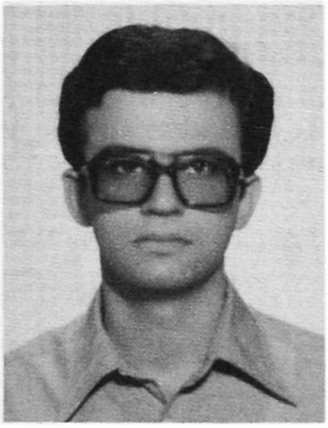
Lower-bound solutions for laterally loaded rectangular plates

Lower-bound Solutions for Laterally Loaded Rectangular Plates

Charge transversale et uniformément repartie des plaques rectangulaires

Traglast von Rechteckplatten mit gleichmässig verteilter Querbelastung

Dimitrios MANOLAKOS
Dr. Eng.
Natl. Techn. Univ. of Athens
Athens, Greece



Dimitrios Manolakos, born 1955, got his naval architecture and marine engineering degree in 1979 and his doctorate of engineering in 1984 at the National Technical University of Athens. For the past 3 years he has been researching into the analysis of ship collisions by use of plastic design methods, at NTUA, Athens.

SUMMARY

This paper presents a lower-bound solution, based on von Mises' yield criterion and not depending on boundary conditions, for a rectangular plate, subjected to a uniformly distributed lateral load.

RÉSUMÉ

Cette étude présente une méthode de calcul de la limite inférieure de la charge d'une plaque rectangulaire, sous l'effet d'une charge transversale uniformément répartie et de conditions au contour quelconques. La méthode est basée sur la condition de plasticité de von Mises.

ZUSAMMENFASSUNG

Dieser Beitrag zeigt eine Methode zur Berechnung unterer Grenzwerte der Traglast von Rechteckplatten mit gleichmässig verteilter Querbelastung und beliebigen Randbedingungen. Der Methode liegt die von Mises-Fliessbedingung zugrunde.

I. INTRODUCTION

In order to accurately assess the safety of ductile structures, it is necessary to find the limit load causing collapse.
The advantages of lower-bound solutions to the collapse loads of plates is well known.
Such solutions for simply supported or fully clamped plates have been reported in several references i.e. WOOD [I], JONES [2],[3].

The purpose of this paper is to extend the study to rectangular plates, under uniform loading, with variable edges fixity.

2. DISCRIPTION OF THE PROBLEM

Consider the case of a rectangular plate with dimensions a x b x t subjected to a uniformly distributed pressure p and made of a material which flows plastically according to the von MISES' yield criterion.

Assume end-fixity coefficients k,m,n,r respectively, as shown in fig. I.

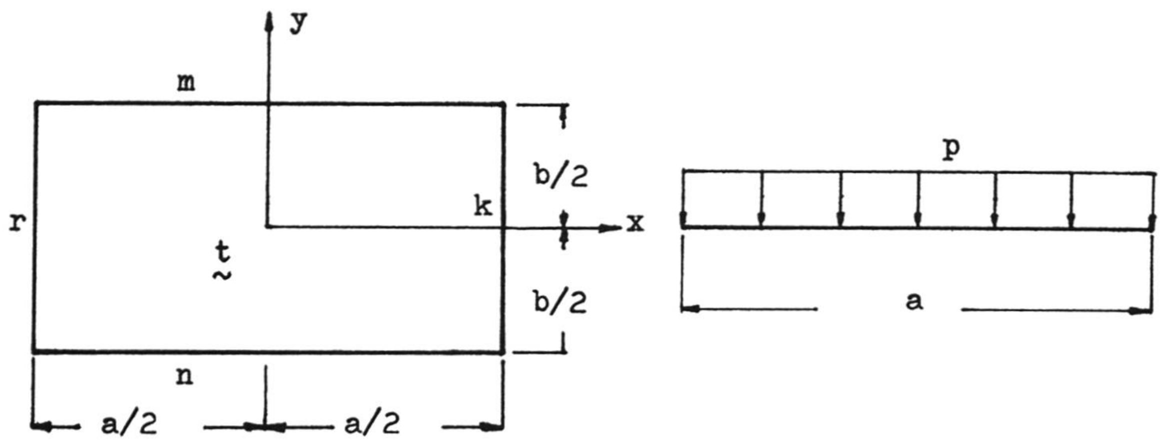


Fig. I Uniformly loaded rectangular plate

In applying the lower-bound theorem of Plasticity it is necessary to find a statically admisible stress field or one satisfying :

a. The equilibrium equation at all points in the plate.
b. The boundary conditions, and
c. The yield condition anywhere within the plate's domaine.

