}{\pi} \quad (27b)$$



K_{cr}
100
90
80
70
60
50
40
30
20
10
K_2
K_1
5
10
15
20
25
30
35
40
45
50
K'

Fig. 5



K_{cr}
700
600
500
400
300
200
100
K_2
K_1
50
100
150
200
250
300
350
400
450
500
550
K'

Fig. 6

4.2. Structure for which the equations $\left(\frac{d^2y}{dx^2}\right)_{x=H} = 0$ en $p = 0$ hold

The structure is thought to be loaded on the top floor only; the dead load of the floors and the columns is supposed to be negligible with reference to the total roof load P.
In this supposition the differential equation is reduced to

$$EJ \frac{d^3y}{dx^3} + \left(P - \frac{12\ EI}{\ell\ L}\right) \frac{dy}{dx} = 0$$

or with

$$a = \frac{P}{EJ} - \frac{12\ I}{J\ \ell L}, \quad \frac{d^3y}{dx^3} + a\frac{dy}{dx} = 0$$

The general solution is :

$$y = C_1 \sin\sqrt{a}\,x + C_2 \cos\sqrt{a}\,x + C_3.$$

From the boundary conditions follows :

for $x = 0$: $y = 0$ or $C_2 + C_3 = 0$

for $x = 0$: $\frac{dy}{dx} = 0$ or $C_1\sqrt{a} = 0$ $(C_1 = 0)$

for $x = H$: $\frac{d^2y}{dx^2} = 0$ or $C_2 \cos\sqrt{a}\,H = 0$.

In order that there should not be a trivial solution, there must be $\cos\sqrt{a}\,H = 0$

or $\sqrt{a}\,H = \frac{\pi}{2}, \frac{3\pi}{2} \ldots$

The smallest value satisfying this is $\frac{\pi}{2}$.

The critical load is then given by

$$P_{cr} = \frac{\pi^2 EJ}{4\ H^2} + \frac{12\ EI}{\ell\ L} \tag{28}$$

Everything happens as if the critical load of the structure is equal to the Euler buckling load, globalized over all columns, increased with a term $\frac{12\ EI}{\ell\ L}$ which denotes the influence of the stiffness of the beams. At a zero stiffness of the beams the Euler load is effectively found again.

If the beams are infinitely stiff the columns buckle locally : $P_{cr} = \frac{\pi^2\ EJ}{\ell^2}$.

Expressed in the Euler buckling load, it becomes : $P_{cr} = \frac{\pi^2\ EJ}{4\ H^2}(2n)^2$

The condition at which the danger of global and local buckling is equal, is given by

$$\frac{IH}{LJ} = \frac{\pi^2(4n^2 - 1)}{48\ n} .$$

The number of stories at which this condition is satisfied, amounts to :

$$n = \frac{6}{\pi^2}\frac{IH}{LJ} + \sqrt{\left(\frac{6}{\pi^2}\frac{IH}{LJ}\right)^2 + \frac{1}{4}} \tag{29}$$

4.3. Structure for which the equations $\left(\frac{dy}{dx}\right)_{x=H} = 0$ and $P = 0$ hold

The conditions at the foundation, as dealt with sub 4.1 still remain valid.
The third boundary condition $\frac{dy}{dx} = 0$ at the point $x = H$ leads to :

$$\left[- u \sqrt{\frac{p}{EI}}\,(h - x)^{1/2}\right]_{x=H} = 0$$

As $(z)_{x=H} = z_1 \neq 0$, there must be $u = C_1 J_{-1/3}(z_1) + C_2 J_{1/3}(z_1) = 0$.

