Statistical model for fatigue analysis of wires, strands and cables

Autor(en): **Castillo, Enrique / Esslinger, Volker / Canteli, Alfonso Fernández**

Objekttyp: **Article**

Zeitschrift: **IABSE proceedings = Mémoires AIPC = IVBH Abhandlungen**

Band (Jahr): **9 (1985)**

Heft P-82: **Statistical model for fatigue analysis of wires, strands and cables**

PDF erstellt am: **17.08.2024**

Persistenter Link: https://doi.org/10.5169/seals-39131

Statistical Model for Fatigue Analysis of Wires, Strands and Cables

Modèle statistique pour l'analyse de la fatigue des fils, des torons et des câbles

Statistisches Modell zur Ermüdungsanalyse von Drähten, Litzen und Kabeln

Enrique CASTILLO
Prof. Dr.
Univ. of Santander
Santander, Spain

Alfonso Fernández CANTELI
Dr.
Univ. of Oviedo
Oviedo, Spain

Volker ESSLINGER
Mechanical Engineer
EMPA
Dübendorf, Switzerland

Bruno THÜRLIMANN
Prof. Dr.
Swiss Federal Institute of Technology
Zurich, Switzerland

SUMMARY
A mathematical model for analyzing fatigue data of wires, strands and cables, based on statistical requirements (compatibility, stability and limit conditions), is presented and applied to test data. The model allows extrapolation of results obtained on specimens restricted by the test length of testing machines to the actual length of cables in cable-stayed bridges and similar structures. Finally, the fatigue strength of cables is analyzed from the strength of a single wire or strand by considering the progressive loss of cross sectional area as a result of successive failures of single wires.

RÉSUMÉ
Un modèle mathématique pour l'analyse des résultats de fatigue des fils, des torons et des câbles de précontrainte est dérivé en tenant compte des exigences statistiques (compatibilité, stabilité, valeurs limites) et est appliqué à des résultats d'essais. Le modèle permet une extrapolation de la longueur testée à la longueur actuelle des câbles des ponts haubanés ou des constructions analogues. La résistance à la fatigue des câbles est analysée à partir de la résistance des fils ou des torons en considérant la perte progressive de la section comme le résultat de la rupture successive des fils qui la composent.

ZUSAMMENFASSUNG
Ein mathematisches Modell zur Analyse von Ermüdungsdaten von Drähten, Litzen und Kabeln unter Berücksichtigung statistischer Anforderungen (Kompatibilität, Stabilität, Grenzwerte) wird hergeleitet und auf Versuchsresultate angewendet. Das Modell gestattet eine Extrapolation der Werte von Probestücken mit beschränkter Länge auf Längen von Kabeln für Schrägseilbrücken und ähnliche Bauwerke. Die Ermüdungsfestigkeit von Kabeln wird aus einer Betrachtung des progressiven Verlustes an Querschnittsfläche infolge von Brüchen in einzelnen Drähten aus der Festigkeit eines Drahtes oder einer Litze hergeleitet.

1. INTRODUCTION

A comprehensive study of fatigue data of reinforcing bars, prestressing wires and strands, and in particular of the influence of the specimen length on the fatigue strength [29] aroused the interest to develop a new statistical model presented in this paper.

Although the fatigue phenomena related to reinforcing and prestressing steels are being studied since several decades, systematic investigations in connection with the design of prestressed structures, cable-stayed bridges and similar structures, started only in the last twenty years. The experience gained over this period allows a researcher to have some feeling for the main features of the statistical behavior of the fatigue strength (endurance limit, scatter, shape of the S-N-curves, etc.). However, an analysis remains essentially restricted to find a mathematical equation fitting the experimental data. This leads to models which cannot be extended beyond the range of the experiments.

On the other hand, an effective model should not include factors of second order, as for instance the stress level in comparison to the stress range. In spite of the statistical significance of the stress level, there is a clear tendency today to neglect it. Wöhler curves based on the stress range with a fixed σ_{max} furnish safe design values for all stress ranges with a lower maximum stress. On the contrary, the parameter specimen "length" is an essential parameter. A model is needed for extending the S-N-curves to any actual length, for instance to the length of a tendon of a cable-stayed bridge, which cannot be tested in full length.

In order to obtain and evaluate fatigue data, several techniques, developed and used in other fields (explosives, biological, medical research, etc.), have been applied. However, a rigorous analysis of their suitability is necessary because these techniques may not be adequate to study the specific problems of fatigue of metals. For instance, this is the case for the well-known up-and-down (staircase) method [25] developed to analyse sensitivity data, i.e. data consisting in two possible alternatives (success or failure), but not suitable for fatigue, where the intermediate information (number of cycles to failure) should not be lost.

These are some of the reasons indicating that new models are required to resolve some serious deficiencies of presently used methods.

2. SINGLE WIRE OR STRAND

2.1 Present Models Used in Fatigue Analysis

The presently used statistical models for fatigue can be included in a general family of S-N-curves (Wöhler curves) defined by the function $E(N;\Delta\sigma)$, Fig. 1, which represents the cdf (cumulative distribution function) of N for a given $\Delta\sigma$, where N is the logarithm of the number of cycles and $\Delta\sigma$ the stress range or a function of it, normally log.

A model can be described with the help of the following functions:

$g(\Delta\sigma)$: Median of N as a function of $\Delta\sigma$
$h(\Delta\sigma)$: Standard deviation of N as a function of $\Delta\sigma$
$F(\Delta\sigma;N)$: Rate of failures for given $\Delta\sigma$ and N or theoretical cdf of $\Delta\sigma$ for given N
μ, σ : Median and standard deviation associated with $F(\Delta\sigma;N)$
N_0 : Limit number of cycles to which a specimen is tested (normally $N_0 = 2\cdot10^6$)

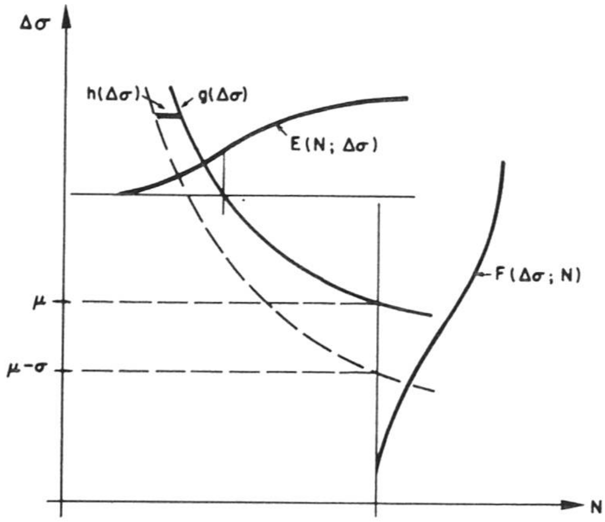


Fig. 1: General Model for Fatigue Analysis

FUNCTION	TYPE OR FAMILY	REFERENCES
g(Δσ)	Linear	5,41,43,47,51,52,53
	Polygonal	29,33,34,35,46,50
	Parabolic	5,51
	Exponential	26,27
	Hyperbolic	
	Others	41,43,55,56
h(Δσ)	Constant	5,50,53
	Linear	29,47
E(N;Δσ)	Normal	4,5,13,29,33,50,53
	Weibull	4,41,43,52,57
	Free	4
F(Δσ;N)	Normal	4,10,24,25,29,33,36,37,38,40,50
	Weibull	
	arcsin	36,37
	Extended normal	2,13,14
Δσ	Natural	2,4,5,10,13,24,36,37,38,40,41,51,55,56
	Logarithmic	5,29,34,35,36,41,43,46,47,50,51,53
N	Logarithmic	all

Table 1: Summary of Models for Fatigue Analysis of Reinforcing Bars, Prestressing Wires and Strands

The differences in the various proposed models or techniques of estimating their parameters are directly related to the type of the selected functions. Table 1 shows a summary of some presently used models in fatigue of steel or reinforcing and prestressing steel.

Some of the models are only partially defined in the sense that the function $E(N;\Delta\sigma)$ is not, or is incompletely established. Others do not take into account the relationship between the functions $E(N;\Delta\sigma)$ and $F(\Delta\sigma;N)$. Such models are either internally incompatible, as explained later, or lead to cdf belonging to extraneous or not common families.

2.2 Deficiencies in the Present Statistical Analysis

The fatigue analysis by means of the reported models can be unsatisfactory because of:

2.2.1 Deficiencies in the Models

(1) The model does not reproduce the real statistical properties of the experimental data. In this category are included those models which assume a linear law for $g(\Delta\sigma)$, a constant law for $h(\Delta\sigma)$ and those showing no endurance limit. Experimental evidence contradicts these assumptions [8, 29, 39], see Fig. 2.

(2) The model fails to fulfil stability or internal compatibility conditions. Due to the fact that the models are implicitly assumed valid for any arbitrary length, the $E(N;\Delta\sigma)$ and $F(\Delta\sigma;N)$ function families must be stable. The stability of the model is guaranteed if, and only if, the $E(N;\Delta\sigma)$ and $F(\Delta\sigma;N)$ functions for different lengths belong to the selected family. For instance, if the fatigue strength of different pieces of material are assumed statistically independent, and $F_{L_o}(x)$ is the cdf of the strength of a piece of length L_o, the statistical theory shows that the cdf, $F_L(x)$, for a piece of length $L = n\,L_o$ is given by

$$F_L(x) = 1 - \left[1 - F_{L_o}(x)\right]^n \qquad (1)$$

For $F_{L_o}(x)$ to belong to a stable family, $F_{nL_o}(x)$ must belong to the same family for all n as well. Whereas normal and arcsin families do not satisfy stability, Weibull and extended normal do. (*).

On the other hand, functions $E(N;\Delta\sigma)$ and $F(\Delta\sigma;N)$ must be compatible, i.e. they must satisfy the relation linking them. Because of the definition of the function $F(\Delta\sigma;N)$, as the rate of failures for given $\Delta\sigma$ and N, the compatibility condition is given by

$$F(\Delta\sigma;N) = E(N;\Delta\sigma) \quad ; \quad \forall\ \Delta\sigma \text{ and } N \qquad (2)$$

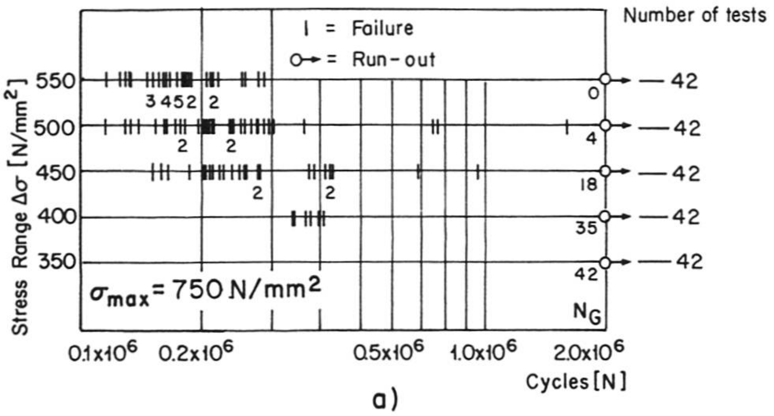
None of the reported models fulfil all these conditions.

2.2.2 Deficiencies in the Estimation of Model Parameters

(1) The run-outs are not optimally handled. In some cases the run-outs are disregarded. In others, they are treated as failures.

(2) The number of cycles to failure is not used. The up-and-down method shows this deficiency since it neglects the important part of the information contained in the broken specimens (number of cycles to failure).

(*) The extended normal family of cdfs is defined by $F(x;\mu,\sigma,n) = 1-(1-\phi[\frac{x-\mu}{\sigma}])^n$ where ϕ is the cdf of the standard normal r.v., and μ, σ and n are parameters.



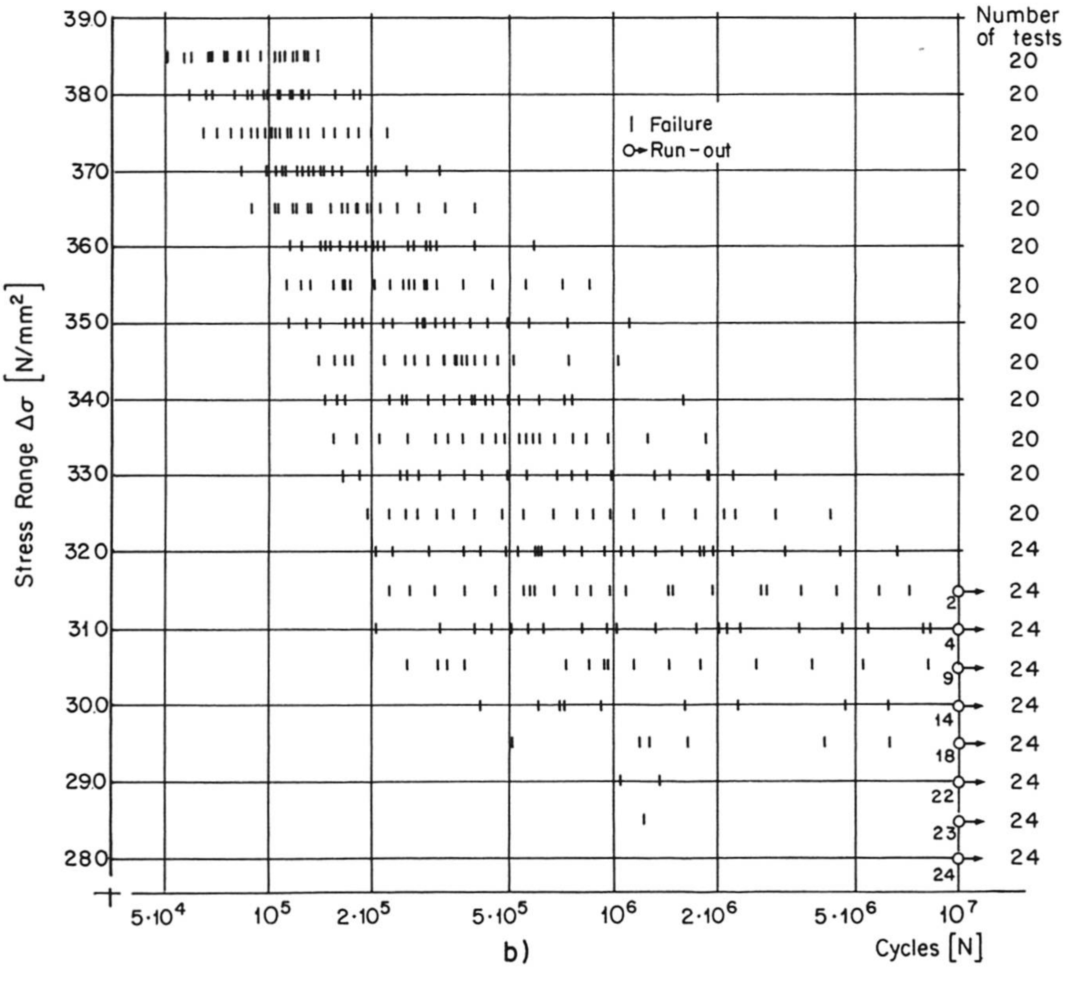


Fig. 2: Experimental Evidence for the Non-Linearity of the g(Δσ)-Function Non-Constant Character of the h(Δσ)-Function and the Existence of an Endurance Limit [8,39].

2.3 Minimal Requirements for a Valid Fatigue Analysis

According to the above, a fatigue analysis has to satisfy the following minimal conditions:

(1) Concerning experimental evidence

- The $g(\Delta\sigma)$-function must be nonlinear.
- The $h(\Delta\sigma)$-function must increase as $\Delta\sigma$ decreases.
- An endurance limit exists, that is a $\Delta\sigma$ value below which there is a zero probability of failure.

(2) Concerning stability and internal compatibility of the model

- The $E(N;\Delta\sigma)$ and $F(\Delta\sigma;N)$ functions must be stable
- The $E(N;\Delta\sigma)$ and $F(\Delta\sigma;N)$ functions must be compatible for any N and $\Delta\sigma$.

