Session IV: Numerical methods

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Numerical Methods

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On the Finite Element Method in the Field of Plasticity

Sur la méthode des éléments finis en plasticité

Zur Methode der finiten Elemente auf dem Gebiet der Plastizität

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SUMMARY
The fundamental aspects of some broadly applicable finite element procedures for the analysis of structures assuming ideal elasto-plastic or rigid-plastic material behaviour are presented and shortly discussed.

RESUME
Les aspects fondamentaux de certains procédés très généraux basés sur la méthode des éléments finis pour l'analyse des structures avec un comportement élasto-plastique ou rigide-plastique du matériau sont présentés et brièvement discutés.

ZUSAMMENFASSUNG
Die Grundprinzipien einiger allgemein anwendbarer, auf die Methode der finiten Elemente sich stützender numerischer Verfahren zur Berechnung elasto-plastischer und starr-plastischer Tragwerke werden geschildert und kurz diskutiert.

1. INTRODUCTION

The main reason for the extraordinary success of the finite element method in structural engineering lies certainly in its very broad applicability to all kind of structure types, loading conditions and material properties. Of course, this is also true in the field of plasticity, where computer based finite element procedures represent powerfull tools for the numerical analysis of complex real life structures.

The aim of the present paper is to present a short state-of-the-art theoretical review of some general finite element procedures assuming ideal elasto-plastic or rigid-plastic material behaviour. In order to confine the discussion to few fundamental questions, no speciic structure type and no specific material will be considered here. It should be clear, however, that much research work in recent years has led to very many different approaches for taking into account plastic deformations, some of them being certainly more straightforward, if possibly less generally applicable, then those discussed here.

It is assumed that the reader is familiar with matrix notation and with the main principles of conventional finite element analysis.

2. FINITE ELEMENT MODELS

Finite element models are used to build parametric fields satisfying prescribed continuity conditions. Parametric fields for the components of the displacement vector $\{u\}$ defining the displacement state and for the stress vector $\{\sigma\}$ defining the stress state within a structure are given in matrix notation by

$$\{u\} = [\Phi]\{U\} , \quad (1)$$
$$\{\sigma\} = [\Psi]\{\Sigma\} , \quad (2)$$

where U- and Σ-components of the global vectors $\{U\}$ and $\{\Sigma\}$ are nodal displacements and stress parameters respectively. The coefficients of the matrices $[\Phi]$ and $[\Psi]$ are shape functions defined piecewise within each element by generally simple analytical functions and satisfying prescribed continuity conditions along the element interfaces.

It is typical of the finite element method to use for the virtual displacements the same assumptions as for the real ones, thus restricting the infinite class of virtual functions considered by conventional virtual work methods to those given by the assumed shape functions of the matrix $[\Phi]$. Denoting virtual quantities with an asterisk the virtual displacement field $\{u^*\}$ is given by

$$\{u^*\} = [\Phi]\{U^*\}, \quad (3)$$

where the U^*'s are virtual displacement parameters.

The strain state within the structure is defined by a strain vector $\{\varepsilon\}$ whose components are obtained from the displacement vector $\{u\}$ applying an operator Δ, i.e. using kinematical strain-displacement relations:

$$\{\varepsilon\} = \{\Delta u\} = [\Delta\Phi]\{U\}. \quad (4)$$

As small displacements shall be assumed, Δ is a linear operator, thus identical relations can be written for the virtual strains $\{\varepsilon^*\}$:

$$\{\varepsilon^*\} = \{\Delta u^*\} = [\Delta\Phi]\{U^*\}. \quad (5)$$

The main problem of finite element structural analysis is that of finding a feasible internal stress distribution $\{\sigma\}$ satisfying equilibrium with the prescribed external loads $\{p\}$. This is often achieved by applying the principle of virtual displacements which says that the internal stresses $\{\sigma\}$ are in equilibrium with the external loads $\{p\}$ when the internal and the external virtual works are equal for all possible values of the virtual displacements $\{u^*\}$:

$$\int_V \{\varepsilon^*\}^T\{\sigma\}\cdot dV = \int_V \{u^*\}^T\{p\}\cdot dV, \tag{6}$$