In order that there should not be a trivial solution, the determinant of the system

$$\begin{cases} C_1 J_{-1/3}(z_o) + C_2 J_{1/3}(z_o) = 0 \\ C_1 J_{-1/3}(z_1) + C_2 J_{1/3}(z_1) = 0 \end{cases}$$

must be zero : $J_{-1/3}(z_o) J_{1/3}(z_1) - J_{-1/3}(z_1) J_{1/3}(z_o) = 0$ (30)

As z_o and z_1 are functions of K and K', eq. (30) may be written in the form

$$F'(K,K') = 0 \tag{31}$$

The smallest value of K, (K_{cr}) at which (31) is satisfied for a given value of K', defines the critical load $P_{cr} = \frac{EI}{H^3} K_{cr}$. (32)

The zero points of the function F' have been calculated numerically; this allowed the drawing of diagram K_{cr}/K', shown in fig. 5 and 6 - curve K_2.

4.4. Structure for which the equations $(\frac{dy}{dx})_{x=H}$ and p = 0 hold.

The first two boundary conditions, mentioned sub 4.2 still remain valid. The third condition becomes : for x = H : $\frac{dy}{dx} = 0$ or $C_2 \sin\sqrt{a}\,H = 0$.

In order that there should not be a trivial solution, there must be

$\sin a H = 0$ or $\sqrt{a}\; H = \pi,\; 2\pi \ldots$

The smallest value satisfying this is π.
The critical load is then given by : $P_{cr} = \frac{\pi^2 EJ}{H^2} + \frac{12\,EI}{\ell\,L}$. (33)

Here again the critical load is found by adding to the Euler buckling load the influence term of the horizontal member stiffness.
It is to be noted that the term,which includes the influence of the member stiffness,is independent of the boundary condition in x = H.
Analogously as sub 4.2 it is possible to calculate the number of stories at which the danger of global buckling of the structure is equal to the danger of its local buckling.
One finds : $n = \frac{6}{\pi^2}\frac{IH}{LJ} + \sqrt{\left(\frac{6}{\pi^2}\frac{IH}{LJ}\right)^2 + 1}$ (34)

4.5. Structures loaded by P ≠ 0 and p ≠ 0

Let us first assume that for x = H the condition $\frac{d^2y}{dx^2} = 0$ is satisfied. If both a uniform load p and a compressive load P act on the structure, the differential equation $EJ\frac{d^3y}{dx^3} + p\left[H + \frac{P}{p} - \frac{12\,EI}{\ell\,Lp} - x\right]\frac{dy}{dx} = 0$
only differs from the one in which P = 0 is supposed

$$EI\frac{d^3y}{dx^3} + p\left[H - \frac{12\,EI}{\ell\,Lp} - x\right]\frac{dy}{dx} = 0$$

in the term $a = H + \frac{P}{p} - \frac{12\,EI}{\ell\,Lp}$ (35)

Let us assume tentatively the critical roof load to be known, i.e. P_{cr_1}.
Everything happens as if the stiffening by the members (term $\frac{12\ EI}{\ell\ L}$) is reduced by the load P_{cr_1}.

The value of K' becomes then

$$K_1' = \left(\frac{12\ IE}{\ell L} - P_{cr_1}\right)\frac{H^2}{EJ} = \frac{12\ IH^2}{\ell LJ} - \frac{P_{cr_1}\ H^2}{EJ} \tag{36}$$

This value of K_1' determines a first critical load p_{cr_1} by using the K/K'-diagram in question.
Thus the critical, uniformly distributed load,which corresponds with a given roof load,is determined.

Next, a second critical roof load P_{cr_2} is taken and the corresponding load p_{cr_2} is calculated, etc.
The points determined are plotted in a P_{cr}/p_{cr}-diagram and joined by a flowing curve.
If it is desired that the coefficient of safety is the same for p and P :

$$p = \frac{p_{cr}}{s} \quad \text{and} \quad P = \frac{P_{cr}}{s}\ , \text{ then is } \frac{P}{p} = \frac{P_{cr}}{p_{cr}}\ .$$