The equilibrium equation is written,[4];

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\, \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -p \qquad ;$$

or introducing the generalized stresses $m_x = M_x/M_P$, $m_y = M_y/M_P$ and

$m_{xy} = M_{xy}/M_P$, the equilibrium equation drops into the form :

$$\frac{\partial^2 m_x}{\partial x^2} + 2\frac{\partial^2 m_{xy}}{\partial x\,\partial y} + \frac{\partial^2 m_y}{\partial y^2} = -\frac{p}{M_P} \qquad (I)$$

The following moment distributions are used to satisfy the equilibrium equation and boundary conditions for the plate:

$$m_x = I + Ax + Bx^2 \qquad (2)$$

$$m_y = I + Cy + Dy^2 \qquad (3)$$

$$m_{xy} = Exy \qquad (4)$$

From the conditions: At $x=a/2$, $m_x=k$ and at $x=-a/2$, $m_x=r$, we obtain from eq.(2):

$$I + \frac{Aa}{2} + \frac{Ba^2}{4} = k \qquad (5)$$

$$I - \frac{Aa}{2} + \frac{Ba^2}{4} = r \qquad (6)$$

From which, easily we can find for the constants A and B the expressions:

$$A = (k-r)/a \qquad (7)$$

$$B = 2(k+r-2)/a^2 \qquad (8)$$

By the same manner, we obtain for constants C and D, respectively:

$$C = (m-n)/b \qquad (9)$$

$$D = 2(m+n-2)/b^2 \qquad (IO)$$

Replacing in eq.(I), we can evaluate the constant E. That is,

$$2B + 2D + 2E = -\frac{p}{M_p} \quad \text{or}$$

$$E = -\frac{p}{2M_p} - B - D \quad \text{or taking into account eqs (8) and (IO)}$$

$$E = -\frac{p}{2M_p} - \frac{2(k+r-2)}{a^2} - \frac{2(m+n-2)}{b^2} \qquad (II)$$

The von MISES' yield criterion, in the generalized form, is written:

$$m_x^2 - m_x m_y + m_y^2 + 3m_{xy}^2 = I \qquad (I2)$$

Substituting the moment distributions (2),(3) and (4) in the yield condition and taking simultaneously into consideration the expressions (7),(8),(9),(IO) and (II), we have:

$$\left\{I+\frac{k-r}{a}x+\frac{2(k+r-2)}{a^2}x^2\right\}^2 - \left\{I+\frac{k-r}{a}x+\frac{2(k+r-2)}{a^2}x^2\right\}\left\{I+\frac{m-n}{b}y+\frac{2(m+n-2)}{b^2}y^2\right\} +$$

$$+\left\{I+\frac{m-n}{b}y+\frac{2(m+n-2)}{b^2}y^2\right\}^2 + 3x^2y^2\left\{-\frac{p}{2M_p} - \frac{2(k+r-2)}{a^2} - \frac{2(m+n-2)}{b^2}\right\}^2 = I \qquad (I3)$$

Assuming that yielding occurs at points (x=a/2,y=b/2), (x=a/2,y=-b/2), (x=-a/2,y=b/2) and (x=-a/2,y=-b/2) and substituting in eq. (I3), the following relations are valid:

$$k^2 - km + m^2 + \frac{3a^2b^2}{I6}E^2 = I \qquad (I4)$$

$$k^2 - kn + n^2 + \frac{3a^2b^2}{I6}E^2 = I \qquad (I5)$$

$$r^2 - mr + m^2 + \frac{3a^2b^2}{I6}E^2 = I \qquad (I6)$$

$$r^2 - nr + n^2 + \frac{3a^2b^2}{I6}E^2 = I \qquad (I7)$$

By superposition of eqs (I4),(I5),(I6) and (I7), we obtain:

$$2(k^2+m^2+n^2+r^2)-(k+r)(m+n)+4\,\frac{3a^2b^2}{I6}\,E^2 = 4\ ;\ \text{or substituting for E}$$

$$\frac{2(k^2+m^2+n^2+r^2)-(k+r)(m+n)}{4}+\frac{3a^2b^2}{I6}\left\{-\frac{p}{2M_p}-\frac{2(k+r-2)}{a^2}-\frac{2(m+n-2)}{b^2}\right\}^2 = I$$