where $\{\varepsilon^*\}$ and $\{u^*\}$ are kinematically compatible, i.e. $\{\varepsilon^*\}$ is derived from $\{u^*\}$ according to Eq. (5). V is the total volume of the structure consisting of several finite elements. Using the parametric virtual displacement field $\{\varepsilon^*\} = [\Delta\Phi]\{U^*\}$, Eq. (6) leads to

$$\{R\} = \{P\}, \tag{7}$$

where the vector $\{R\}$ of the internal nodal reaction forces due to the stress state $\{\sigma\}$ and the global vector $\{P\}$ of the external nodal loads are defined as follows:

$$\{R\} = \int_V [\Delta\Phi]^T\{\sigma\}\cdot dV, \tag{8}$$

$$\{P\} = \int_V [\Phi]^T\{p\}\cdot dV . \tag{9}$$

Eq. (7) represents a set of generalized equilibrium equations between the internal nodal forces $\{R\}$ and the external nodal loads $\{P\}$ leading, in general, to an only approximate satisfaction of the microscopic equilibrium conditions. To solve the problem of finding $\{\sigma\}$, however, the material behaviour has to be taken into account.

3. IDEAL ELASTO-PLASTIC STRESS-STRAIN RELATIONS

If the stresses are sufficiently small the material is assumed to behave perfectly elastically. The stress-strain relations are then given by Hooke's law

$$\{\sigma\} = [D](\{\varepsilon\} - \{\varepsilon_o\}), \tag{10}$$

where $[D]$ is the material dependent, symmetric and positive-definite "elasticity" matrix. The ε_o's are initial strains (e.g. due to temperature change) which are not directly associated with stresses. For simplicity initial strains shall not be considered here.

Eq. (10) is assumed to be valid only if the following yield conditions are satisfied:

$$f_k(\{\sigma\}) < c_k \qquad (k = 1 \text{ to } K), \tag{11}$$

where the f_k's are generally non-linear functions of the stress components. The c_k's are positive material constants. In the stress space the equations

$$f_k(\{\sigma\}) = c_k \qquad (k = 1 \text{ to } K) \tag{12}$$

piecewise define the yield surface of the material (see Fig. 1). This can, of course, in some cases be defined by a single non-linear function (K = 1). In order to take into account strain hardening or softening effects the c_k's are sometimes assumed to be functions of stress-strain history. For simplicity this shall not be considered here, i.e. ideal elasto-plastic material behaviour with constant c_k's shall be assumed.

If the stresses increase so much as to reach one of the surfaces (Eq. 12) delimiting the yield surface the relations between $\{\sigma\}$ and $\{\varepsilon\}$ change, and in fact it is only possible to give tangential relations between stress increments $d\{\sigma\}$ and strain increments $d\{\varepsilon\}$, the total stresses $\{\sigma\}$ being path dependent functions of the total strains $\{\varepsilon\}$ (non-conservative material behaviour).

It is then convenient to think of the strain increment $d\{\varepsilon\}$ as a sum of an "elastic" increment $d\{\varepsilon_{el}\}$ and a "plastic" increment $d\{\varepsilon_{pl}\}$:

$$d\{\varepsilon\} = d\{\varepsilon_{el}\} + d\{\varepsilon_{pl}\} , \tag{13}$$

where $d\{\varepsilon_{el}\}$ produces a stress increment $d\{\sigma\}$ according to Hooke's law, while $d\{\varepsilon_{pl}\}$ acts exactly as the initial strains $\{\varepsilon_0\}$ of Eq. (10), i.e. is not associated with any stress changes:

$$d\{\sigma\} = [D]d\{\varepsilon_{el}\} = [D](d\{\varepsilon\} - d\{\varepsilon_{pl}\}). \tag{14}$$

According to the theory of plasticity the plastic strain increment vector $d\{\varepsilon_{pl}\}$ has to be perpendicular to the yield surface, i.e. parallel to the gradient $\{grad\ f_k\}$ of the function $f_k(\{\sigma\})$ for $\{\sigma\}$ given by $f_k(\{\sigma\}) = c_k$, and pointed towards the outside of the allowable stress domain (see Fig. 1):

$$d\{\varepsilon_{pl}\} = \{grad\ f_k\}\cdot d\alpha \quad , \tag{15}$$

where $d\alpha$ is an arbitrary non-negative constant which can be determined by requiring the stress increment $d\{\sigma\}$ to satisfy the k'th yield condition exactly, i.e. to be parallel to the yield surface:

$$\{grad\ f_k\}^T d\{\sigma\} = 0. \tag{16}$$

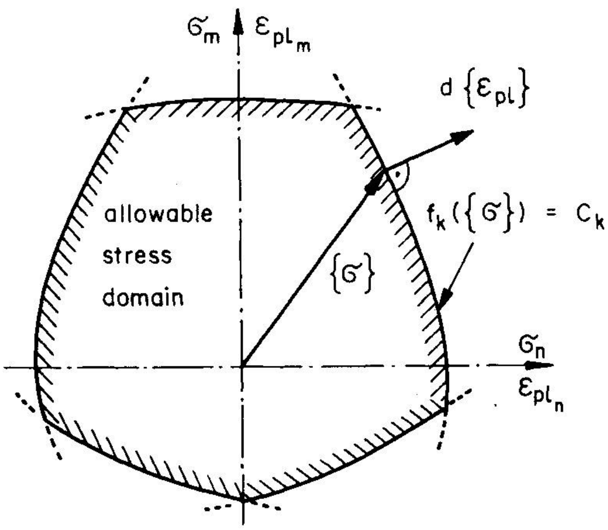

Fig. 1: Yield Surface defined by a set of non-linear yield conditions

From Eqs. (14), (15) and (16) simple algebra leads to a tangential relation between $d\{\sigma\}$ and $d\{\varepsilon\}$ similar to Hooke's law:

$$d\{\sigma\} = [D_T]d\{\varepsilon\}, \tag{17}$$

where $[D_T]$ is a symmetric, positive-semidefinite "tangential" matrix satisfying

$$[D_T]d\{\varepsilon_{pl}\} = [D_T]\{grad\ f_k\} = 0. \tag{18}$$

Eq. (17), however, is not really valid for any $d\{\varepsilon\}$. If unloading takes place, i.e. if $d\{\varepsilon\}$ is such that a purely elastic stress increment $d\{\sigma\} = [D]d\{\varepsilon\}$ would point towards the inside of the allowable stress domain:

$$\{grad\ f_k\}^T [D]d\{\varepsilon\} < 0, \tag{19}$$

then the material is assumed to behave elastically again ($[D_T] = [D]$).

If, after the stress vector $\{\sigma\}$ has reached the yield surface satisfying the single k'th yield condition exactly, the strains are increased any further, other yield conditions might become satisfied exactly, the stress vector $\{\sigma\}$ reaching "edges" or "corners" of the yield surface. The procedure explained above has then to be generalized for taking into account simultaneously more than one of the conditions (15), (16) and (19).

Of course, all this quite complicates elasto-plastic analysis and it is certainly an advantage if the material behaviour can be described by just a few non-linear yield conditions, possibly by a single one.

4. ELASTO-PLASTIC INCREMENTAL ANALYSIS

In finite element elasto-plastic analysis the primary unknown of the problem is generally chosen to be the displacement state of the structure described by the parametric field of Eq. (1).

As long as the material behaves elastically ($\{\sigma\} = [D]\{\varepsilon\}$) the internal nodal reaction forces $\{R\}$ of the structure can be expressed as linear function of the unknown nodal displacement parameters $\{U\}$:

$$\{R\} = \int_V [\Delta\Phi]^T \{\sigma\} \cdot dV = [K]\{U\} \;, \tag{20}$$

the global linear elastic stiffness matrix $[K]$ being defined by

$$[K] = \int_V [\Delta\Phi]^T [D][\Delta\Phi] \cdot dV. \tag{21}$$

The U's are then found by solving the system of linear equilibrium equations

$$[K]\{U\} = \{P\}. \tag{22}$$

However, when, due to high stress levels in some parts of the structure plastic strains occur, the relations between $\{R\}$ and $\{U\}$ become non-linear. It is then necessary to increase the external loads $\{P\}$ in steps $\Delta\{P\}$ and to find a new stress distribution after each load increase satisfying equilibrium (7) while taking into account the elasto-plastic stress-strain relations discussed above. The most widely accepted iterative algorithm to do so can be described as follows:

a. Initialize $\{U\} := \{P\} := 0$

b. Increase $\{P\} := \{P\} + \Delta\{P\}$ and $\{U\} := \{U\} + \Delta\{U\}$, where $\Delta\{U\}$ is obtained from the solution of the following system of linear equations:

$$[\tilde{K}]\Delta\{U\} = \Delta\{P\}, \tag{23}$$

$[\tilde{K}]$ being a approximation of the stiffness matrix valid for the current load step as explained below.