On plotting this straight line in the diagram P_{cr}/p_{cr} , the intersection point A of this straight line with the curve P_{cr}/p_{cr} determines the critical values sought

$$P^r_{cr} = (P_{cr})_A \quad , \quad p^r_{cr} = (p_{cr})_A\ .$$

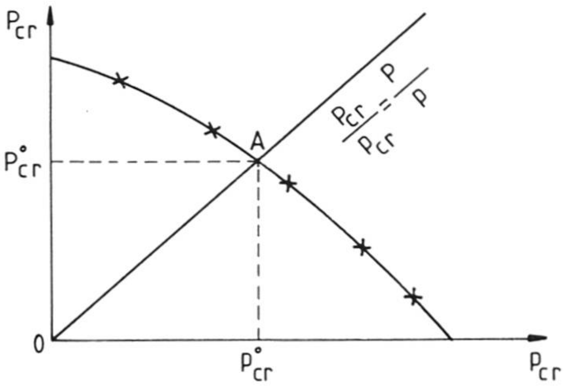


Fig.7

For an easy construction of the diagram in fig. 7, first P^o_{cr} is calculated in the supposition p = 0 (formula 28)

$$P^o_{cr} = \frac{\pi^2\ EJ}{4H^2} + \frac{12\ EI}{\ell\ L}$$

and p^o_{cr} in the supposition P = 0 (formula 26)

$$p^o_{cr} = \frac{EJ}{H^3}\ K_{cr}$$

After that, the corresponding value of p_{cr}, is calculated for 3 or 4 values of P_{cr}, included between P^o_{cr} and 0.

This curve may be determined very easily with the aid of diagram 5.
The constructed curve may also be replaced approximately by a straight line which joins the points $B(0,P^o_{cr})$ and $C(p^o_{cr},0)$.

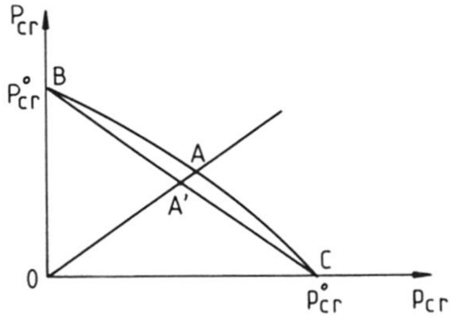


Fig. 8

Analogously an intersection point A' is now determined (see fig. 8).
The straight line BC is represented by

$$P_{cr} = P^o_{cr} - \frac{P^o_{cr}}{p^o_{cr}} \cdot p_{cr} \; .$$

The intersection point A' has then as coordinates

$$P^r_{cr} = \frac{P}{p} \frac{P^o_{cr}}{\frac{P}{p} + \frac{P^o_{cr}}{p^o_{cr}}} \qquad (37)$$

$$p^r_{cr} = \frac{P^o_{cr}}{\frac{P}{p} + \frac{P^o_{cr}}{p^o_{cr}}} \qquad (38)$$

If for x = H the equation $\frac{dy}{dx} = 0$ holds as boundary condition, the argumentation continues to be right.
Then is calculated : $P^o_{cr} = \frac{\pi^2 EJ}{H^2} + \frac{12\ EI}{\ell\ L}$ and $p^o_{cr} = \frac{EJ}{H^3} K_{cr}$

Intermediate points can again be calculated as above, by reading K_{cr} in the diagram of figures 5 and 6.

5. INFLUENCE OF NOT CONTINUOUS ACTING OF THE MEMBER MOMENT

In consequence of the not continuous acting of the member moment, a nodal point can suffer a certain relaxation. This means that the node undergoes an additional rotation. As it concerns here a correction on the basic rotations, it may be assumed, with regard to the calculation of this correction, that at the midpoint between two successive nodal points, both in the members and in the columns, there are hinges. The sum of the member moments at the place of column i comes originally to

$$\left(\frac{6\ EI}{L_i} + \frac{6\ EI}{L_{i-1}} \right) \varphi \; .$$

When this member moment acts on the node of the column, it is equalized pro rata of the distributing coefficients. The moment $\frac{6\ EI}{L_i} \varphi$ is reduced to

$$\frac{6\ EI}{L_i} \varphi - \left(\frac{6\ EI}{L_i} + \frac{6\ EI}{L_{i-1}} \right) \varphi \frac{\frac{I}{L_i}}{\frac{2 I_i}{\ell} + \frac{I}{L_i} + \frac{I}{L_{i-1}}}$$