This equation gives for load p :

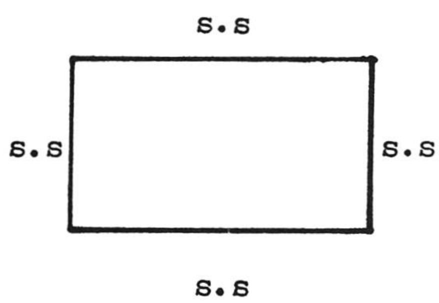
$$p = \frac{4M_p}{b^2}\left\{\frac{I}{\sqrt{3}\,\beta}\sqrt{4-2(k^2+m^2+n^2+r^2)+(k+r)(m+n)} - \frac{k+r-2}{\beta^2}-(m+n-2)\right\} \qquad (I8)$$

where:
$\beta = a/b$

The above eq.(I8) is a lower-bound solution for the load p, not depended on the end-fixity conditions. Next paragraph presents some of the main cases on the uniform loading of a rectangular plate.

3. PARTICULAR CASES OF LOWER-BOUND LOAD

3.I Simple supported plates at all edges (Fig. 2)

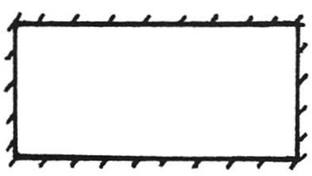


Eq.(I8) gives for k=m=n=r=0 :

$$p = \frac{8M_p}{b^2}\left(I+\frac{I}{\sqrt{3}\beta}+\frac{I}{\beta^2}\right) \qquad (I9)$$

Fig. 2 All edges simple supported

3.2 Fully clamped plates at all edges (Fig.3)



Eq.(I8) gives for k=m=n=r=-I :

$$p = \frac{I6M_p}{b^2}\left(I+\frac{I}{\beta^2}\right) \qquad (20)$$

Fig. 3 Fully clamped plate

3.3 Two edges simple supported, two edges fully clamped (Fig. 4)

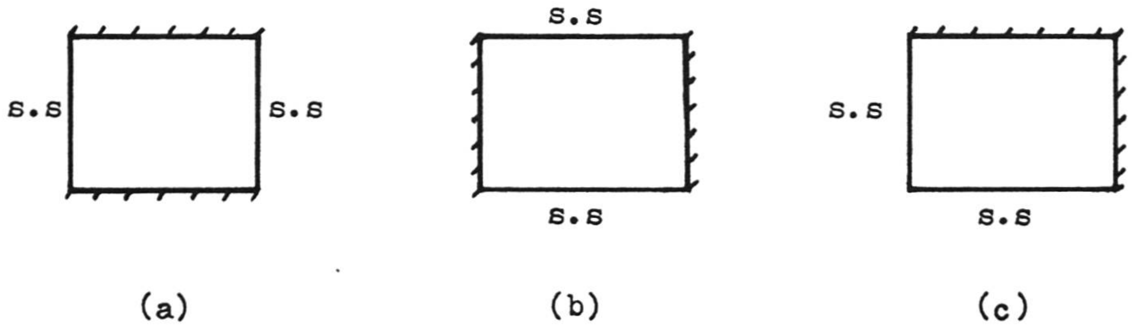


Fig. 4 Plates with 2 edges simple supported and 2 edged clamped

a. Long sides clamped (Fig. 4a):

It is k=r=0 and m=n=-I. Substituting in eq.(I8), we obtain:

$$p = \frac{8M_P}{b^2}\left(2 + \frac{1}{\beta^2}\right) \qquad (2I)$$

b. Short sides clamped (Fig. 4b):

Substituting in eq.(I8) for k=r=-I and m=n=0, we have:

$$p = \frac{8M_P}{b^2}\left(1 + \frac{2}{\beta^2}\right) \qquad (22)$$

c. Clamped two adjacent sides (Fig. 4c):

Substituting in eq.(I8) for k=m=0 and n=r=-I or for k=m=-I and n=r=0 or for k=n=0 and m=r=-I or for k=n=-I and m=r=0, in any case we find the following lower-bound load:

$$p = \frac{4M_P}{b^2}\left(3 + \frac{1}{\sqrt{3}\,\beta} + \frac{3}{\beta^2}\right) \qquad (23)$$