(3) Concerning model parameter estimation

- The method should take into account:
  A specific treatment for run-outs.
  All the information, i.e. the number of cycles to failure for broken specimens.

2.4 Derivation of Proposed Model

As mentioned before, the model is wholly defined as soon as the function $E(N;\Delta\sigma)$ is chosen. The selection of the $E(N;\Delta\sigma)$ function for the proposed model is based on physical considerations together with stability, limit and compatibility conditions. The procedure is summarized in Fig. 3. It will be done simultaneously for the analysis of the random variable (r.v.) number of cycles to failure, N, for a given stress range, $\Delta\sigma$, and for the analysis of the r.v. stress range, $\Delta\sigma$, associated with a given N.

The role of N as a r.v. is physically clear. Experimentally it is obtained by just fixing the stress range and measuring the number of cycles to failure. On the contrary, the nature of $\Delta\sigma$ for a given N needs explanation. Given a specimen of length L_o, a one-to-one correspondence between $\Delta\sigma$ and N exists. If it were possible to test repeatedly the same piece at different $\Delta\sigma$-levels, the S-N-curve for that specimen could be established (see Fig. 4). Therefore, given the specimen, a $\Delta\sigma_o$ can be associated with every N_o.

2.4.1 Physical Considerations

For a wire of length L composed of n imaginary pieces of length L_o the fatigue failure takes place in the weakest link (Fig. 5), i.e.

$$x_{nL_o} = \min(X_1, X_2, \ldots, X_n) \tag{3}$$

where x_{nL_o} represents either the number of cycles to failure associated with a given stress range $\Delta\sigma$ or the stress range corresponding to a given number of cycles for the wire of length nL_o. This holds because, if a wire of length nL_o is considered, the number of cycles to failure, given $\Delta\sigma$, will be the minimum of the corresponding N-values for all its constituent pieces (see Fig. 6).

The flaws in the material causing fatigue failure can be systematically or independently distributed along the wire. In the first case there is a statistical dependence between neighboring pieces [12], but independence can be assumed for distant ones. In the second the statistical independence can be assumed throughout.

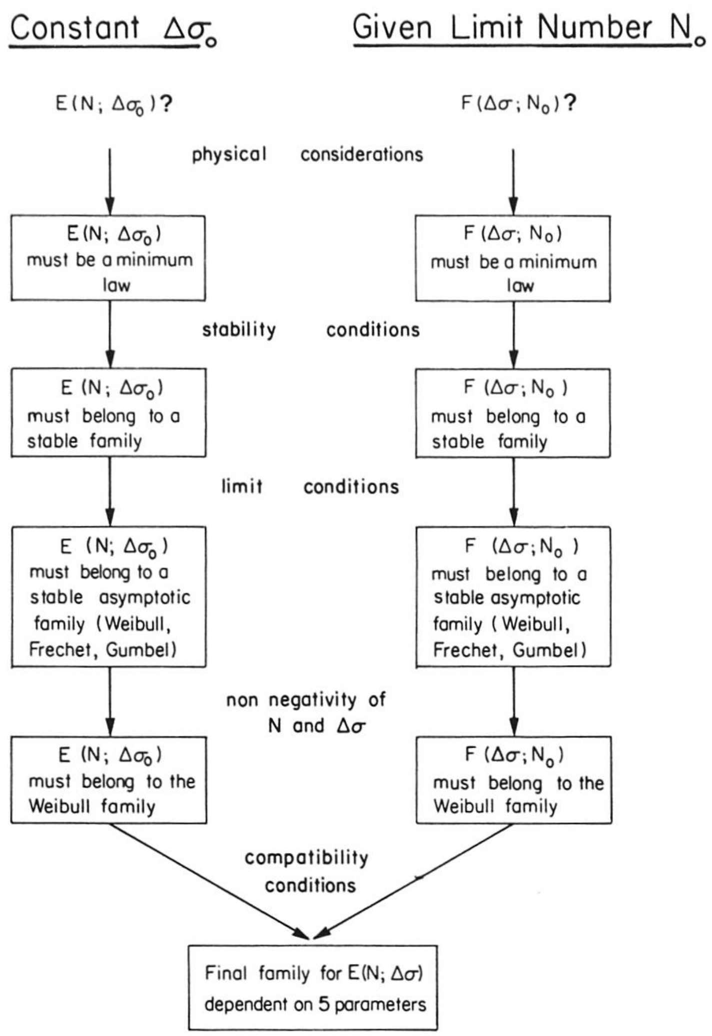


Fig. 3: Illustration of the Selection Procedure for $E(N;\Delta\sigma)$

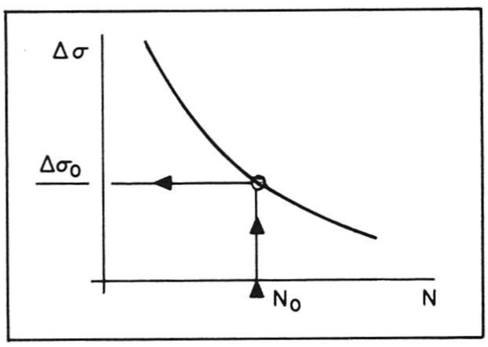


Fig. 4: One-to-one Correspondence between $\Delta\sigma$ and N for Given Specimen

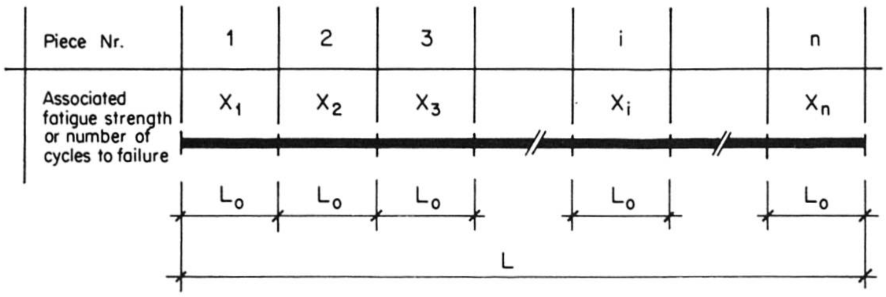


Fig. 5: Schematic Representation of an Element Composed of n Imaginary Pieces and Associated Fatigue Strength

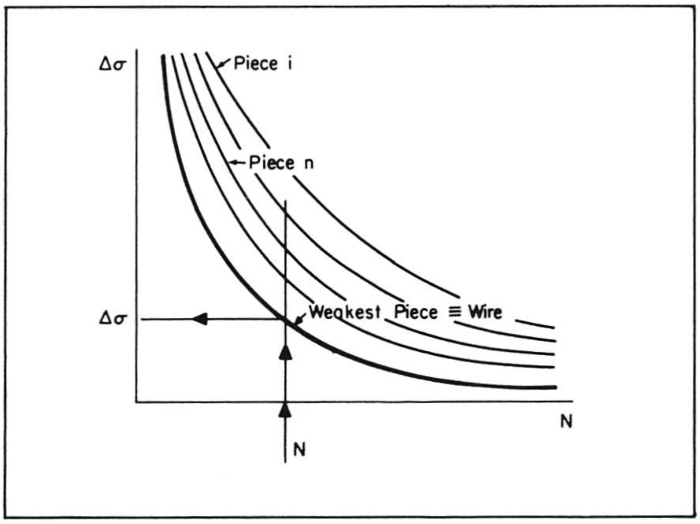


Fig. 6: Schematic Representation of the One-to-One Correspondence between $\Delta\sigma$ and N for a Wire Consisting of n pieces

If the fabrication process is regular and the storage and manipulation process is homogeneous, there are physical reasons to assume the same distribution function for the fatigue strength of the different pieces.

2.4.2 Stability Condition

The stability condition implies that the functions $E(N;\Delta\sigma)$ and $F(\Delta\sigma;N)$ must belong to a stable family.

2.4.3 Limit Condition

If the length of the imaginary constituent pieces of a wire goes to zero, or the number of pieces goes to infinity, the families of the above E and F functions must be asymptotic. In the case of independence or asymptotic independence (long elements) between the number of cycles to failure or the stress range for different pieces, the only asymptotic families are Weibull, Frechet and Gumbel. Because N and $\Delta\sigma$ must be positive, only Weibull satisfies all the requirements [21, 31, 54].

This implies the cdf $E(N;\Delta\sigma)$ for pieces of length L_o respectively L are given by

$$E^*(N;\Delta\sigma,L_o) = 1 - \exp\left[-\left[\frac{N - N_o(\Delta\sigma,L_o)}{N_a(\Delta\sigma,L_o)}\right]^{b(\Delta\sigma,L_o)}\right] \quad (4)$$

and

$$E^*(N;\Delta\sigma,L) = 1 - \exp\left[-\left[\frac{N - N_o(\Delta\sigma,L)}{N_a(\Delta\sigma,L)}\right]^{b(\Delta\sigma,L)}\right] \quad (5)$$

where $N_o(\Delta\sigma,L)$, $N_a(\Delta\sigma,L)$ and $b(\Delta\sigma,L)$ are functions of $\Delta\sigma$ and L to be determined. In the case of statistical idependence between the pieces, expressions (1), (4) and (5) result in

$$E^*(N;\Delta\sigma L) = 1 - \exp\left[-\frac{L}{L_o}\left[\frac{N - N_o(\Delta\sigma,L_o)}{N_a(\Delta\sigma,L_o)}\right]^{b(\Delta\sigma,L_o)}\right] \quad (6)$$

Identification of Eqs. (5) and (6) leads to

$$N_o(\Delta\sigma,L) = N_o(\Delta\sigma,L_o) = N_o(\Delta\sigma) \quad (7)$$

$$b(\Delta\sigma,L) = b(\Delta\sigma,L_o) = b(\Delta\sigma) \quad (8)$$

$$N_a(\Delta\sigma,L) = N_a(\Delta\sigma,L_o)\left[\frac{L_o}{L}\right]^{1/b(\Delta\sigma)} = N_a^o(\Delta\sigma)\left[\frac{L_o}{L}\right]^{1/b(\Delta\sigma)} \quad (9)$$

where $N_a^o(\Delta\sigma) = N_a(\Delta\sigma,L_o)$.

The expressions (7) and (8) prove that $N_o(\Delta\sigma,L)$ and $b(\Delta\sigma,L)$ do not depend on L.

Following a similar process with $\Delta\sigma$ instead of N leads to:

$$F^*(\Delta\sigma;N,L) = 1 - \exp\left[-\frac{L}{L_o}\left[\frac{\Delta\sigma - \Delta\sigma_o(N,L_o)}{\Delta\sigma_a(N,L_o)}\right]^{a(N,L_o)}\right] \quad (10)$$

and

$$\Delta\sigma_o(N,L) = \Delta\sigma_o(N,L_o) = \Delta\sigma_o(N) \quad (11)$$

$$a(N,L) = a(N,L_o) = a(N) \quad (12)$$

$$\Delta\sigma_a(N,L) = \Delta\sigma_a^o(N)\left[\frac{L_o}{L}\right]^{1/a(N)} \quad (13)$$

where $\Delta\sigma_a^o(N)$ stands for $\Delta\sigma_a(N,L_o)$. The expressions (11) and (12) show that $\Delta\sigma_o(N,L)$ and $a(N,L)$ do not depend on L.

2.4.4. Compatibility Condition

The compatibility requirement is given by Eq. (2), which in this case becomes

$$F^*(\Delta\sigma;N,L) = E^*(N;\Delta\sigma,L) \quad ; \; \forall\Delta\sigma \text{ and } N \tag{14}$$

Considering Eqs. (4) and (7) to (13), Eq. (14) goes over to

$$\left[\frac{N - N_o(\Delta\sigma)}{N_a^o(\Delta\sigma)}\right]^{b(\Delta\sigma)} = \left[\frac{\Delta\sigma - \Delta\sigma_o(N)}{\Delta\sigma_a^o(N)}\right]^{a(N)} \tag{15}$$

This functional equation gives the necessary condition which the functions $N_o(\Delta\sigma)$, $N_a^o(\Delta\sigma)$, $b(\Delta\sigma)$ and $\Delta\sigma_o(N)$, $\Delta\sigma_a^o(N)$ and $a(N)$ have to fulfil.

A simplification of equation (15) is obtained by making

$$b(\Delta\sigma) = a(N) = A \tag{16}$$

where A is constant.

In this case the general solution of the resulting functional equation (see Appendix) gives

$$E^*(N;\Delta\sigma,L) = 1 - \exp\left[-\frac{L}{L_o}\left(\frac{(N-B)\,(\Delta\sigma - C)}{D} + E\right)^A\right] \tag{17}$$

The percentile curves can be obtained by making $E^*(N;\Delta\sigma,L)$ equal to a constant P, i.e. their analytical equations are

$$(N - B)(\Delta\sigma - C) = D\left[\left[-\frac{L_o}{L}\log(1 - P)\right]^{1/A} - E\right] \tag{18}$$

The equation (18) corresponds to equilateral hyperbolas, with asymptotes (Fig. 7):

$$N = B$$
$$\Delta\sigma = C \tag{19}$$

It is possible for both limit values, B and C, to become zero.

The resulting model satisfies the three conditions established by the experimental evidence:

- The $g(\Delta\sigma)$-function is not linear (hyperbola)
- The $h(\Delta\sigma)$-function increases with decreasing $\Delta\sigma$
- There exists an endurance limit given by $\Delta\sigma = C$

Additionally, the model shows the existence of a zero-perceptile curve (S-N threshold curve), which does not coincide with the asymptotes, beyond which failure does not occur. Physical evidence seems to support this theoretical result.

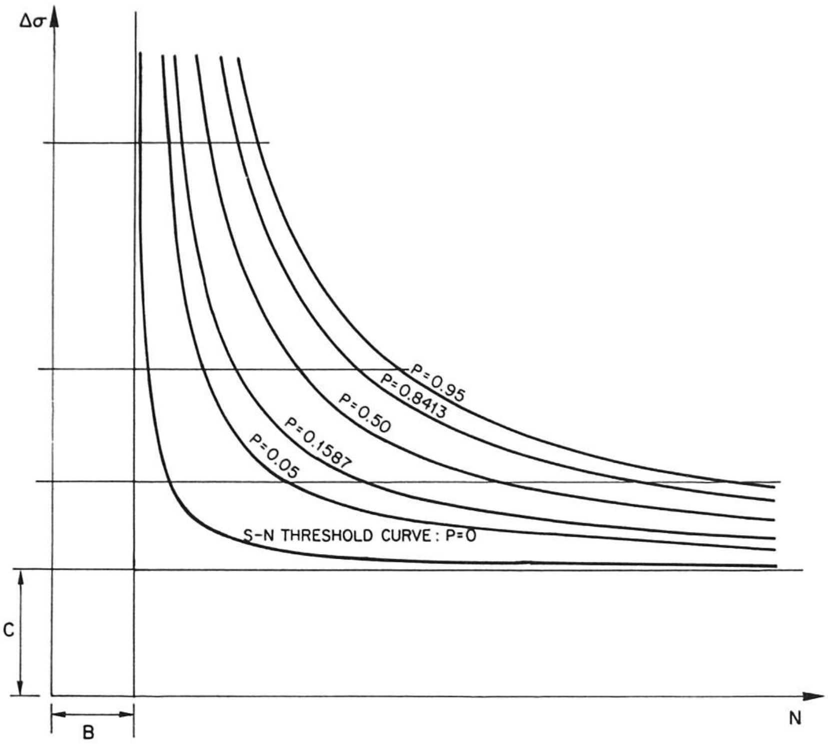


Fig. 7: Percentile Curves Given by the Proposed Model

The solution of the functional equation (15) together with (16) furnishes a model depending on five parameters A, B, C, D and E which have the following meaning:

A : Weibull shape or slope parameter (Fig. 8)
B : Asymptotic N limit (Fig. 7)
C : Endurance limit, i.e. $\Delta\sigma$ value below which a zero probability to get failures exists for $N \rightarrow \infty$ (Fig. 7)
D : Scale fitting parameter obtained for an arbitrarily chosen reference length L_o
E : Constant defining the S-N threshold curve below which a zero probability of fatigue failure exists.