Instead of $\frac{6\ EI}{L_i} \varphi$ there now acts from the member to the left of the column i a member moment

$$\frac{6\ EI}{L_i} \varphi \frac{2\ I_i}{2\ I_i + I\ell\left(\frac{1}{L_i} + \frac{1}{L_{i-1}} \right)}$$

upon the column i.
So the point is to reduce the stiffness of the member by multiplying by a coefficient β_i.
For the member moment, resulting from the member to the right of the column i, it is proved analogously that $\frac{I}{L_{i+1}}$ has to be multiplied by the same coefficient β_i :

$$\beta_i = \frac{1}{1 + \frac{I\ell}{2I_i}\left(\frac{1}{L_i} + \frac{1}{L_{i+1}}\right)} \qquad (39)$$

The summation term $\frac{1}{L}$ now becomes :

$$\frac{2}{L} = \beta_o \frac{1}{L_1} + \beta_1 \left(\frac{1}{L_1} + \frac{1}{L_2}\right) + \beta_2 \left(\frac{1}{L_2} + \frac{1}{L_3}\right) + \dots + \beta_n \frac{1}{L_n} . \qquad (40)$$

6. EXTENSION TO A SYMMETRIC SPACE STRUCTURE

So far only sway buckling of a plane structure in its plane has been considered. Let us now turn more generally to a construction, as shown in fig. 9, composed of a serie of columns, joined by beams according to the y-direction. The moment of inertia of these beams per field can be different, but the moment of inertia of the stories per field is assumed to be constant.

It is further supposed that the floors are infinitely stiff in the horizontal plane and that buckling occurs according to the y-direction. A possible rotation of the structure is left out of consideration here; this problem still needs to be worked out further.

In these suppositions all columns suffer the same deflection at the place of a floor.

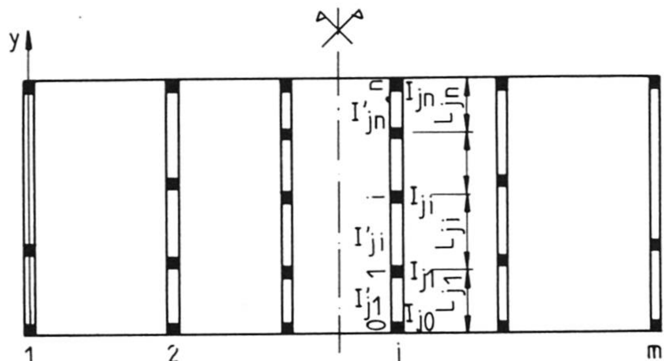


Fig. 9

The construction is built up of m plane frames.

In the frame j there are n+1 columns (0,1, ... i ... n) whose moments of inertia are denoted by the notations : I_{jo}, I_{j1}, ... I_{jn}.

Between two successive columns i-1 and i the moment of inertia of the member is denoted by I'_{ji}, the span by L_{ji}.

The equilibrium condition (5) for any given column of the system is :

$$EI_{ji} \frac{d^3y}{dx^3} + \left[p_{ji} (H-x) + P_{ji} - \frac{6\, EI'_{ji}}{\ell} \left(\frac{1}{L_{ji}} + \frac{1}{L_{ji+1}}\right)\right] \frac{dy}{dx} + D_{ji} = 0.$$

The addition of these equilibrium conditions, written for each column of the system, produces again the differential equation (7) :

$$EJ \frac{d^3y}{dx^3} + \left[p (H-x) + P - \frac{12\, E}{\ell} \frac{I}{L}\right] \frac{dy}{dx} = 0$$

with

$$J = \sum_1^m \sum_o^n I_{ji} \; ; \; p = \sum_1^m \sum_o^n p_{ji} \; ; \; P = \sum_1^m \sum_o^n P_{ji} \; ; \; \frac{I}{L} = \frac{1}{2} \sum_1^m \sum_o^n \left(\frac{I'_{ji}}{L_{ji}} + \frac{I'_{ji+1}}{L_{ji+1}}\right) .$$