c. Determine the internal nodal reactions $\{R\}$ according to Eq. (8) from the actual stress state, $\{\sigma\}$ obtained from the incremented strain state $\{\varepsilon\} = [\Delta\Phi]\{U\}$ corresponding to the new $\{U\}$.

d. If, within a prescribed tolerance $\{R\} = \{P\}$ repeat from b.

e. Otherwise apply the nodal loads $\{P\}-\{R\}$ representing unbalanced residual nodal forces obtained from the difference between the external loads $\{P\}$ and the internal reactions $\{R\}$.

A corresponding displacement increase $\Delta\{U\}$ is found by solving the system of linear equations:

$$[\tilde{K}]\Delta\{U\} = \{P\}-\{R\} \tag{24}$$

f. Increase $\{U\} := \{U\}+\Delta\{U\}$ and repeat from c.

As in most cases a limit load is to be found rather than the response of the structure to a prescribed load, the external loads $\{P\}$ have to be increased until an equilibrium stress state can not be found anymore or until the displacements in some parts of the structure grow beyond prescribed "collapse" limits.

Two main questions arise. The first one concerns the stiffness matrix $[\tilde{K}]$ of Eqs. (23) and (24), which, ideally, should describe the relation between $\{R\}$- and $\{U\}$-increments within a load step for the partially plastified structure. Often $[\tilde{K}]$ is approximated by the linear elastic stiffness matrix $[K]$, the method described here being then often called (somehow improperly) the "initial stress method". Fig. 2 shows its basic principle when applied to a single degree of freedom system for a single load step ($\Delta P = P$). Sometimes a better approximation for $[\tilde{K}]$ is used taking into account the changes in stiffness caused by the plastified zones of the structure.

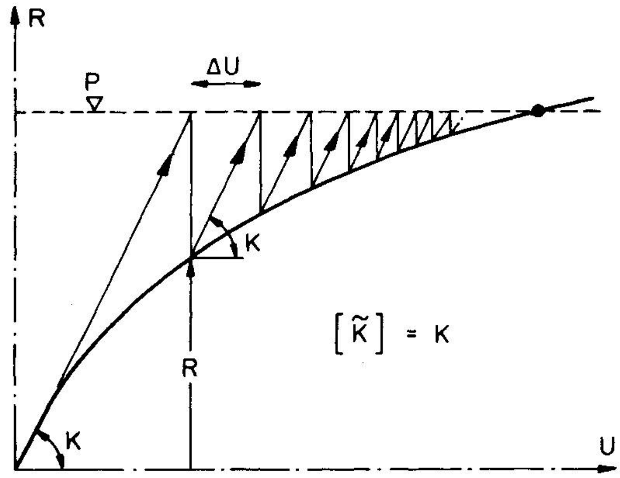

Fig. 2: Initial stress method for a single degree of freedom system

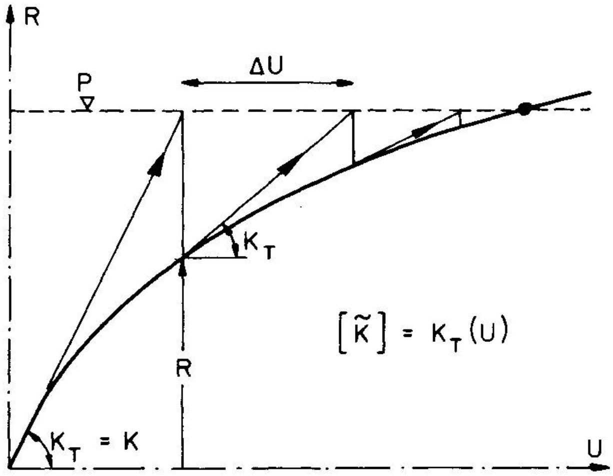

Fig. 3: Newton-Raphson method for a single degree of freedom system

A frequent choice for $[\tilde{K}]$ is the tangent stiffness matrix $[K_T]$ relating infinitesimal $d\{R\}$- and $d\{U\}$-increments at the beginning of a load step or during the iterations within a load step:

$$d\{R\} = [K_T]d\{U\}. \tag{25}$$

The $[K_T]$-matrix can be obtained, like $[K]$ in Eq. (21), from a sum of the contributions of each single element using instead of $[D]$ the elasto-plastic tangential $[D_T]$-matrix defined in Eq. (17):

$$[K_T] = \int_V [\Delta\Phi]^T [D_T][\Delta\Phi]\cdot dV. \tag{26}$$

If $[K_T]$ is used, the solution procedure described above corresponds to the so-called "Newton-Raphson-Method" for the solution of non-linear systems of coupled equations. Fig. 3 shows its basic principle. Obviously a much faster convergence is obtained when using $[K_T]$ instead of $[K]$, however, the computational effort needed at each step will increase very much. In fact not only the numerical evaluation of $[K_T]$ is time consuming, but also a totally new solution of the Eqs. (23) or (24) is needed each time $[K_T]$ is changed, which is not the case when using allways the same elastic stiffness matrix $[K]$. An obvious possibility would be to evaluate $[K_T]$ (or some more or less crude approximation of it) only from time to time, thus using the same $[K_T]$-matrix for several steps. It should be noted, however, that convergence (quite contrary to geometrically non-linear problems) can in many cases be obtained using allways the same linear elastic stiffness $[K]$.

A second question concerns the way the internal reactions $\{R\}$ or their increases $\Delta\{R\}$, which, of course, can also be obtained from a sum of element contributions, are evaluated from the stress increments $\Delta\{\sigma\}$ caused by the strain increments $\Delta\{\varepsilon\}$ associated with $\Delta\{U\}$. Obviously, $\Delta\{\varepsilon\}$ not being infinitesimal, the use of the incremental relations between $d\{\sigma\}$ and $d\{\varepsilon\}$ derived above (Eq. (17)), will, in general, involve some approximations. Details should not be discussed here, it should be noted, however, that as long as a stress distribution can be found which satisfies equilibrium, i.e. leading to $\{R\} = \{P\}$ violations of the elasto-plastic incremental stress-strain relations are not too disturbing. In fact from the lower-bound theorem of the plasticity theory one knows that the stress distribution obtained can only underestimate the limit load, thus leading to a safe design.

5. RIGID-PLASTIC ANALYSIS

If rigid-plastic material behaviour is assumed the statical (or lower-bound) and the kinematical (or upper-bound) theorems of the theory of plasticity represent powerful tools for the evaluation of a limit load factor λ multiplying given external loads $\{p\}$ and possibly of the shape of the collapse mechanism.

According to the statical theorem a stress state $\{\sigma\}$ has to be found which satisfies equilibrium with the external loads $\lambda\{p\}$ as well as the yield conditions everywhere within the structure. The limit load is then found by maximizing λ.

By introducing a finite element parametric stress field (Eq. (2)) the internal reactions $\{R\}$, which have to equal $\lambda\{P\}$ in order to satisfy equilibrium (7), can be evaluated as linear functions of the unknown nodal stress parameters $\{\Sigma\}$:

$$\{R\} = \int_V [\Delta\Phi]^T\{\sigma\}\cdot dV = [E]\{\Sigma\} = \lambda\{P\}, \qquad (26)$$

where $[E]$ is a global "equilibrium"-matrix obtained, as usual, by a sum of element contributions and definded by

$$[E] = \int_V [\Delta\Phi]^T[\Psi]\cdot dV. \qquad (27)$$

The stress parameters $\{\Sigma\}$ will also have to satisfy yield conditions. These will have to be checked in Q discrete "checkpoints" throughout the structure, where the stress components assume the values $\{\sigma_q\} = [\Psi_q]\{\Sigma\}$ ($q = 1$ to Q). Allthough the use of the non-linear yield conditions (Eq. (11)) is possible, it is certainly convenient in rigid-plastic analysis to use linear ones, thus introducing polyedrical yield surfaces (see Fig. 4), even if a larger number of inequalities may become necessary. Linear yield conditions are given by:

$$\{f_k\}^T\{\sigma_q\} \leq c_{kq} \qquad (k = 1 \text{ to } K;\ q = 1 \text{ to } Q) \qquad (28)$$