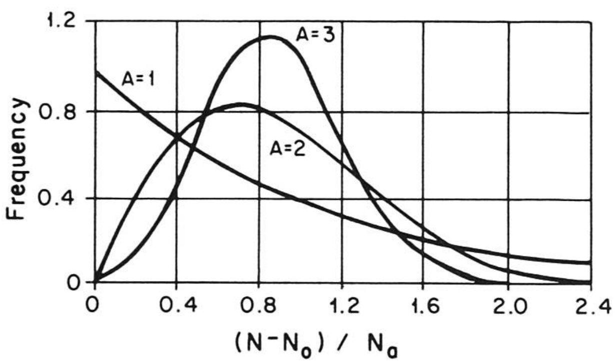


Fig. 8: Typical Weibull-Distribution Curves

Figure 9 shows the theoretical influence of length on the median and the 0.1587-quantile curves. For $L/L_o \rightarrow \infty$ the hyperbolas degenerate into the zero-

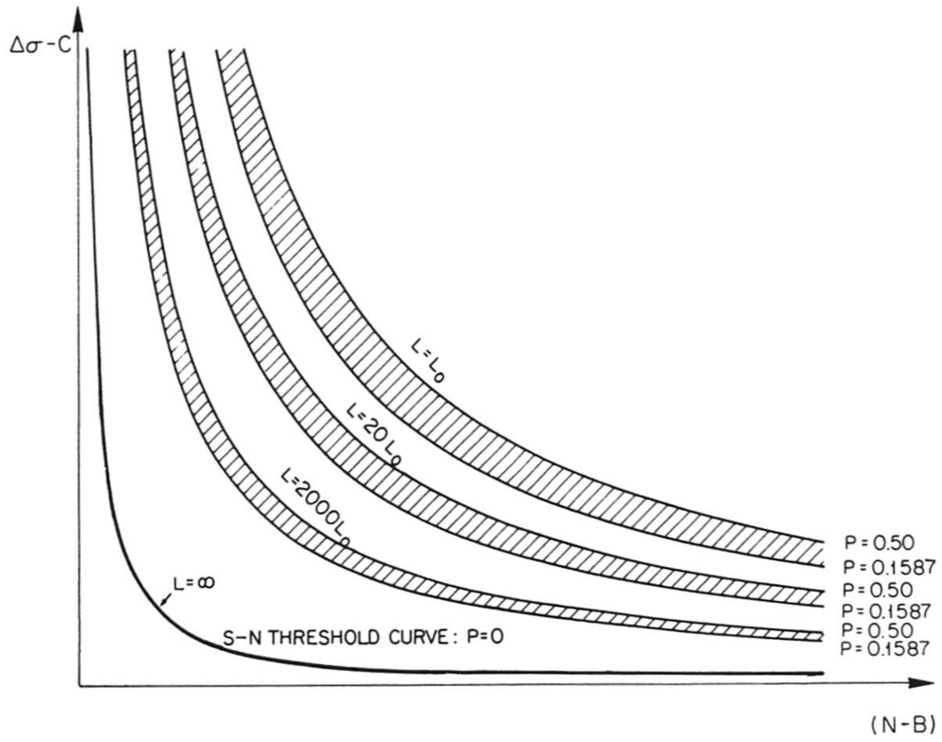


Fig. 9: Theoretical Influence of Length on the Median and 0.1587-Quantile Curves

percentile hyperbola. This implies a zero scatter for every $\Delta\sigma$. Therefore, for long elements, the $\Delta\sigma$ for design can be obtained from the S-N threshold curve.

2.5 Estimation of the Model Parameters

Data from fatigue tests is generally reported in censored form, because the testing strategy, due to time and economical reasons, only permits specimens to fail below a specified limit number of cycles N_o. In this sense, the data can be termed statistically incomplete, and specific methods are required, to include run-out data.

For the present model, the E-M-algorithm [22] is used. This algorithm is based upon the maximization of the likelihood function and operates in an iterative way. Each iteration of the E-M-algorithm involves two steps, which are called the expectation step (E-step) and the maximization step (M-step).

In the first step the q run-outs of a given stress level $\Delta\sigma_i$ are assigned to expected values, N_j^* (j=1,2,...,q), of the q order statistics conditioned to the censoring value N_o (see Fig. 10), i.e.

$$N_j^* = \int_{N_o}^{\infty} N \frac{q!}{(j-1)!\,(q-j)!} \left[F_{N_o}(N)\right]^{j-1} \left[1-F_{N_o}(N)\right]^{q-j} f_{N_o}(N)\,dN \qquad (20)$$

with

$$F_{N_o}(N) = 1 - \exp\left[-\frac{L_i}{L_o}\left[\left(\frac{(N-B)(\Delta\sigma_i-C)}{D}+E\right)^A - \left(\frac{(N_o-B)(\Delta\sigma_i-C)}{D}+E\right)^A\right]\right] \qquad (21)$$

where L_i and $\Delta\sigma_i$ are the length and the stress range for the i-th specimen, and $f_{N_o}(N)$ is the derivative of $F_{N_o}(N)$.

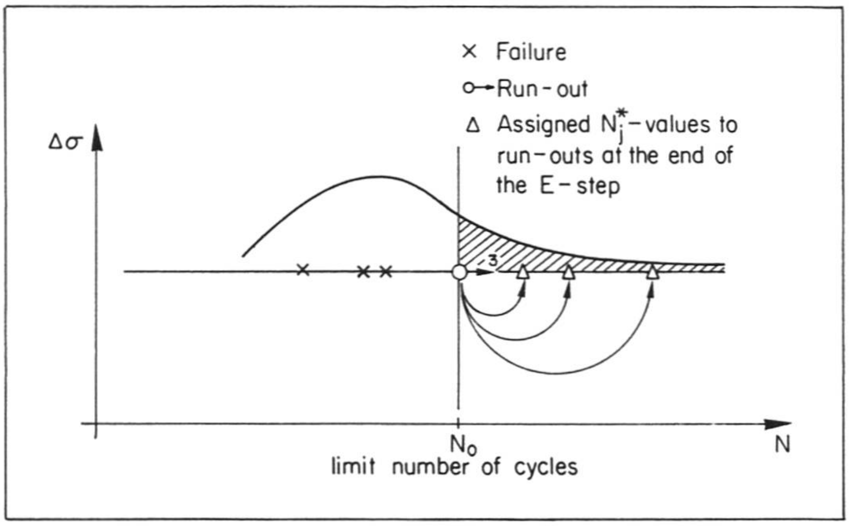


Fig. 10: Illustration of the E-step in the E-M-Algorithm

The second step is performed by the maximum likelihood method as if all the data were not censored (the run-outs are assumed real failures with assigned number of cycles, which for the first iteration are identified with the limit number of cycles N_o), i.e. from Eq. (17) the following log-likelihood function is maximized with respect to A,B,C,D and E:

$$V = \sum_{i=1}^{n} \left[- \frac{L_i}{L_o} \left[\frac{(N_i - B)(\Delta\sigma_i - C)}{D} + E \right]^A + \log \left[\frac{L_i}{L_o} \frac{A}{D} \left[\frac{(N_i - B)(\Delta\sigma_i - C)}{D} + E \right]^{A-1} (\Delta\sigma_i - C) \right] \right] \quad (22)$$

where n is the sample size and N_i is the assigned number of cycles for run-outs or the real number of cycles to failure for the complete data, for the i-th specimen.

In Fig. 11, a diagram of the E-M-algorithm is presented.

2.6 Considerations to the Planning of Tests

The proposed model is also useful for establishing testing strategies [23]. In Fig. 12 the median curves for two different lengths are schematically represented and the areas covered by the up-and-down method [24,25,36] are shadowed. The solid parts of the curves are the zones from which the data are obtained and where the curve is fitted. The dashed parts of the curves represent the extrapolated zones of the model. Due to the large extent of these extrapolated areas as compared to the tested zone this method cannot be recommended for an over-all description of the S-N field. On the other hand, the reliability of the endurance limit based on long specimens is higher than that for short specimens, because of the proximity of the data to that limit.

When a S-N-curve reliable over its entire range is needed, the investigation of a wider interval of Δσ-values is necessary. Short specimens require larger testing areas than long ones (higher stress-ranges and greater number of cycles) in order to achieve the same reliability (Fig. 13).

On the other hand, the statistical information implied by the failure of one specimen of length nL_o at N cycles is equivalent to the information associated with the failure of one specimen of length L_o at N cycles plus that associated with n-1 specimens of the same length L_o censored at N. This means that, ir-

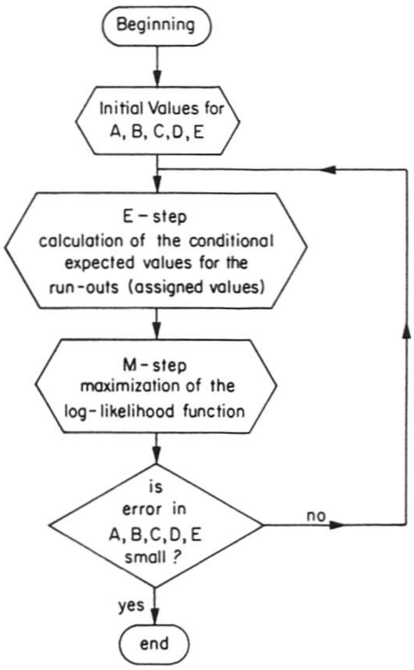


Fig. 11: Flow Chart for E-M-Algorithm

respective of economical considerations, the significance of the statistical information supplied by long samples is much larger and reliable than that given by short samples.

2.7 Examples of application

The validity of the model was tested with fatigue data of prestressing wires and strands used in prestressed structural members and in tendons of cable-stayed bridges. The tests were carried out at the Swiss Federal Testing Laboratories (EMPA-Dübendorf) and the results are reported in [29]. In the following, $\Delta\sigma$ will represent the logarithm of the stress range.

2.7.1 Application to Prestressing Wires

Specimens of three different lengths (140 mm, 1960 mm and 8540 mm), series 1 to 3, ranging from the very short usual specimen length to the record length ever reported, were tested. They seem to cover a reasonable length range for testing the model. Initially, the model parameters were calibrated for the three lengths separately. The resulting values of the parameters and derived characteristic values are given in Table 2, and the corresponding S-N-curves shown in Figs. 14, 15 and 16.

A comparison of the characteristic values for $N_0 = 2 \cdot 10^6$ cycles obtained by the proposed method and three variants [24,25,36] of the up-and-down method from [29] is given in Table 3.

A first look at these results calls the attention to the fact that the median values for the proposed method lie always around 1 to 5 % below the medians

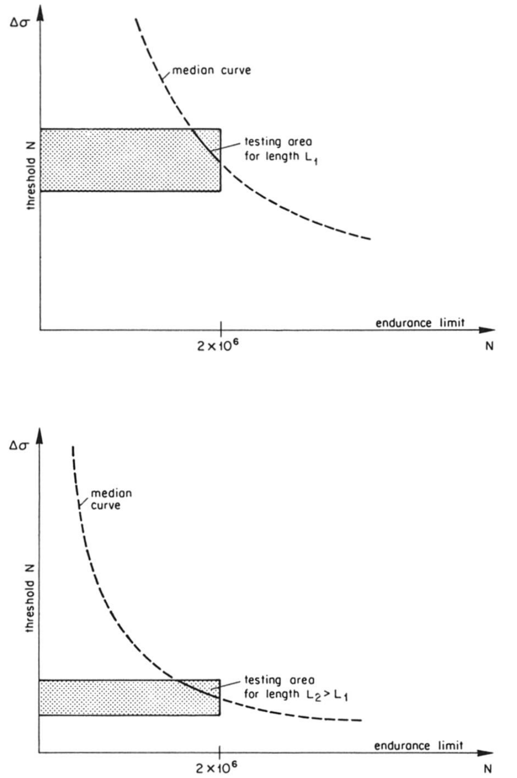


Fig. 12: Schematic Representation of the Areas Investigated by the Up-and-Down Method

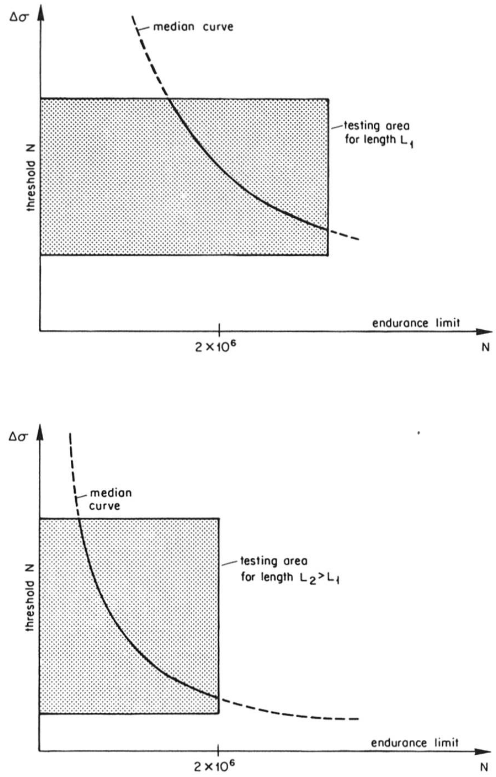


Fig. 13: Schematic Representation of the Areas to be Investigated in Order to Have Equally Reliable S-N-Curves over its Entire Range

Series	1	2	3
Length	140 mm	1960 mm	8540 mm
A	2.99	4.99	10.2
B (Threshold N)	11.46 (95103)	9.55 (14004)	9.69 (16215)
C (Endurance Limit)	5.76 (316.6)	5.45 (231.9)	5.46 (234.1)
D	0.862	1.508	1.091
E	-0.095	-0.092	-0.092
Quantile 0.05	361.2	281.9	283.1
Quantile 0.1587	380.7	295.3	289.2
Median	417.8	316.3	297.4
Quantile 0.8413	460.3	336.2	304.0
Quantile 0.95	489.4	348.2	307.6

Table 2: Prestressing Wire:
Parameters and Quantiles for $N = 2 \cdot 10^6$ Cycles Calculated Using the Proposed Model for Series 1 to 3 Separately

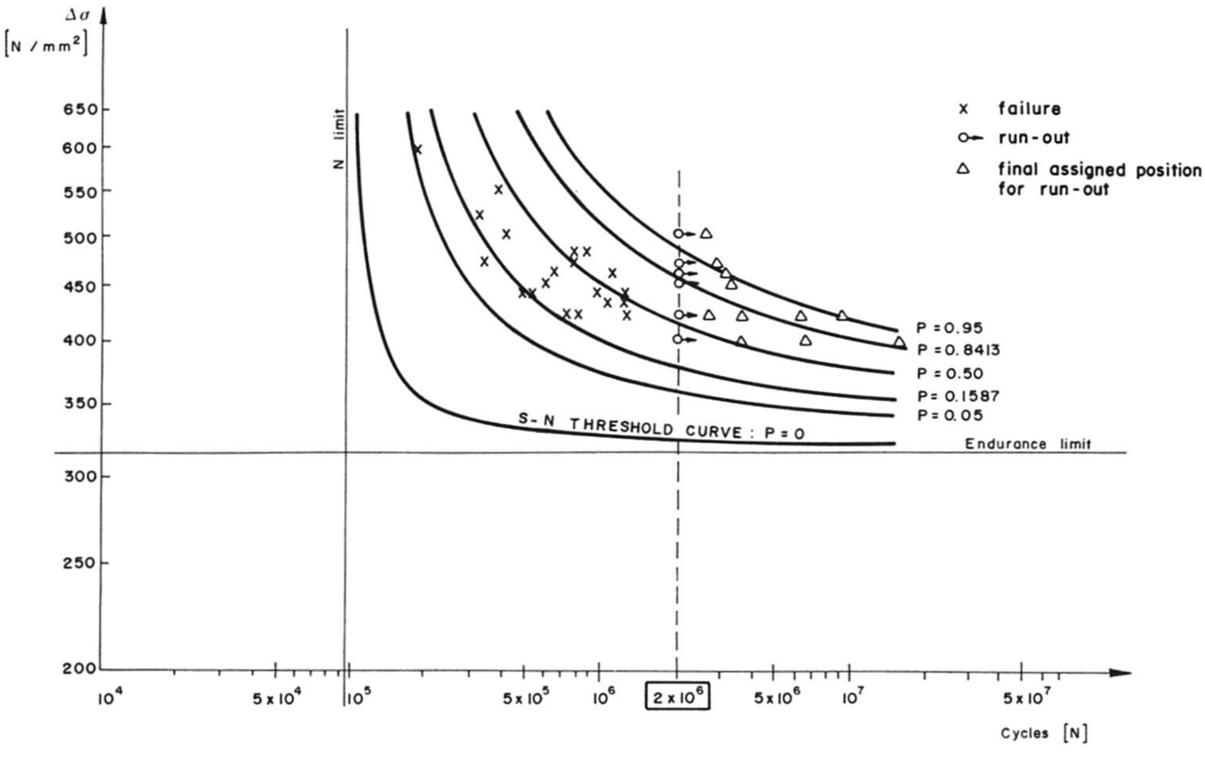


Fig. 14: S-N-Curves for Prestressing Wire of 140 mm Length Based on Test Data Series 1 (32 Specimens)