If the relaxation is to be taken into account, then the coefficient β_{ji} can be introduced :

$$\beta_{ji} = \frac{2\,I_{ji}}{2\,I_{ji} + I'_{ji}\,\dfrac{\ell}{L_{ji}} + I'_{ji+1}\,\dfrac{\ell}{L_{ji+1}}}$$

$$\frac{I}{L} = \frac{1}{2} \sum_{j=1}^{m} \sum_{i=o}^{n} \beta_{ji} \left(\frac{I'_{ji}}{L_{ji}} + \frac{I'_{ji+1}}{L_{ji+1}} \right) .$$

Consequently, the proposed method of calculation also holds good with regard to construction symmetrically designed in plan view.

7. EQUIVALENT SLENDERNESS

The validity of the developed theory is limited to the elastic field. With relatively stiff structures of small slenderness, however, there may occur plastic buckling. An exact calculation in this field has not been made yet, to our knowledge. In order to get something practical we first determine the critical load of a free-standing column, subjected to a load P at the top and to a load p per unit length, uniformly distributed over the height of the column.

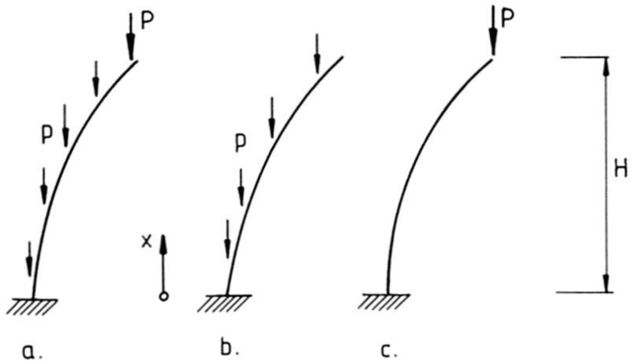


Fig. 10

Let i be the moment of inertia of the column. Let us first assume $\frac{d^2y}{dx^2} = 0$ as boundary condition in x = H.
When only the load p is applied, the elastic critical load is given by the expression :

$$p'^{o}_{cr}\,H = \frac{Ei}{H^2}\;7{,}83 \tag{41}$$

When only the load P is applied, the elastic critical load is given analogously by

$$P'^{o}_{cr} = \frac{\pi^2\,Ei}{4\,H^2} \tag{42}$$

When both p and P are different from zero, a $p'_{cr}\,H/P'_{cr}$ -diagram may be constructed, as explained in 4.5.
The approximation may also be maintained by a straight line.
If it is assumed here as well that

$$\frac{P}{pH} = \frac{P'_{cr}}{p'_{cr}H} = \frac{P'^{o}_{cr}}{p'^{o}_{cr}\,H} = \gamma \tag{43}$$

then the real critical elastic load is given by

$$P'^{r}_{cr} = \gamma\,\frac{P'^{o}_{cr}}{\gamma + \dfrac{P'^{o}_{cr}}{p'^{o}_{cr}\,H}} = \gamma \cdot p'^{r}_{cr}\,H \tag{44}$$

Introducing (41) and (42) in (44) then one obtains :

$$P'^{r}_{cr} = \gamma \frac{1}{\gamma + 0{,}315} \frac{\pi^2 Ei}{4H^2} = \gamma\ p'^{r}_{cr}\ H \qquad (45)$$

From this the critical stress is derived ($\lambda^2 = H^2\ A/i$ and A is the sectional area of the column) :

$$\sigma'^{r}_{cr} = \frac{P'^{r}_{cr} + p'^{r}_{cr}\ H}{A} = \frac{\gamma + 1}{\gamma + 0{,}315} \frac{\pi^2 E}{4\lambda^2} \qquad (46)$$