or, for all conditions together at a checkpoint q

$$[f]^T\{\sigma_q\} \leq \{c_q\} \qquad (q = 1 \text{ to } Q), \qquad (29)$$

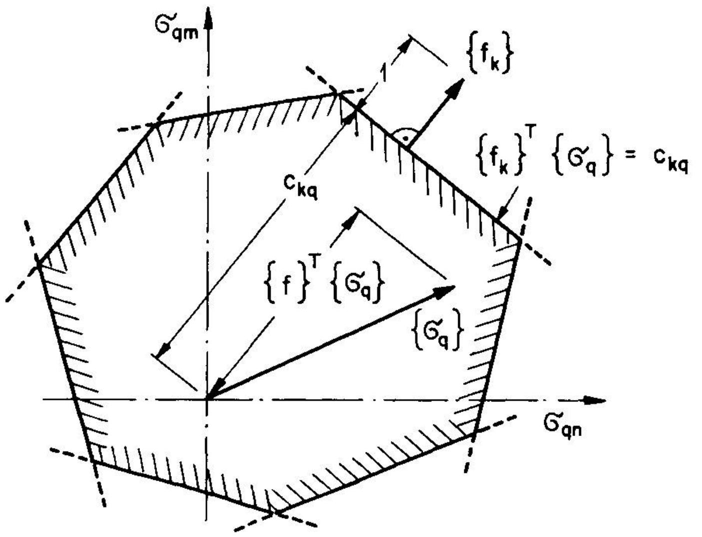

Fig. 4: Yield surface defined by a set of linear yield conditions

where c_{kq} represents the resistance of the structure at a checkpoint q for a stress direction $\{f_k\}$.

From the optimality condition $\lambda \rightarrow$ maximum, from the equilibrium equations (26) and from the linearized yield conditions (29) the following linear program for the unknowns λ and $\{\Sigma\}$ is found (see also Fig. 5):

$$\lambda \rightarrow \text{maximum}$$

$$\{\sigma\} = -\{P\}\lambda + [E]\{\Sigma\} \qquad (30)$$

$$0 \leq c_{kq} - [f]^T[\Psi_q]\{\Sigma\} \quad (q = 1 \text{ to } Q).$$

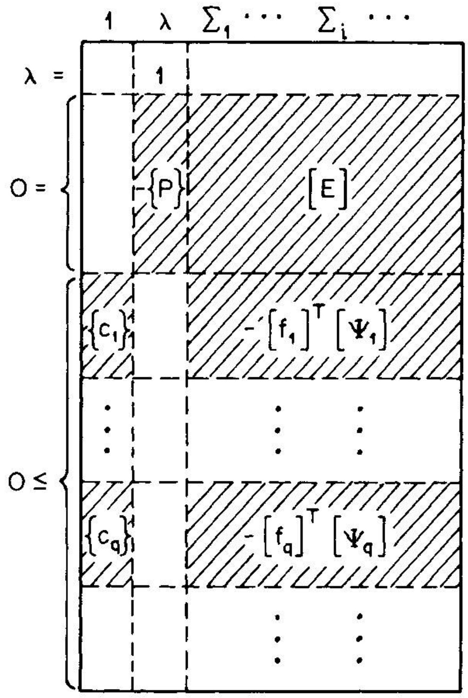

Fig. 5: Tableau of the linear program (30): $\lambda \rightarrow$ maximum

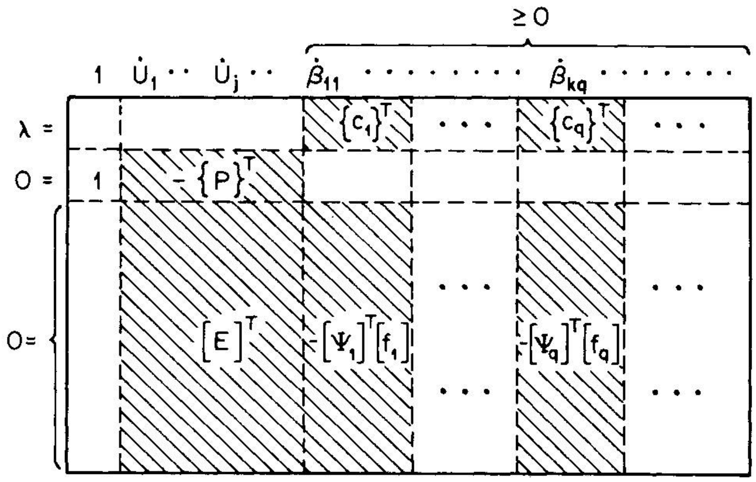

Fig. 6: Tableau of the linear program (31): $\lambda \rightarrow$ minimum

From the kinematical (or upper-bound) theorem the following linear program, whose derivation shall not be given here is found (see also Fig. 6):