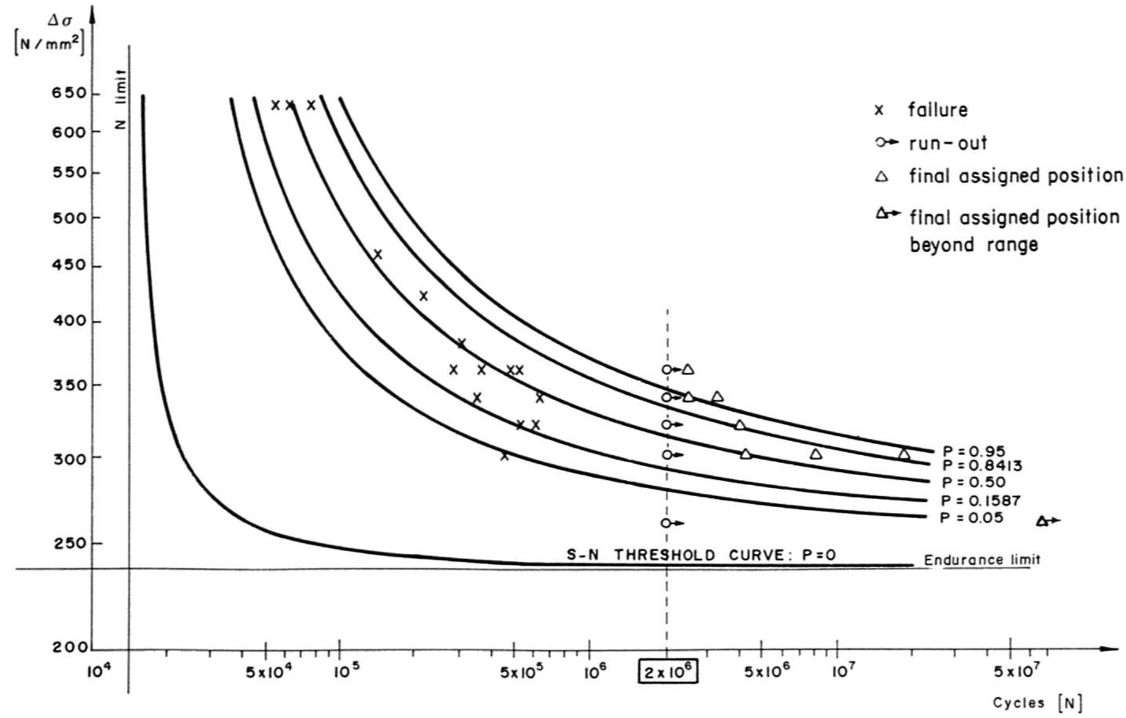


Fig. 15: S-N-Curves for Prestressing Wire of 1960 mm Length Based on Test Data Series 2 (26 Specimens)

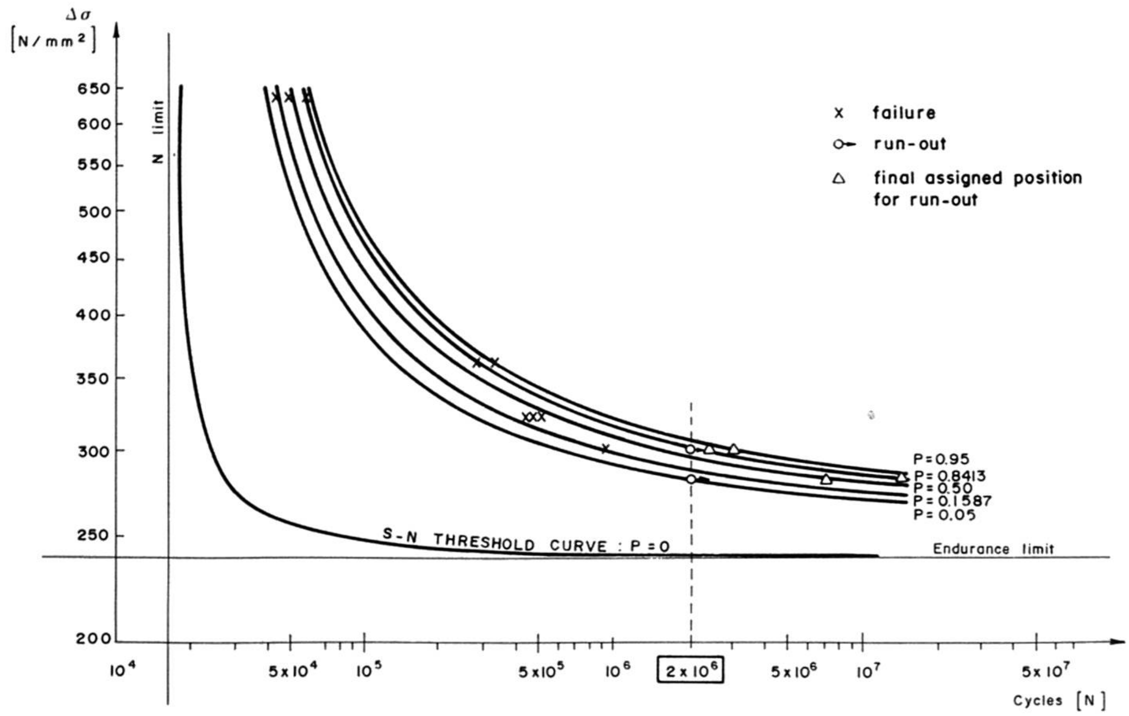


Fig. 16: S-N-Curves for Prestressing Wire of 8540 mm Length Based on Test Data Series 3 (14 Specimens)

Method of evaluation	Prooflength (mm)	Median	Standard Deviation s	Quantile 0.05
Dixon/Mood	Series 1 140	420.0	9.0	405.0
	Series 2 1960	330.0	30.2	280.0
	Series 3 8540	303.3	8.1	290.0
Deubelbeiss	Series 1 140	420.0	17.0	392.0
	Series 2 1960	330.0	39.0	267.0
	Series 3 8540	305.0	18.0	275.0
Hück	Series 1 140	420.0	10.1	404.0
	Series 2 1960	330.1	31.5	280.0
	Series 3 8540	305.0	-	-
Proposed method	Series 1 140	417.8	37.1 (*)	361.2
	Series 2 1960	316.3	21.0 (*)	281.9
	Series 3 8540	297.4	8.2 (*)	283.1

(*) $s_{approx} = \Delta\sigma_{0.50} - \Delta\sigma_{0.1587}$

Table 3: Prestressing Wire:
Comparison of Characteristic Values for $N = 2 \cdot 10^6$ Cycles
Calculated for Series 1 to 3 Using Different Methods

given by the other medhods. This is the natural consequence of linearity implied by the up-and-down method in the neighborhood of the limit number of cycles.

A second look reveals important discrepancies between the standard deviation from the proposed and the other three methods. Moreover, only the proposed method shows a decreasing standard deviation with increasing length, corresponding to physical evidence. The results for large series reported in [2] and [8] show the inconsistency of the very low value for s in the first series obtained by the first three methods. This was discussed in detail in [29].

A comparison of the parameter values given in Table 2 reveals the following facts:

- The value of A increases with length. On the contrary, the value of D seems to vary randomly. The agreement between parameters B and C for series 2 and 3 can be judged as excellent.
- The good agreement between the endurance limits for series 2 and 3 does not include series 1. This could be a consequence of the fact that in series 1

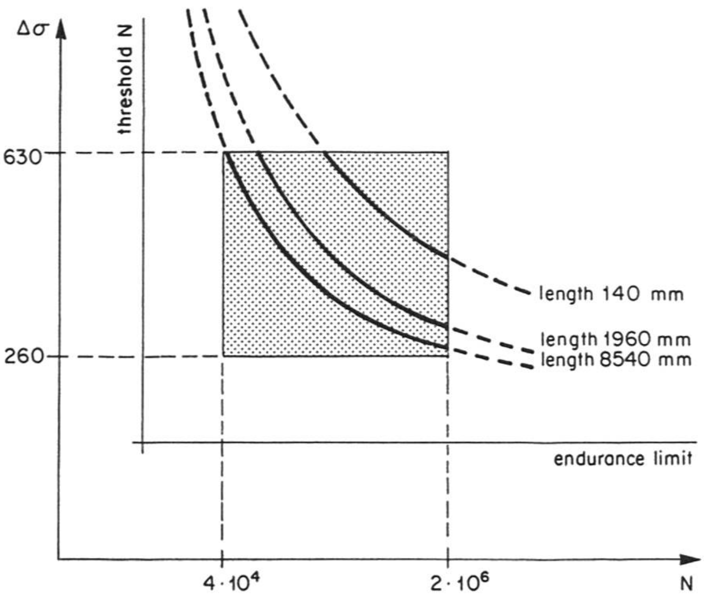


Fig. 17: Prestressed Wire:
Schematic Representation of the Investigated
S-N-Field in Series 1 to 3

all specimens were tested at stress ranges considerably above the endurance limit corresponding to a shorter part of the S-N-curve (see Figs. 14 and 17). Furthermore, series 1 was conducted at high frequency (133 Hz), contrarily to series 2 and 3, which were tested at low frequency (6 Hz).

Table 4 shows a comparison between the calculated values from series 1 and 3 and the predicted values for the same series on the basis of the calculated values from series 2. Whereas both sets of values agree in the case of series 3, differences arise in series 1, specially for the endurance limit. This confirms the reasons given in chapter 2.6 "Considerations to the Planning of Tests" about the questionability of tests at high frequency with short specimens if the studies range of N is not sufficiently extended.

Due to the fact that the parameter estimation method allows the simultaneous use of specimens with different lengths, the parameters were also calculated from the test results of all three series together.

For Figs. 18, 19 and 20 the parameter values from series 1, 2 and 3 together were used to predict the S-N-curves for the three specimen lengths. Also shown are the test results of the series with the corresponding length.

Table 5 compares the resulting values with those calculated from series 2 only. They are in close agreement.

2.7.2 Application to Prestressing Strands

Specimens of four different lengths (490 mm, 1100 mm, 1960 mm and 3860 mm), series 4 to 7 were tested. For the lengths 1100 and 3860 mm the up-and-down technique was applied.

The model parameters were only calculated for all series together. The resulting and derived characteristic values are given in Table 6 and the corresponding S-N-curves are shown in Figs. 21, 22, 23 and 24. A comparison of the characteristic values for the series 5 and 7, obtained by the proposed method and three variants [24,25,36] of the up-and-down method from [29], is given in Table 7. These were the only two series for which the fatigue strength for $N_0 = 2 \cdot 10^6$ cycles was studied by the up-and-down method.

	Series 1 Specimen Length 140 mm		Series 3 Specimen Length 8540 mm	
	Predicted from series 2	Calculated	Predicted from series 2	Calculated
Quantile 0.05	318.2	361.2	271.2	283.1
Quantile 0.1587	344.1	380.7	280.7	289.2
Median	386.6	417.8	295.4	297.4
Quantile 0.8413	428.9	460.3	309.2	304.0
Quantile 0.95	455.3	489.4	317.4	307.6
Endurance Limit	231.9	316.6	231.9	234.1
Threshold N	14004	95103	14004	16215

Table 4: Prestressing Wire:
Calculated and Predicted Characteristic Values for $N = 2 \cdot 10^6$
Cycles for Series 1 and 3

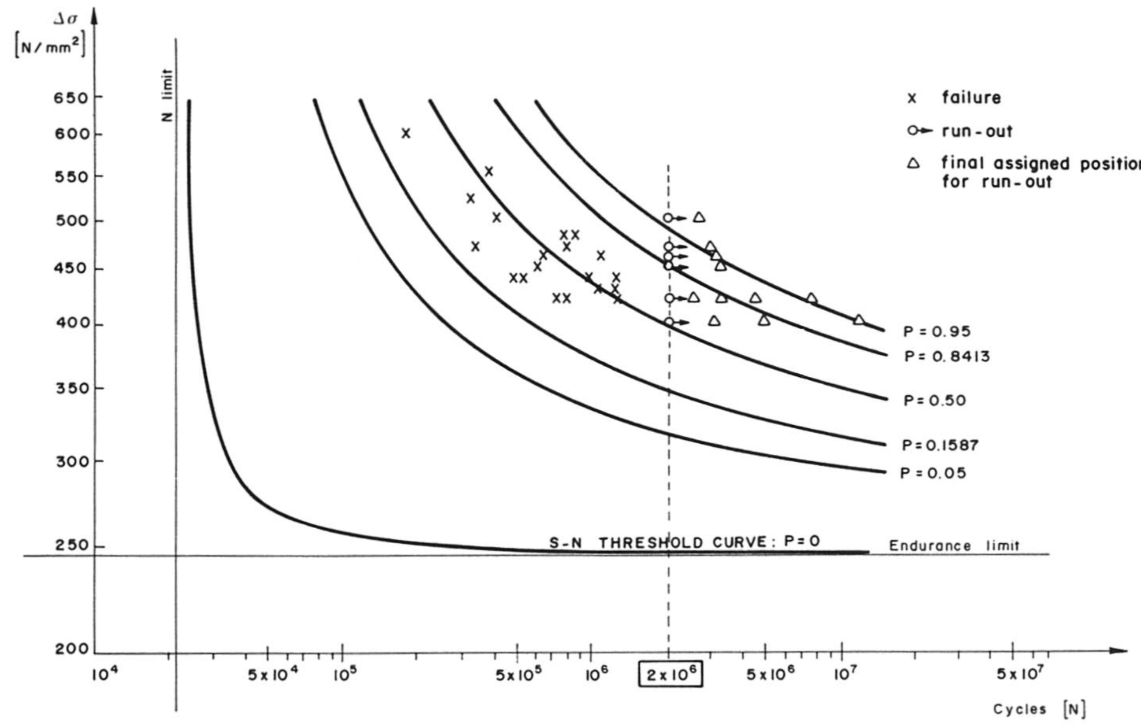


Fig. 18: S-N-Curve for Prestressing Wire of 140 mm Length Calculated from
Series 1, 2 and 3. Test Data from Series 1