If this critical stress is compared with the critical stress,calculated for the global structure, an equivalent slenderness may be derived
(A represents now the sum of the sectional areas of all columns) :

$$\lambda = \sqrt{\frac{\gamma + 1}{\gamma + 0{,}315} \frac{\pi^2 E}{4} \frac{A}{P^{r}_{cr} + p^{r}_{cr}\ H}} \qquad (47)$$

If the third boundary condition is $\left(\frac{dy}{dx}\right)_{x=H} = 0$, then an equivalent slenderness may be derived in a similar way :

$$\lambda = \sqrt{\frac{\gamma + 1}{\gamma + 0{,}522} \pi^2 E \frac{A}{P^{r}_{cr} + p^{r}_{cr}\ H}} \qquad (48)$$

If either p = 0 or P = 0, then the equivalent slenderness is determined by :

$$P^{o}_{cr} = P'^{o}_{cr} \qquad \text{or} \qquad p^{o}_{cr} = p'^{o}_{cr}$$

One obtains then :

for $\left(\frac{d^2y}{dx^2}\right)_{x=H} = 0$ $\quad p = 0 \ :\ \frac{\pi^2 E}{4\lambda^2} = \frac{P^{o}_{cr}}{A}$ or $\lambda = \sqrt{\frac{\pi^2 E}{4} \frac{A}{P^{o}_{cr}}}$

$\quad P = 0 \ :\ \frac{7{,}83\ E}{\lambda^2} = \frac{p^{o}_{cr}\ H}{A}$ or $\lambda = \sqrt{7{,}83\ E \frac{A}{p^{o}_{cr}\ H}}$

for $\left(\frac{dy}{dx}\right)_{x=H} = 0$ $\quad p = 0 \ :\ \frac{\pi^2 E}{\lambda^2} = \frac{P^{o}_{cr}}{A}$ or $\lambda = \sqrt{\pi^2\ E \frac{A}{P^{o}_{cr}}}$

$\quad P = 0 \ :\ \frac{18{,}9\ E}{\lambda^2} = \frac{p^{o}_{cr}\ H}{A}$ or $\lambda = \sqrt{18{,}9\ E \frac{A}{p^{o}_{cr}\ H}}$.

With the slenderness thus calculated, the reduction coefficient φ is read in the φ/λ diagram, valid with regard to a single column, subjected to a compressive load. The permissible critical buckling stress is than : $\bar{\sigma}_{cr} = \varphi\ .\ \bar{\sigma}$
The dash over σ points to "permissible" stress.

8. APPLICATION

Given a water tower, as depicted in figure 11. The dead load of the reservoir is estimated at 6000 kN; the effective water capacity is 5000 kN.
E = 30.000.000 kN/m^2.

The dead load of the floors and the columns amounts to : $0,80 \times 0,80 \times 25 \times 4 \times 30$

$$+ 0,13 \times 6,8^2 \times 25 \times 5$$
$$+ 0,28 \times 0,37 \times 25 \times 5,2 \times 4 \times 5$$
$$= 2940,76 \text{ kN.}$$

The weight of the outer walls is :
$0,28 \times 5,2 \times 18 \times (30-5\text{x}0,5) \times 4 = 2882,88$ kN.

The useful overload (10 kN/m^2) is :

$(6,80 - 0,56)^2 \times 10 \times 5 = 1946,88$ kN

The load per unit height is then :

$$p = \frac{7770,52}{30} = 259 \text{ kN/m .}$$

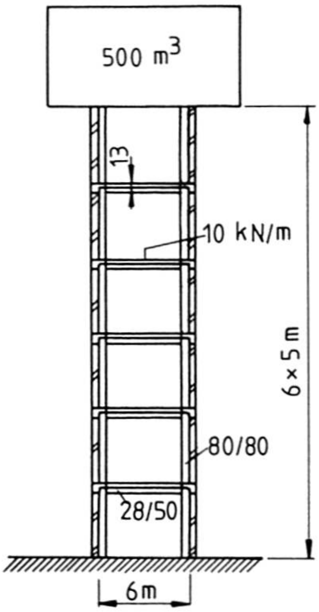


Fig. 11

The following calculations are made (member : rectangular section) : (19) (33) (32) :