3.4 Three edges simple supported, one edge clamped (Fig. 5)

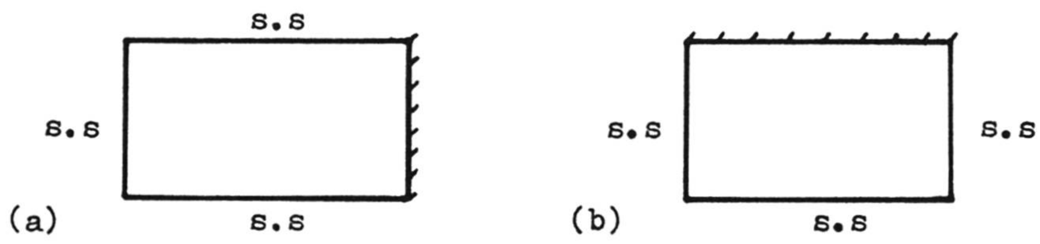


Fig. 5 Plates with 3 edges simple supported and I edge clamped

a. A short side clamped (Fig. 5a):

Substituting in eq.(I8) for k=-I and m=n=r=0 or for r=-I and k=m=n=0 it is:

$$p = \frac{4M_p}{b^2}\left(2 + \frac{2}{\sqrt{3}\,\beta} + \frac{3}{\beta^2}\right) \qquad (24)$$

b. A long side clamped (Fig. 5b):

Substituting in eq.(I8) for k=n=r=0 and m=-I or for k=m=r=0 and n=-I, we obtain for lower-bound load:

$$p = \frac{4M_p}{b^2}\left(3 + \frac{2}{\sqrt{3}\,\beta} + \frac{2}{\beta^2}\right) \qquad (25)$$

3.5 One side simple supported, three sides fully clamped (Fig. 6)

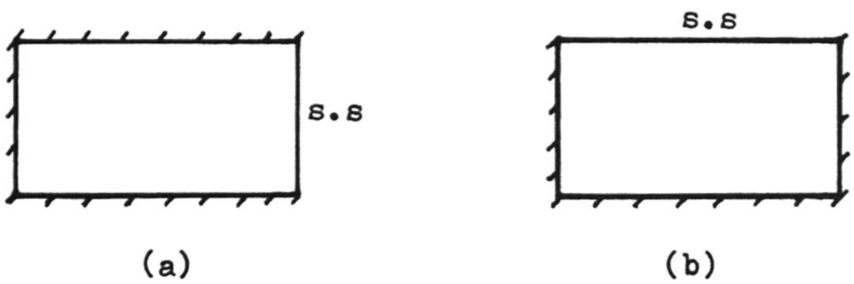


Fig. 6 Plates with 3 edges clamped and I edge simple supported

a. A short side simple supported (Fig. 6a):

It is k=0 and m=n=r=-I or r=0 and k=m=n=-I. Both cases give:

$$p = \frac{4M_p}{b^2}\left(4 + \frac{3}{\beta^2}\right) \qquad (26)$$

b. A long side simple supported (Fig. 6b):

It is m=0 and k=r=n=-I or n=0 and k=r=m=-I, with lower-bound load

$$p = \frac{4M_p}{b^2}\left(3 + \frac{4}{\beta^2}\right) \qquad (27)$$

For β = I, we have the corresponding expressions for square plates.

4. CONCLUSIONS

A unified theoretical lower-bound solution for uniformly loaded rectangular plates, based on the von MISES' yield criterion and not depended on ends fixity, has been presented. The suggested theory provides with a useful formulation, covering any combination of edges conditions and verifies the special cases of simply supported or fully built-in plates successfully.

NOTATION

a, b	length and breadth of a rectangular plate, respectively.
t	thickness of a rectangular plate.
p	lateral pressure distributed uniformly over the entire midplane of a plate.
M_p	$\sigma_y t^2/4$
σ_y	yield stress.
M_x, M_y, M_{xy}	plate bending and twisting moments per unit length in x-y reference system.
m_x, m_y, m_{xy}	generalized stresses.
β	a/b

REFERENCES

I. WOOD R.H., Plastic and elastic design of slabs and plates. The Ronald Press, New York, I96I.

2. JONES N., A lower bound to the static collapse pressure of a fully clamped rectangular plate. Department of Ocean Engineering, Report 7I-20, MIT, Cambridge, Mass., I97I.

3. JONES N., Review of the plastic behavior of beams and plates. Int. Shipbuilding Progress, Vol. I9, I972, pp. 3I3-327.

4. HODGE P.G., Plastic analysis of structures. Mc-Graw-Hill, N.Y., I959.