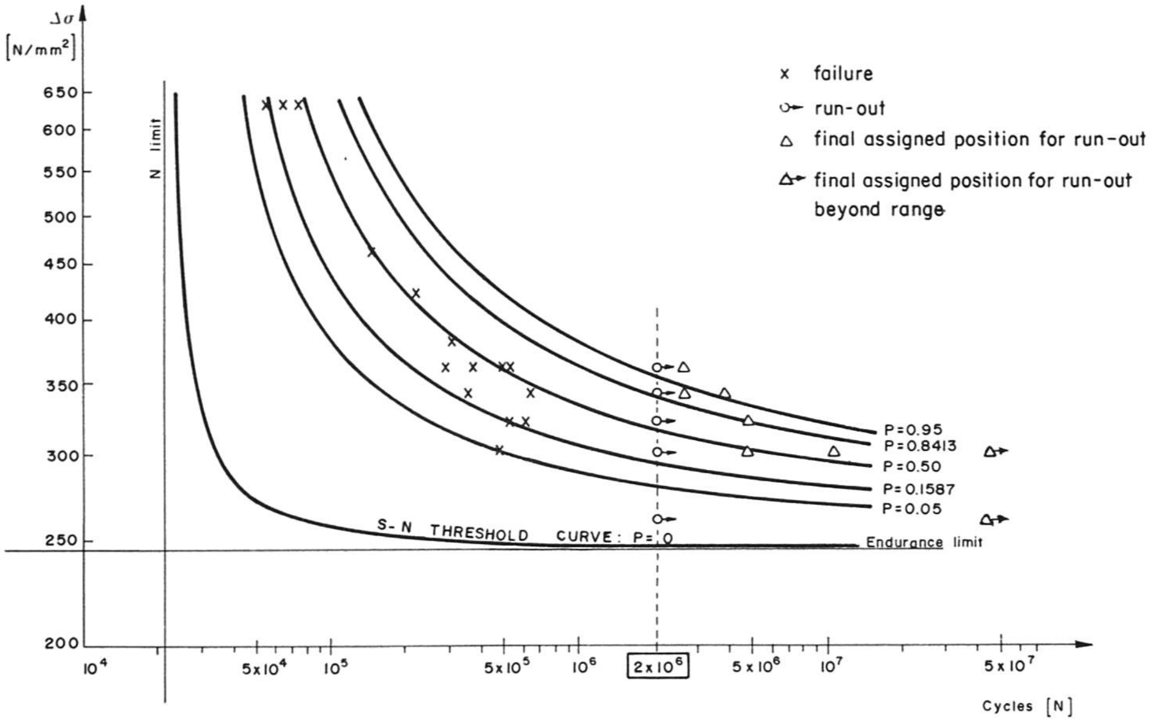


Fig. 19: S-N-Curves for Prestressing Wire of 1960 mm Length Calculated from Series 1, 2 and 3. Test Data from Series 2

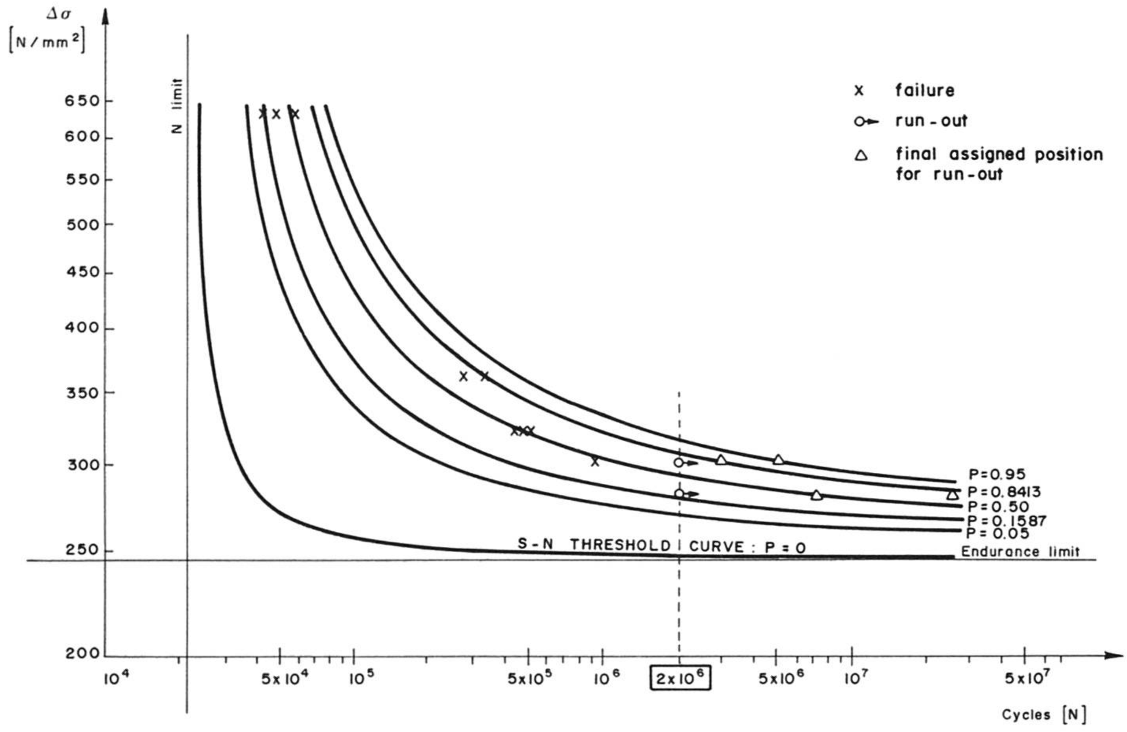


Fig. 20: S-N-Curves for Prestressing Wire of 8540 mm Length Calculated from Series 1, 2 and 3. Test Data from Series 3

Parameters	All Series Together	Series 2 only
A	4.09	4.99
B (Threshold N)	9.97 (21341)	9.55 (14004)
C (Endurance Limit)	5.48 (240.3)	5.45 (231.9)
D	1.274	1.508
E	-0.085	-0.092
Quantile 0.05	281.9	281.9
Quantile 0.1587	295.5	295.
Median	318.2	316.3
Quantile 0.8413	340.9	336.2
Quantile 0.95	355.3	348.2

Table 5: Prestressing Wire:
Comparison of Parameters and Quantiles for $N = 2 \cdot 10^6$ Cycles and Reference Length 1960 mm Calculated from Series 1 to 3 and from Series 2 only

Analogous conclusions as for the wire can be drawn for the strand. The median values agree well for all methods. As expected, the median values for the three variants of the up-and-down method lie above the one for the proposed method.

Furthermore, the proposed method is the only one for which the standard deviation again decreases with increasing length. Its values for the studied length seems to be somehow less than those for wire although for longer lengths the differences are not significant.

2.8 Conclusions

- Present models used to analyze fatigue data of wires or strands fail to reproduce the physical evidence and/or show some internal inconsistencies.
- A new model taking into account physical evidence as well as conditions of stability, compatibility and limit is developed. The model is based either on independence or asymptotic independence of the strengths of neighboring pieces.
- In the parameter estimation procedure the E-M-algorithm is applied in order to make use of the information accumulated in run-out tests.
- Since the length is included as a parameter, the estimation of the model parameters can be based on data from specimens with different lengths. An application to the problem of fatigue of long prestressing wires or strands is possible.
- The model analyzes the fatigue strength over the entire range of number of cycles, i.e. broadens the fatigue strength concept to any number of cycles different from the standard $2 \cdot 10^6$.

PARAMETER	VALUES
A	5.19
B (Threshold N)	10.07 (23647)
C (Endurance Limit)	5.37 (214.02)
D	1.132
E	-0.092
Quantile 0.05	253.2
Quantile 0.1587	263.0
Median	278.1
Quantile 0.8413	292.1
Quantile 0.95	300.5

Table 6: Prestressing Strand:
Parameter Values and Quantiles for $N = 2 \cdot 10^6$ Cycles and Reference Length 1100 mm Calculated from Series 4 to 7

METHOD OF EVALUATION	SPECIMEN LENGTH	MEDIAN	STANDARD DEVIATION s	QUANTILE 0.05
DIXON/MOOD	Series 5: 1100	280.0	17.1	252.0
	Series 7: 3860	260.0	17.1	232.0
DEUBELBEISS	Series 5: 1100	280.0	20.0	247.0
	Series 7: 3860	263.0	21.0	229.0
HÜCK	Series 5: 1100	279.8	19.3	249.0
	Series 7: 3860	259.8	53.0	179.0
PROPOSED METHOD	Series 5: 1100	278.1	15.1(*)	253.2
	Series 7: 3860	252.9	10.8(*)	235.0

(*) s_{approx} = Quantile 0.50 - Quantile 0.1587

Table 7: Prestressing Strand:
Comparison of Characteristic Values for $N = 2 \cdot 10^6$ Cycles Calculated for Series 5 and 7 Using Different Methods

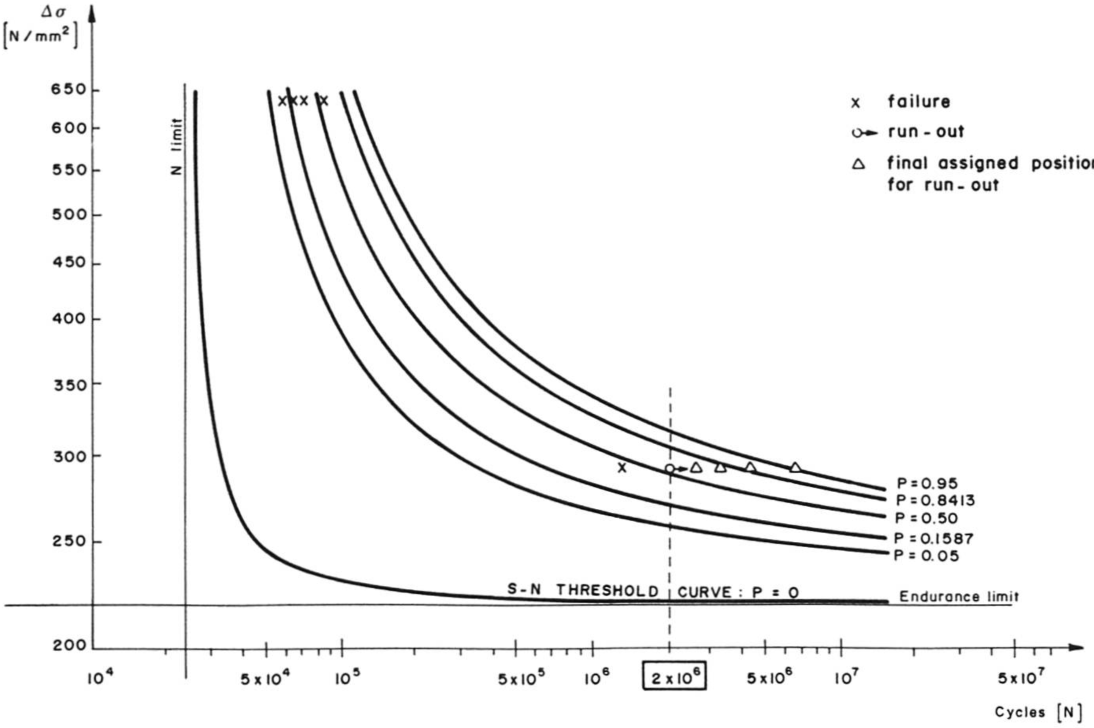


Fig. 21: S-N-Curves for Prestressing Strand of 490 mm Length Calculated from Series 4 to 7. Test Data from Series 4 (10 Specimens)

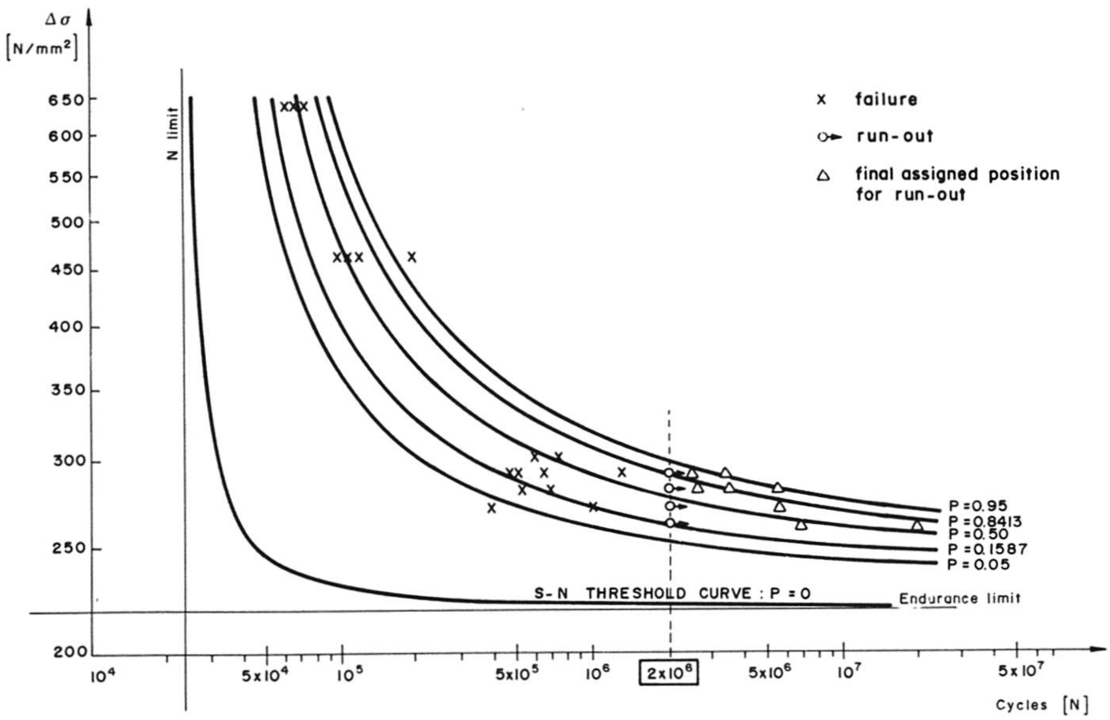


Fig. 22: S-N-Curves for Prestressing Strand of 1100 mm Length Calculated from Series 4 to 7. Test Data from Series 5 (28 Specimens)

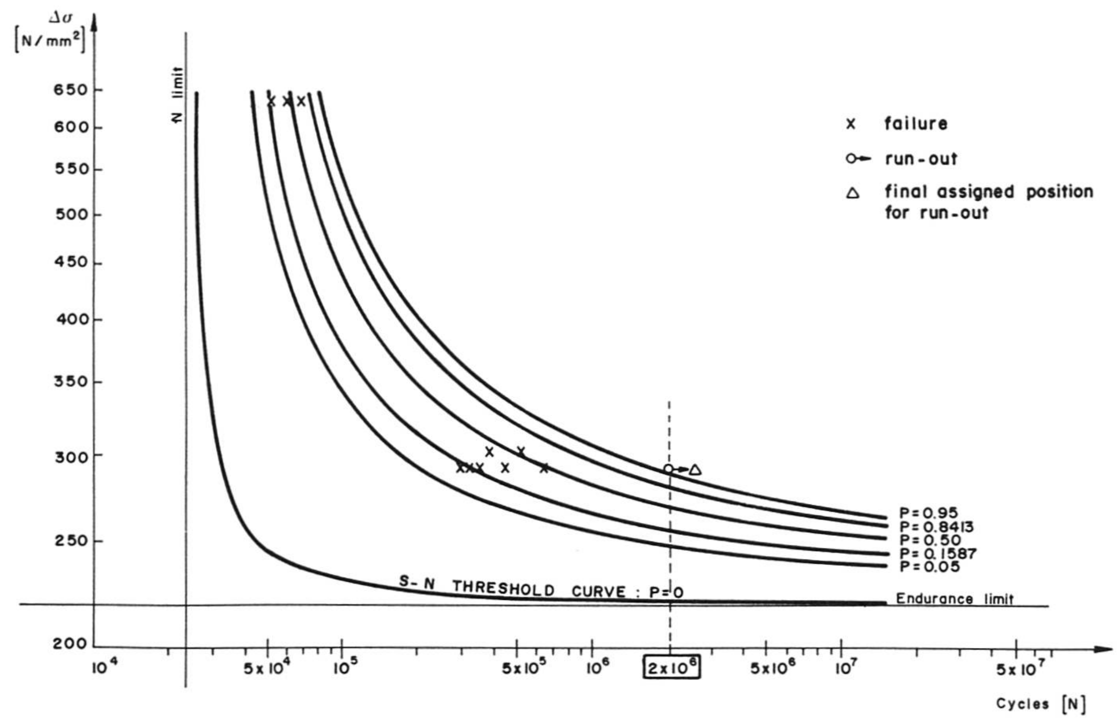


Fig. 23: S-N-Curves for Prestressing Strand of 1960 mm Length Calculated from Series 4 to 7. Test Data from Series 6 (14 Specimens)