$$I = \frac{0,50^3 \times 0,28}{12} = 0,0029 \text{ m}^4$$

$$J = 4 \times \frac{0,80^4}{12} = 0,1365 \text{ m}^4$$

$$\frac{1}{L} = \frac{1}{6} + \frac{1}{6} = 0,3333 \text{ m}^{-1}$$ (relaxation here has practical no influence)

$$(19) : K' = \frac{12 \times 0,0029 \times 30^2 \times 0,3333}{0,1365 \times 5} = 15,295$$

$$(33) : P^o_{cr} = \frac{\pi^2 \times 3 \times 10^7 \times 0,1365}{30^2} + \frac{12 \times 3 \times 10^7 \times 0,0029}{5} 0,3333 = 114.490 \text{ kN}$$

$$(32) : p^o_{cr} = \frac{45 \times 3 \times 10^7 \times 0,1365}{30^2} = 204.750 \text{ kN}$$

For different values of P_{cr} the corresponding critical load $p_{cr}H$ is found according to formula (36).

P_{cr}	K'_i	K_i	$p_{cr}H$
0			204.750
20000	15,295 - 4,396 = 10,899	≃ 38,5	175.170
40000	15,295 - 8,791 = 6,504	≃ 30	136.500
60000	15,295 - 13,187 = 2,108	≃ 22,5	109.200
114490			0

$$\frac{P}{pH} = \frac{11.000}{7770,52} = 1,415 = \frac{P^r_{cr}}{p^r_{cr}\ H}$$

$$P_{cr} = 1,415\ p_{cr}H$$

$P^r_{cr} = 83.000$ kN $\quad s = 7,54$

$p^r_{cr}\ H = 60.000$ kN $\quad s = 7,72$

From the above it appears that the coefficients of safety are practically equal. The slight difference is due to reading errors.
According to the approximation formulas (37) and (38) one finds :

P^{r}_{cr} = 82.060 kN

P^{r}_{cr} H = 57.990 kN

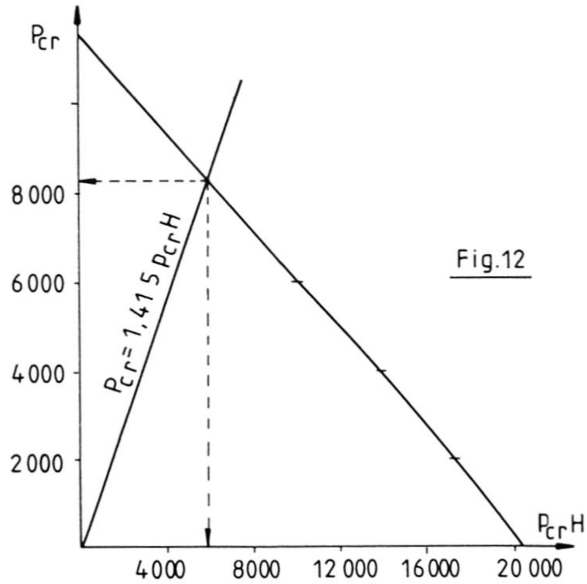


Fig.12

CONCLUSION

The very complex problem of sway buckling of a framed structure has been reduced to a simple and clear method for calculation. The case in which the roof load is comparable to that of the normal roofs as well as the case in which extra heavy roof loads are to be considered (e.g. a water tower) has been condensed into an explicit formula.

Without any difficulty the safety of the structure with reference to sway buckling can be calculated. At this operation the coefficient of safety for the compressive load may, if desired be increased, for instance, with regard to that for the floors. This would be justified in the case of a water tower, when the intermediate floors are by no means to be taken into consideration for storage accomodation.

In this case it would also be possible to reduce the overload on the floors and, at the same time, to apply an equal coefficient of safety for both floor load and roof load.

The concept "equivalent slenderness" also allows the checking of plastic buckling.

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