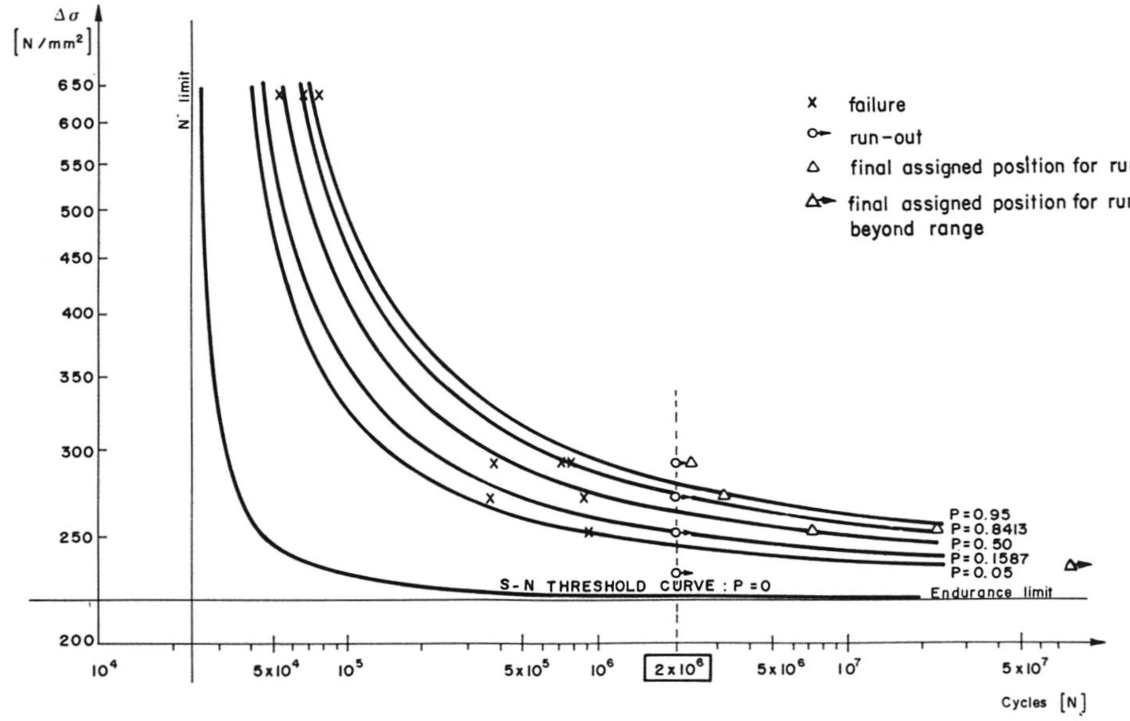


Fig. 24: S-N-Curves for Prestressing Strand of 3860 mm Length Calculated from Series 4 to 7. Test Data from Series 7 (15 Specimens)

- The application of the model in the analysis of fatigue data from an extensive experimental program carried out at the Swiss Federal Testing Laboratories (EMPA-Dübendorf) on prestressing wires and strands with different length [29] has shown its validity and usefulness.
- The results confirm that the median and the standard deviation of the fatigue strength for a given limit of cycles decrease with increasing length of the specimen. For an infinite length they reach the endurance limit and zero respectively.
- Comparison of the standard deviation values obtained by the proposed and up-and-down methods confirms the unreliability of the up-and-down method for the estimation of the standard deviation. On the contrary, the medians obtained from all methods agree well.
- From the point of view of parameter estimation the testing of long specimens offers clear advantages.
- The model has shown sensitivity to threshold values, so a reliable estimation of the entire S-N-field involves testing specimens at both extreme values of the stress range (high and low $\Delta\sigma$).

3. CABLES

Long cables made-up of many parallel wires or strands are used in cable-stayed bridges or similar structures. The specialized literature [3, 32] presents the fatigue strength of the cables as a governing criterion for the design of such structures. Therefore, the knowledge of the statistical behavior under fatigue becomes essential.

The high costs involved in testing large cables lead to the testing of only short component elements, normally about one meter long strand and around 200 mm long wire specimens. From the results of such tests the fatigue strength of the entire cable has to be estimated. In such an extrapolation the redundance i.e. the presence of parallel wires or strands in the cable, and the difference between the specimen length and the total length of the cable must be taken into account.

A new statistical model for analyzing the fatigue strength of a single wire or strand was developed in the first part of the paper. In the following part, the fatigue strength of a long cable is analyzed from the strength of a single element (wire or strand) by considering the progressive loss of the cross sectional area as a result of successive failures of single elements.

For studying the static strength of large cables some models can be found in the existing literature [6, 7, 9, 11, 15 to 20, 30, 41 to 45, 48, 49] but for the fatigue strength no general model is yet available.

3.1 Statistical Model

For analyzing the fatigue strength of a cable, a model based on the following assumptions is developed:

(1) The cable is made-up of m parallel elements (wires or strands) of identical length L.

(2) Failure in the elements is caused by fatigue. Static failure is excluded. This implies either small stress range increments, with respect to the static strength, caused by the failure of one element or a sufficiently large number of elements m.

(3) No transfer of stress by bond, friction, etc. between the elements is possible, i.e. the elements are completely independent over their entire length.
(4) An element with actual length L can be considered composed of n pieces with a reference length L_o:

$$L = nL_o \tag{23}$$

(5) The fatigue failure of one element is associated with the first failure appearing in the n pieces, i.e. with the minimum number of cycles to failure.
(6) The fatigue behavior of a single element is assumed to correspond to the model given in the first part, i.e. the cdf of N for given $\Delta\sigma$ for an element is given by eq. (17):

$$E^*(N;\Delta\sigma,L) = 1 - \exp\left[-n\left[\frac{(N-B)\ (\Delta\sigma-C)}{D} + E\right]^A\right] \tag{24}$$

where, as in chapter 2.7, $\Delta\sigma$ represents the logarithm of the stress range.

It is worthwhile remembering that knowledge of expression (2) is equivalent to knowledge of the whole S-N-field for any length L.
(7) As a consequence of the successive failure of elements the stress range of the unfailed elements progressively increases, provided the applied loads remain constant which is the case for a cable of an actual structure. Since the S-N-curve holds for one-step tests, i.e. tests with $\Delta\sigma$ = constant, a so-called cumulative damage hypothesis is needed. In the following, it is assumed that the previous damage history in the elements is transferred to the new stress range in such a way that the probability of failure remains the same [28].
(8) The number of cycles to failure for the different elements is assumed to be an independent or quasi-independent random variable.
(9) The dynamic effect due to the momentary stress increment produced by the failure of an element on the rest of the elements is neglected.
(10) The failure of the cable is defined as failure of the k-th element. Fig. 25 shows the schematic representation of a system with m elements of length L subjected to a stress range $\Delta\sigma_1$ and the number of cycles to failure $N_{(1)}$, $N_{(2)}, \dots , N_{(m)}$, ranged in increasing order.

At the beginning, the stress range in one element is equal to $\Delta\sigma_1$. As soon as the number of cycles reaches the value $N_{(1)}$, the first element fails and, as a consequence, the new stress range in the remaining elements becomes equal to $\Delta\sigma\, m/(m-1)$. According to the cumulative damage hypothesis the process goes on, as if the cable were subjected to a stress range equal to $\Delta\sigma_1\, m/(m-1)$ from the beginning and for an equivalent number of cycles (see Fig. 26). This process repeates itself until a stress range equal to $\Delta\sigma_1\, m/(m-k+1)$, corresponding to failure of the k-th element is attained.

According to the assumption (7) the equivalent number of cycles for two different $\Delta\sigma$-levels is given by the cumulative damage hypothesis, i.e. the probabilities of failure for both levels coincide. This condition, according to Eq. (24) becomes for the ℓ-failure (see Fig. 27):

$$(N_{(\ell)}^{\Delta\sigma_1} - B)\ (\Delta\sigma_1 - C) = (N_{(\ell)}^{\Delta\sigma_j} - B)\ (\Delta\sigma_j - C) \tag{25}$$

where the pairs $(N_{(\ell)}^{\Delta\sigma_1}, \Delta\sigma_1)$ and $(N_{(\ell)}^{\Delta\sigma_j}, \Delta\sigma_j)$ correspond to the same damage state.

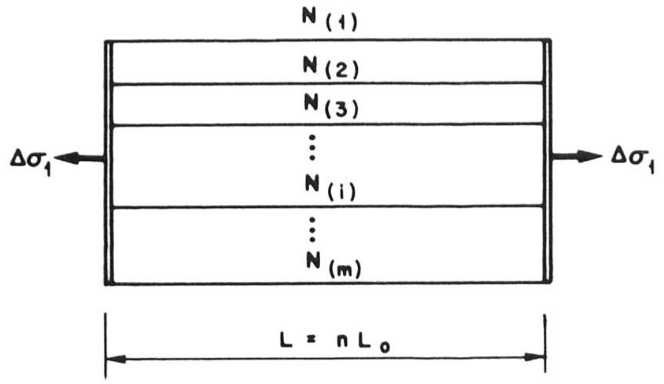


Fig. 25: Schematic Representation of a Cable with m Elements

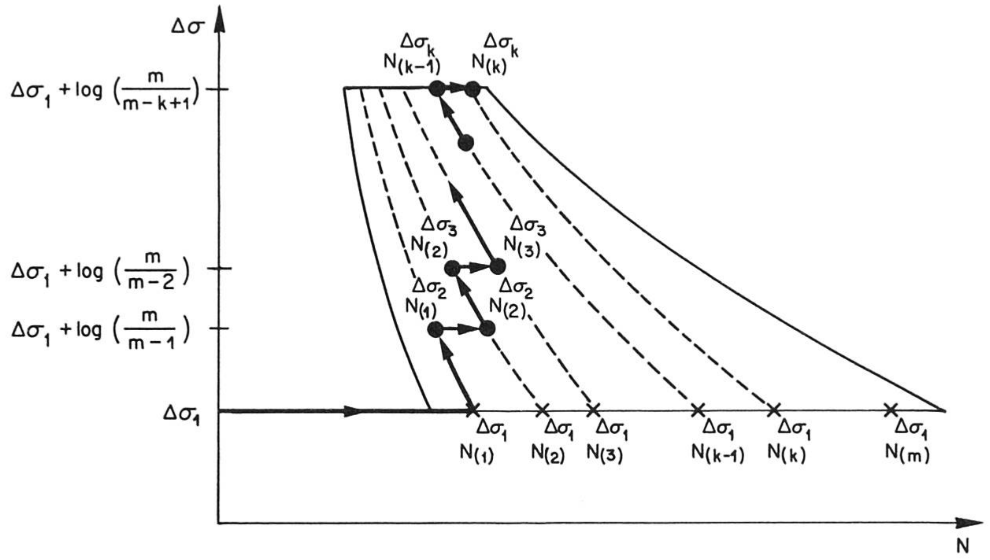


Fig. 26: Schematic Increase of the Stress Range and the Number of Cycles to the Defined Point of Failure of the Cable

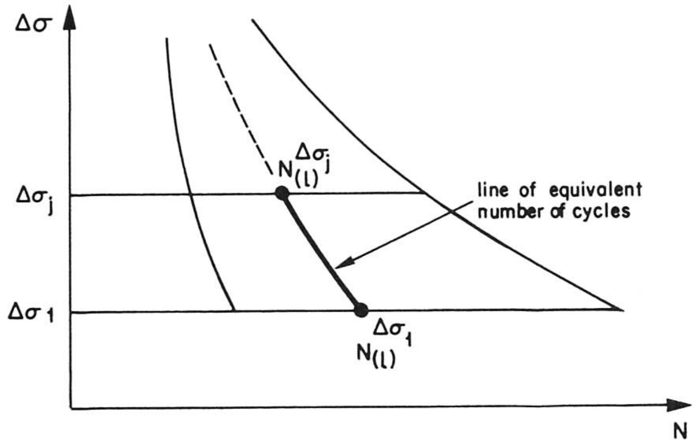


Fig. 27: Illustration of the Cumulative Damage Hypothesis

The number of cycles, N_k^*, reached at the defined point of failure of the cable, i.e. failure of the k-th element, is given by (see Fig. 26):

$$\exp(N_k^*) = \exp(N_{(1)}) + \sum_{j=2}^{k} \left[\exp(N_{(j)}^{\Delta\sigma_j}) - \exp(N_{(j-1)}^{\Delta\sigma_j})\right] \tag{26}$$

which, taking into account Eq. (31), becomes

$$\exp(N_k^*) = \exp(N_{(1)}) + \sum_{j=2}^{k} Q_j\left[\exp(N_{(j)}^{\Delta\sigma_1} P_j) - \exp(N_{(j-1)}^{\Delta\sigma_1} P_{j-1})\right] \tag{27}$$

where

$$P_j = (\Delta\sigma_1 - C)/(\Delta\sigma_j - C) \tag{28}$$

$$Q_j = \exp\left[B(1 - P_j)\right] \tag{29}$$

$$\Delta\sigma_j = \log\left[m/(m-j+1)\right] + \Delta\sigma_1 \tag{30}$$

The equation (27) allows to calculate the cdf of N_k^* only by numerical or simulation techniques. Since these techniques can be cumbersome, it would be of interest, from a practical point of view, to give upper and lower bonds for N_k^*. These bounds are given by (see Fig. 26):

$$N_{(k)}^{\Delta\sigma_k} \leq N_k^* \leq N_{(k)}^{\Delta\sigma_1} \tag{31}$$

Equation (31) allows the study of the r.v. N_k^* using the analysis of the order statistics which is easier to handle. The cdf of the k-order statistics $N_{(k)}^{\Delta\sigma_1}$ and $N_{(k)}^{\Delta\sigma_k}$ are given by [21, 31]:

$$G_k(N;\Delta\sigma_p) = \sum_{j=k}^{m} \binom{m}{j} \left[1-\exp\left[-n\left[\frac{(N-B)\ (\Delta\sigma_p-C)}{D} + E\right]^A\right]\right]^j \exp\left[(j-m)n\left[\frac{(N-B)\ (\Delta\sigma_p-C)}{D} + E\right]^A\right] \tag{32}$$

for p = 1 and k, respectively. Because of Eq. (31) they give upper and lower bounds for the cdf of N_k^*.

3.2 Cables Made-up of Strands

If the cable is made-up of m^* strands of length L, the strand itself contains r wires, and the estimation of the model parameters has been performed for a reference length L_1 and, as usual, only the first wire failure in a strand is considered, the proposed model can also be applied.

Since the cdf of the fatigue strength of one strand with r wires and length L_1 is identical to that for a single wire with length rL_1, and since a cable composed of m^* strands, each of them with r wires, can be assimilated to a cable-made-up by rm^* wires, we need just to make

$$L_o = rL_1 \tag{33}$$

and

$$m = rm^* \tag{34}$$

in using the present model.

3.3 Asymptotic Solutions

Due to the presence of two parameters m and n, the following asymptotic cases can be considered:

(1) Very long cables
(2) Large number of wires or strands in the cable ($m \rightarrow \infty$)

3.3.1 Very Long Cables

In this case all wires or strands fail, practically, at the endurance limit. Therefore, the design must be based on this value (see Fig. 9).

3.3.2 Large Number of Wires or Strands

In this case, if k remains constant, the bounds converge to each other and either of them can be used for a practical solution. On the other hand, the cdf tends to be [31]:

$$G_k(N;\Delta\sigma) = 1-\exp\left[-nm\left[\frac{(N-B)\ (\Delta\sigma-C)}{D}+E\right]^A\right]\sum_{t=0}^{k-1}\frac{1}{t!}(nm)^t\left[\frac{(N-B)\ (\Delta\sigma-C)}{D}+E\right]^{At} \qquad (35)$$

This function has been represented in Fig. 28.

3.4 Example of Application

In order to illustrate the results and their application to practical cases the design of a cable showing the following characteristics is presented:

Length L = 200 m
m = 295 prestressing wires Ø 7 mm
Admissible loss of cross sectional area: 5 %; approx. k = 15
Selected probability of failure: P = 0.05
Number of cycles for design: $N_{design} = 10^7$ cycles

Assuming that wires of the quality defined in chapter 2.7 are composing the cable, the calculated values, found for a reference length L_0 = 1960 mm, in Table 5 are used for the design:

A = 4.09
B = 9.97
C = 5.48
D = 1.274
E = -0.085

From Fig. 29 for P = 0.05 and k = 15 the value

$$nm\left(\frac{(N-B)\ (\Delta\sigma-C)}{D}+E\right)^A = 5.4$$

with

n = 200000/1960
m = 295 wires
$N = 10^7$ cycles

is obtained.

This value gives the 0.05-quantile curve for the defined failure of the cable (5 % loss of section). From this value and for given N, the corresponding

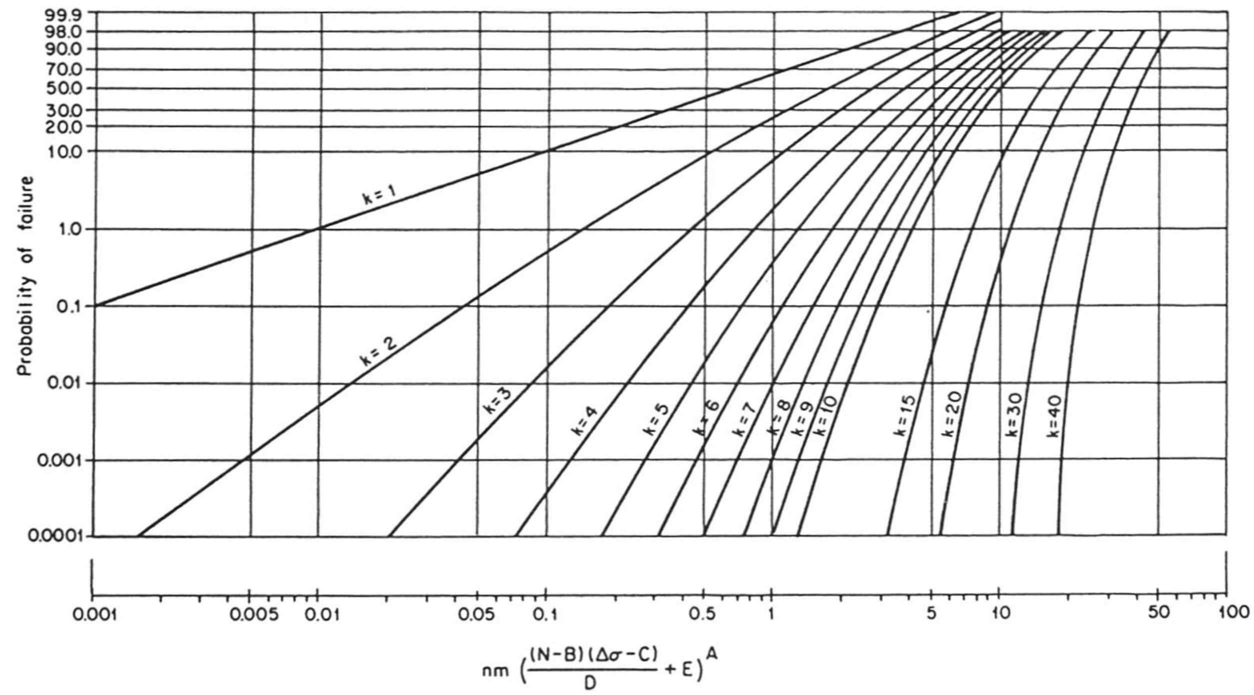


Fig. 28: Cable with Large Number of Elements m:
Asymptotic Cumulative Distribution Function for the Failure of the k-th Element

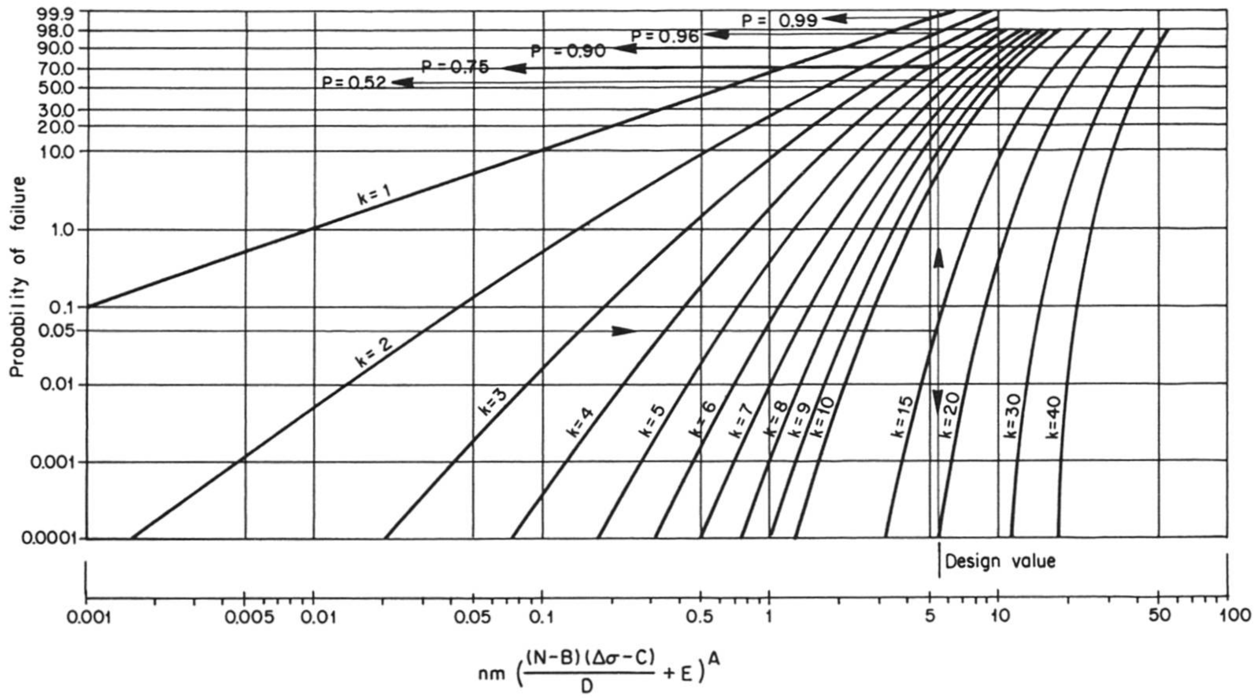


Fig. 29: Example of Application:
Design Value for Selected Probability of Failure P = 0.05 and Failure of k Elements

$\Delta\sigma$ for design can be calculated (see Fig. 30):

$$\Delta\sigma = 250\ \text{N/mm}^2$$

It is interesting to note from Fig. 29 that high probabilities of failure up to the fifth wire are associated with this design value. This observation is supported by the fact that actual cables or cable specimens tested in the laboratory always show some broken wires even for relatively low stress levels.

If a more precise statistical analysis is desired, the S-N-field can be obtained just by repeating the same procedure for different probabilities of loss rates (k-values). Fig. 31 shows the 0.05- and the 0.95-quantile curves for k = 1 and k = 15.

3.5 Conclusions

- The application of the proposed model is extended to the design of cables made-up of parallel elements (wires or strands). Two bounds for the fatigue failure of a cable are obtained from fatigue data of single elements. As the number of elements increases, the upper and lower bounds converge to the exact solution.
- The median and the standard deviation of the fatigue strength of a cable for a given loss of wires, k, and a given number of cycles to failure, N, decrease with the increasing number of wires, m, till the endurance limit value and zero, respectively.
- The endurance limit and the threshold value for N coincide for wires and parallel wire cables, or strands and cables formed by strands.
- For cables with a large number of elements, m, or for long cables, i.e. large number n, the design values are very close to the endurance limit and threshold N.

APPENDIX

In this appendix the general solution of equation

$$\frac{N - N_o(\Delta\sigma)}{N_a^o(\Delta\sigma)} = \frac{\Delta\sigma - \Delta\sigma_o(N)}{\Delta\sigma_a^o(N)} \qquad (A.1)$$

is obtained by using the following theorem [1]:

THEOREM: All solutions of the equation

$$\sum_{k=1}^{n} f_k(x) g_k(y) = 0 \qquad (A.2)$$

with variables in arbitrary sets and function values in an arbitrary field can be written in the form

$$f_k(x) = \sum_{i=1}^{r} a_{ki}\, \phi_i(x) \; ; \quad g_k(x) = \sum_{j=r+1}^{n} b_{kj}\, \psi_j(y) \; ; \; k = 1,2,\dots,n \qquad (A.3)$$

where r is an integer between 0 and n, and $\phi_1, \phi_2, \dots, \phi_r$, on one hand and

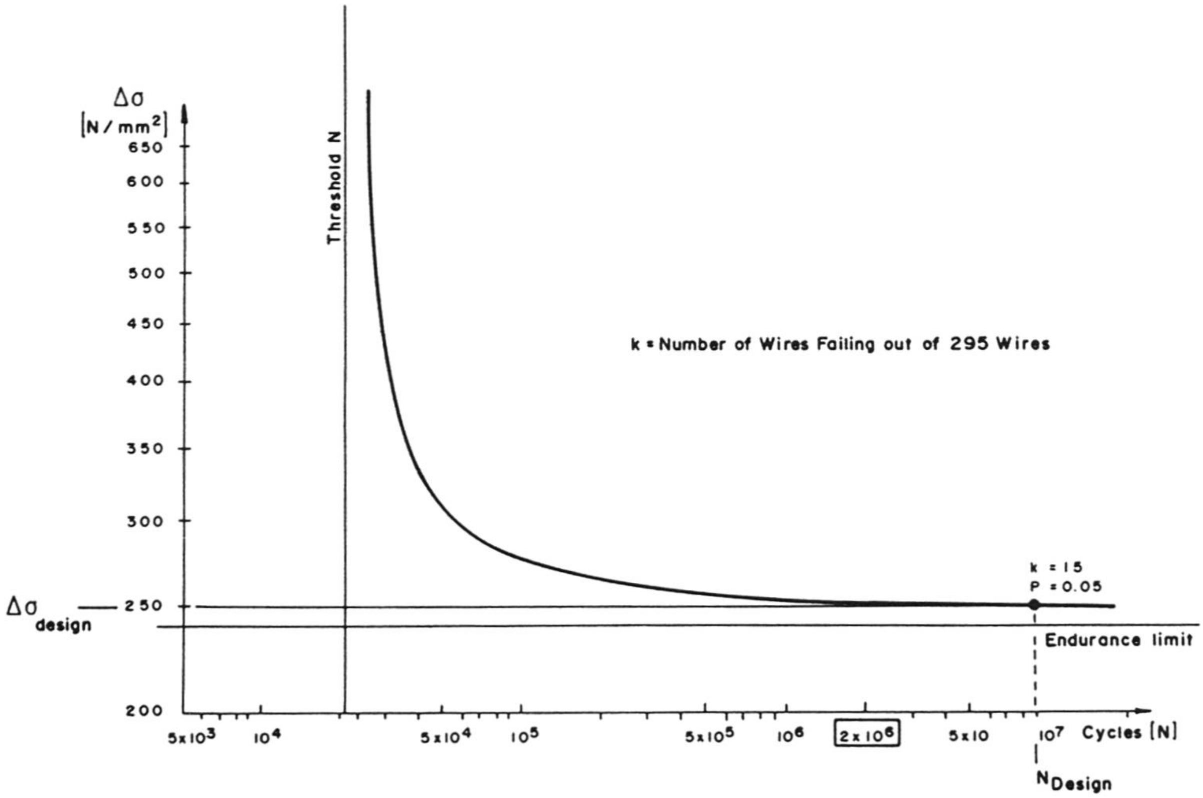


Fig. 30: Example of Application: 0.05-Quantile Curve for Cable Failure

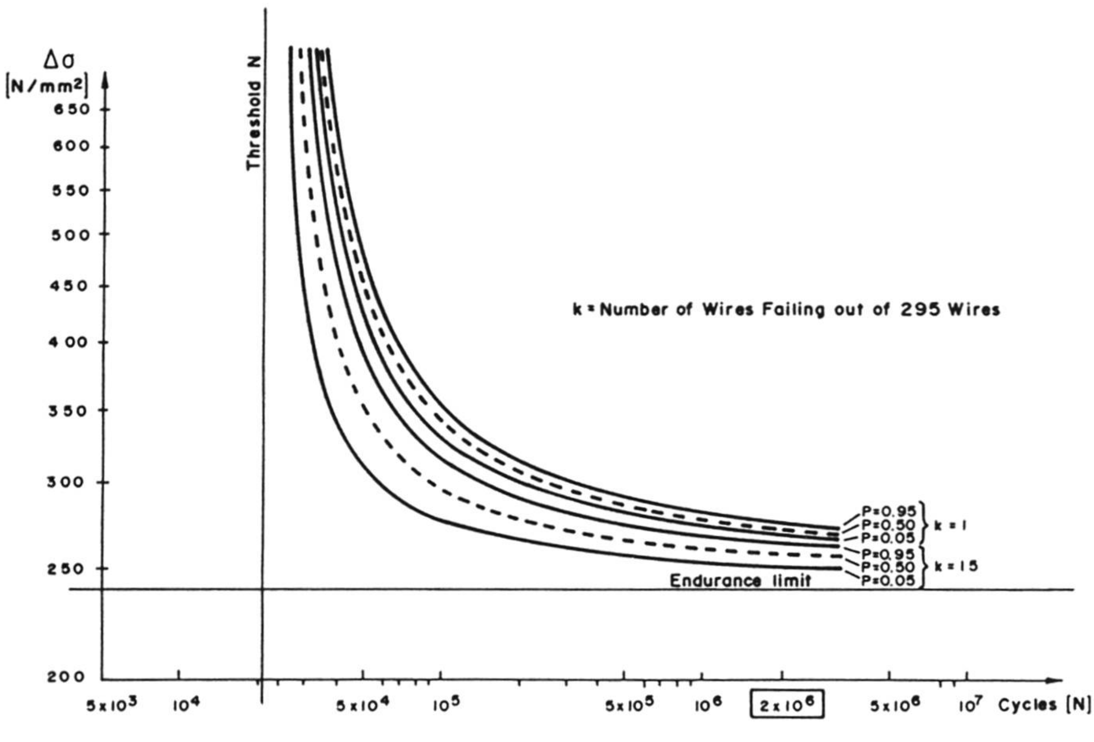


Fig. 31: Different Quantile-Curves for Two Loss Rates of the Cable

$\psi_{r+1}, \psi_{r+2},\dots,\psi_n$ on the other are arbitrary systems of mutually linearly independent functions. The constants a_{ki}, b_{kj} (k=1,2,...,n; i=1,2,...,r; and j=r+1,r+2,...,n) satisfy

$$\sum_{k=1}^{n} a_{ki}b_{kj} = 0 \;;\; i=1,2,\dots,r \;;\; j=r+1,r+2,\dots,n \tag{A.4}$$

but are otherwise arbitrary. Inversely, all systems of functions of the form (A.3) with (A.4) satisfy (A.2).

Remarks. For r=0 or r=n, $\Sigma_1^0 = \Sigma_{n+1}^0 = 0$ per definition and then no condition (A.4) is necessary. Not all ϕ_i, ψ_j necessarily figure in (A.3). For instance it is possible that $a_{kr} = 0$ (k=1,2,...,n).

The solution of (A.1) is equivalent to the solution of equation

$$\sum_{k=1}^{n} f_k(x)g_k(y) = 0 \tag{A.5}$$

where x and y stand for $\Delta\sigma$ and N respectively and

$$f_1(x) = 1/N_a^o(x) \quad ; \quad g_1(y) = y \tag{A.6}$$

$$f_2(x) = N_o(x)/N_a^o(x) \quad ; \quad g_2(y) = -1 \tag{A.7}$$

$$f_3(x) = -x \quad ; \quad g_3(y) = 1/\Delta\sigma_a^o(y) \tag{A.8}$$

$$f_4(x) = 1 \quad ; \quad g_4(y) = \Delta\sigma_o(y)/\Delta\sigma_a^o(y) \tag{A.9}$$

For r=1 or 3 there is no solution of (A.5), and for r=2, the theorem gives (expressions (A.3) and (A.4)):

$$f_1(x) = a_{11}\phi_1(x)+a_{12}\phi_2(x) = 1/N_a^o(x) \tag{A.10}$$

$$f_2(x) = a_{21}\phi_1(x)+a_{22}\phi_2(x) = N_o(x)/N_a^o(x) \tag{A.11}$$

$$f_3(x) = a_{31}\phi_1(x)+a_{32}\phi_2(x) = -x \tag{A.12}$$

$$f_4(x) = a_{41}\phi_1(x)+a_{42}\phi_2(x) = 1 \tag{A.13}$$

$$g_1(y) = b_{13}\psi_3(y)+b_{14}\psi_4(y) = y \tag{A.14}$$

$$g_2(y) = b_{23}\psi_3(y)+b_{24}\psi_4(y) = -1 \tag{A.15}$$

$$g_3(y) = b_{33}\psi_3(y)+b_{34}\psi_4(y) = 1/\Delta\sigma_a^o(y) \tag{A.16}$$

$$g_4(y) = b_{43}\psi_3(y)+b_{44}\psi_4(y) = \Delta\sigma_o(y)/\Delta\sigma_a^o(y) \tag{A.17}$$

$$\begin{pmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \end{pmatrix} \begin{pmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \\ b_{33} & b_{34} \\ b_{43} & b_{44} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad (A.18)$$

From (A.12) to A.15) it can be written without loss of generality

$$\phi_1(x) = x \qquad (A.19)$$

$$\phi_2(x) = 1 \qquad (A.20)$$

$$\psi_3(y) = y \qquad (A.21)$$

$$\psi_4(y) = 1 \qquad (A.22)$$

and taking into account (A.12) to (A.15) and (A.19) to (A.22) it results

$$\left.\begin{array}{l} a_{42} = -a_{31} = b_{13} = -b_{24} = 1 \\ a_{32} = a_{41} = b_{14} = b_{23} = 0 \end{array}\right\} \qquad \begin{array}{r}(A.23)\\(A.24)\end{array}$$

Substituting now (A.23) in (A.18) one gets

$$\begin{pmatrix} a_{11} & a_{21} & -1 & 0 \\ a_{12} & a_{22} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ b_{33} & b_{34} \\ b_{43} & b_{44} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad (A.25)$$

from which

$$a_{11} = b_{33} \ ; \ a_{21} = -b_{34} \ ; \ a_{12} = -b_{43} \ ; \ a_{22} = b_{44} \qquad (A.26)$$

Finally, substitution of (A.13) and (A.26) into (A.10), (A.11), (A.16) and A.17) and some algebra give

$$N_a^o(\Delta\sigma) = 1/(a_{11}\Delta\sigma + a_{12}) \qquad (A.27)$$

$$N_o(\Delta\sigma) = (a_{21}\Delta\sigma + a_{22})/(a_{11}\Delta\sigma + a_{12}) \qquad (A.28)$$

$$\Delta\sigma_a^o(N) = 1/(a_{11}N - a_{21}) \qquad (A.29)$$

$$\Delta\sigma_o(N) = (-a_{12}N + a_{22})/(a_{11}N - a_{21}) \qquad (A.30)$$

which are the location and scale parameters of the Weibull laws.

Substitution of (A.27) to (A.30) into (A.1) leads to

$$\frac{N - N_o(\Delta\sigma)}{N_a^o(\Delta\sigma)} = \frac{\Delta\sigma - \Delta\sigma_o(N)}{\Delta\sigma_a^o(N)} = \frac{(N - a_{21}/a_{11})(\Delta\sigma + a_{12}/a_{11})}{1/a_{11}} + \frac{a_{21}a_{12}}{a_{11}} - a_{22} \qquad (A.31)$$

which can also be written as

$$\frac{N - N_o(\Delta\sigma)}{N_a^o(\Delta\sigma)} = \frac{\Delta\sigma - \Delta\sigma_o(N)}{\Delta\sigma_a^o(N)} = \frac{(N - B)(\Delta\sigma - C)}{D} + E \qquad (A.32)$$

where the new constants are

$$B = a_{21}/a_{11} \qquad (A.33)$$

$$C = -a_{12}/a_{11} \qquad (A.34)$$

$$D = 1/a_{11} \qquad (A.35)$$

$$E = a_{21}a_{12}/a_{11} - a_{22} \qquad (A.36)$$

REFERENCES

1. ACZEL J., Lectures on Functional Equations and their Applications. Academic Press, New York, 1966.
2. ANDRÄ W., SAUL R., Die Festigkeit, insbesondere Dauerfestigkeit langer Paralleldrahtbündel. Die Bautechnik 4, 1979, pp. 128-130.
3. ANDRÄ W., SAUL R., Versuche mit Bündeln aus parallelen Drähten und Litzen für die Nordbrücke Mannheim-Ludwigshafen und das Zeltdach in München. Die Bautechnik 9, 10 and 11, 1974, pp. 289-298, 332-340 and 371-373.
4. ASTM, A Guide for Fatigue Testing and the Statistical Analysis of Fatigue Data. ASTM Special Technical Publication, N. 91 A, 1963.
5. ASTM, Standard Practice for Statistical Analysis of Linear and Linearized Stress-Life (S-N) and Strain-Life (E-N) Fatigue Data. ASTM, E 739-80, 1980.
6. BATDORF S.B., Tensile Strength of Unidirectionally Reinforced Composites-I. J. Reinforced Plastics and Composites, 1, pp. 153-164, 1982.
7. BATDORF S.B., GHAFFANIAN R., Tensile Strength of Unidirectionally Reinforced Composites-II. J. Reinforced Plastics and Composites, 1, pp. 165-176, 1982.
8. BIRKENMAIER M., NARAYANAN R., Fatigue Resistance of Large High Tensile Steel Stay Tendons. IABSE Colloquium, Fatigue of Steel and Concrete Structures, Lausanne, pp. 663-672, 1982.
9. BIRNBAUM Z.W., SAUNDERS S.C., A Statistical Model for Life-length of Materials. J. Amer. Statist. Assoc., 53, pp. 151-159, 1958.
10. BÜHLER H., SCHREIBER W., Lösung einiger Aufgaben der Dauerfestigkeit mit dem Treppenstufen-Verfahren. Archiv für Eisenhüttenwesen, 28. Jahrgang, Heft 3, pp. 153-156, 1957.
11. CASTILLO E., ASCORBE A., FERNANDEZ CANTELI A., Static Progressive Failure in Multiple Tendons. A Statistical Approach. 44th Session of ISI, Madrid, 1983.
12. CASTILLO E., FERNANDEZ CANTELI A., ASCORBE A., MORA E., Aplicación de los modelos de series temporales al análisis estadístico de la resistencia de tendones de puentes atirantados. Anales de Ingenieria Mecánica, to be published.
13. CASTILLO E., FERNANDEZ CANTELI A., ASCORBE A., MORA E., The Box-Jenkins Model and the Progressive Fatigue Failure of Large Parallel Element Stay-Tendons. ASI-NATO, Statistical Extremes and Applications, Lisbon, 1983.
14. CASTILLO E., FERNANDEZ CANTELI A., MORA E., ASCORBE A., Influencia de la longitud en la resistencia a fatiga de tendones de puentes atirantados. Anales de Ingenieria Mecánica, to be published.
15. COLEMAN B.D., Time Dependence of Mechanical Breakdown in Bundles of Fibers. I: Constant Total Load. J. Appl. Phys., 28, pp. 1058-1064, 1957.
16. COLEMAN B.D., Time Dependence of Mechanical Breakdown in Bundles of Fibers. II: The Infinite Ideal Bundle under Linearly Increasing Loads. J. Appl. Phys., 28, pp. 1065-1067, 1957.
17. COLEMAN B.D., Time Dependence of Mechanical Breakdown in Bundles of Fibers. III: The Power Law Breakdown Rule. Trans. Soc. Rheol., 2, pp. 195-218, 1958.
18. COLEMAN B.D. Time Dependence of Mechanical Breakdown in Bundles of Fibers. IV: Infinite Ideal Bundle under Oscillating Loads. J. Appl. Phys., 29, pp. 1091-1099, 1958.

19. COLEMAN B.C., Statistics and Time Dependence of Mechanical Breakdown in Fibers. J. Appl. Phys., 29, pp. 968-983, 1958.

20. DANIELS H.E., The Statistical Theory of the Strength of Bundles of Threads. I. Proc. R. Soc. Lond. A, 183, pp. 404-435, 1945.

21. DAVID H.A., Order Statistics. John Wiley & Sons, New York, 1980.

22. DEMPSTER A.P., LAIRD N.M., RUBIN D.B., Maximum Likelihood from Incomplete Data via the EM-Algorithm. JRSS Series B, pp. 1-38, 1977.

23. DENGEL D., Einige grundlegende Gesichtspunkte für die Planung und Auswertung von Dauerschwingversuchen. Materialprüfung 13, Nr. 5, 1971, pp. 145-151.

24. DEUBELBEISS E., Dauerfestigkeitsversuche mit einem modifizierten Treppenstufenverfahren. Materialprüfung 16, Nr. 8, 1979, pp. 240-244.

25. DIXON W.S., MOOD A.M., A Method for Obtaining and Analyzing Sensivity Data. J. Amer. Statist. Assoc., 43, 1948, pp. 109-126.

26. EPSTEIN B., Truncated Life Tests in the Exponential Case. Annals Math. Statist. 25, 1954, pp. 555-564.

27. EPSTEIN B., SOBEL M., Some Theorems Relevant to Life Testing from an Exponential Distribution. Annals Math. Statist. 25, 1954, pp. 373-381.

28. FERNANDEZ CANTELI A., Statistical Interpretation of the Miner-Number Using an Index of Probability of Total Damage. IABSE Colloquium Fatigue of Steel and Concrete Structures, Lausanne 1982.

29. FERNANDEZ CANTELI A., ESSLINGER V., THÜRLIMANN B., Ermüdungsfestigkeit von Bewehrungs- und Spannstählen. Bericht Nr. 8002-1, Institut für Baustatik und Konstruktion, ETH Zürich, 1984.

30. GABRIEL K., Anwendung von statistischen Methoden und Wahrscheinlichkeitsbetrachtungen auf das Verhalten von Bündeln und Seilen als Zugglieder aus vielen und langen Drähten. Weitgespannte Flächentragwerke, 2. Internat. Symposium, Stuttgart 1979.

31. GALAMBOS J., The Asymptotic Theory of Extreme Order Statistics. John Wiley & Sons, 1978.

32. HAJDIN N., Vergleich zwischen den Paralleldrahtseilen und verschlossenen Seilen am Beispiel der Eisenbahnschrägseilbrücke über die Save in Belgrad. IVBH-Vorbericht zum 10. Kongress in Tokyo, September 1976, pp. 471-475.

33. HELGASON T., HANSON J.M., Fatigue Strength of High-yield Reinforcing Bars. National Cooperative Highway Research Program, Report 164, 1976.

34. HILMES J.B., EKBERG C.E., Statistical Analysis of Fatigue Characteristics of Under-reinforced Prestressed Concrete Flexural Members. Report IOWA Engineering Experiment Station, Iowa State Univ., May 1965.

35. HIRT M.A., Neue Erkenntnisse auf dem Gebiet der Ermüdung und deren Berücksichtigung bei der Bemessung von Eisenbahnbrücken. Der Bauingenieur, 52, 1977, pp. 255-262.

36. HÜCK M., Ein verbessertes Verfahren für die Auswertung von Treppenstufenversuchen. Z. Werkstofftech., 14, pp. 406-417, 1983.

37. IfB, Richtlinien für Zulassungs- und Ueberwachungsprüfungen an Spannstählen. Institut für Bautechnik, Berlin, Fassung Juli 1979.

38. MAENNIG W.W., Bemerkungen zur Beurteilung des Dauerschwingverhaltens von Stahl und einige Untersuchungen zur Bestimmung des Dauerfestigkeitsbereichs. Materialprüfung, 12, Nr. 14, April 1970, pp. 124-131.

39. MAENNIG W.W., Untersuchungen zur Planung und Auswertung von Dauerschwingversuchen an Stahl in Bereichen der Zeit- und der Dauerfestigkeit. Fortschr.-Ber. VDI-Z, Reihe 5, Nr. 5, VDI-Verlag, Düsseldorf 1967.

40. MAENNIG W.W.,Vergleichende Untersuchung über die Eignung der Treppenstufen-Methode zur Berechnung der Dauerschwingfestigkeit. Materialprüfung, 13, Nr. 1, January 1971, pp. 6-11.

41. PHOENIX S.L., The Asymptotic Time to Failure of a Mechanical System of Parallel Members. SIAM J. Appl. Math., 34, 1978, pp. 227-246.

42. PHOENIX S.L., SMITH R.L., A Comparison of Probabilistic Techniques for the Strength of Fibrous Materials under Local Load-Sharing among Fibers. Int. J. Solids Structures, 19, 1983, pp. 479-496.

43. PHOENIX S.L., TIERNEY L.J., A Statistical Model for the Time Dependent Failure of Unidirectional Composite Materials under Local Elastic Load-Sharing among Fibers. Eng. Fract. Mech., 18, 1983, pp. 193-215.

44. PHOENIX S.L., WU E.M., Statistics for the Time Dependent Failure of Kevlar-49/epoxy Composites: Micromechanical Modeling and Data Interpretation. Mech. Compos. Mater., 1983, pp. 135-215 .

45. SCOP P.M. ARGON A.S., Statistical Theory of Strength of Laminated Composites II. J. Compos. Mater., 1969, pp. 30-47.

46. Norm SIA 161: Stahlbauten. Schweiz. Ingenieur- und Architekten-Verein, Zürich, 1979.

47. SIEBKE H., Beschreibung einer Bezugsbasis zur Bemessung von Bauwerken auf Betriebsfestigkeit. Schweissen und Schneiden, 32, Heft 8, 1980, pp. 304-314.

48. SMITH R.L., A Probabilistic Model for Fibrous Composite with Local Load-Sharing. Proc. R. Soc. Lond., A 372, 1980, pp. 539-553.

49. SMITH R.L. Asymptotic Distribution for the Failure of Fibrous Materials under Series-parallel Structure and Equal Load-Sharing. J. Appl. Mech., 103, 1981, pp. 75-82.

50. SPINDEL J.E., BOARD B.R., HAIBACH E., The Statistical Analysis of Fatigue Test Results. ORE (Office for Research and Experiments of the Internat. Union of Railways), Utrecht, June 1979.

51. TIDE R.H.R., VAN HORN D., A Statistical Study of the Static and Fatigue Properties of High Strength Prestressing Strand. Fritz Engineering Lab., Report Nr. 309.2, Lehigh Univ., June 1966.

52. TIERNEY L., Asymptotic Bounds on the Time to Fatigue Failure of Bundles of Fibers under Local Load-Sharing. Adv. Appl. Prob., 14, 1982, pp. 95-121.

53. TILLY G.P., MOSS D.S., Long Endurance of Steel Reinforcement. IABSE Colloquium Fatigue of Steel and Concrete Structures, Lausanne 1982, pp. 229-238.

54. TRADINIK W., KROMP K., PABST R.F., Ueber die Universalität des Weibull-Ansatzes. Materiaprüfung, 23, Nr. 2, Februar 1981.

55. WARNER R.F., Fatigue of Partially Prestressed Concrete Beams. IABSE Colloquium Fatigue of Steel and Concrete Structures, Lausanne 1982, pp. 431-438.

56. WARNER R.F., HULSBOS C.L., Fatigue Properties of Prestressing Strand, PCI Journal 11, Nr. 2, 1966, pp. 25-46.

57. WEIBUL W., Zur Abhängigkeit der Festigkeit von der Probegrösse. Ingenieur-Archiv XXVIII, Band 1959, pp. 360-362.