$$\begin{aligned}
\lambda &= \sum_q \{c_q\}^T\{\dot{\beta}_q\} \rightarrow \text{minimum}, \\
0 &= 1 - \{P\}^T\{\dot{U}\}, \\
\{\sigma\} &= [E]^T\{\dot{U}\} - \sum_q [\Psi_q]^T[f]\{\dot{\beta}_q\} \\
\{\dot{\beta}_q\} &\geq \{\sigma\} \qquad (q = 1 \text{ to } Q),
\end{aligned} \qquad (31)$$

where the $\dot{U}$'s are nodal displacement velocity parameters and the $\dot{\beta}_{kq}$'s are generalized strain velocity parameters somehow related to the $d\alpha$'s introduced in Eq. (15) (see Ref. [4]).

The linear programs (30) and (31) are "dual" to each other. The same load factor λ will therefore be obtained. As expected the value of λ only depends on the mathematical model, not on the method of solution used (statical or kinematical approach). A lower bound of the true value of λ will be obtained if the assumed Ψ-functions and the linear inequalities (29) guarantee that microscopic equilibrium conditions and yield conditions are nowhere violated. An upper-bound (at least for the linearized yield condition used) will be obtained if kinematical compatibility conditions are satisfied exactly. In many cases, however, a bound for λ will not be found, but just an approximation of it.

By solving one of the linear programs (30) or (31) the solution of the other one is also known. Numerical values not only for λ but also for the Σ-, $\dot{U}$- and $\dot{\beta}$-parameters are therefore obtained. The displacement velocity parameters $\{\dot{U}\}$ describe the collapse mechanism. The stress parameters $\{\Sigma\}$ define a corresponding state of admissible stresses. However, because this is defined in an unique way, only in the regions and in the directions in which plastic flow occurs, the values of the Σ-parameters will generally not be very meaningful as large portions of the structure may remain rigid during collapse. The $\dot{\beta}$-parameters provide informations on the distribution of plastic flow during collapse.

The procedure described here, while being at least in principle generally applicable, has the disadvantage of being a two-field procedure as independent parametric assumptions both for the stresses and for the displacements have to be introduced. In fact the criteria for chosing these parametric fields are not always clear. Moreover, it would certainly be an advantage not to have any equilibrium equations in the linear program (30), which would be the case if stress assumptions satisfying a priori equilibrium conditions could be found.

This is in some cases possible if parametric finite element fields for stress functions (like Airy's for plate stretching problems) are introduced. The stress components building the vector $\{\sigma\}$ are then derived, generally by differentation, from the stress functions leading to:

$$\{\sigma\} = [\Psi]\{\Sigma\} + \lambda\{\bar{\sigma}\}, \qquad (32)$$

where the columns of the $[\Psi]$-matrix corresponding to the nodal stress function parameters $\{\Sigma\}$ represent homogeneous stress states while $\lambda\{\bar{\sigma}\}$ represents an inhomogeneous stress state satisfying equilibrium with the external loads. From the statical theorem the following linear program is then obtained:

$\lambda \rightarrow$ maximum,

$$\{\sigma\} \leq \{c_q\} - [f]^T\{\bar{\sigma}_q\}\lambda - [f]^T[\Psi_q]\{\Sigma\}, \qquad (q = 1 \text{ to } Q). \qquad (33)$$

Of course from the kinematical theorem a corresponding dual linear program could also be derived which would show that the $\dot{U}$'s, i.e. the collapse mechanism, will not be obtained by this approach (but the $\dot{\beta}$'s will).

An important advantage of the stress function approach is that equilibrium conditions can, in many cases, be satisfied exactly, thus being possible to obtain a true lower-bound of the load factor λ. There are, however, also some drawbacks:

not for all kinds of structure stress functions exist (e.g. not for framed structures); the stress distribution $\{\bar{\sigma}\}$ is easily found if only surface loads along the structure boundaries are present, which is often the case for plate-stretching and rotationally symmetric problems but almost never for plate-bending and shell problems; the assumed parametric fields for the stress functions have often (e.g. in the case of Airy's function) to satisfy stringent continuity conditions at the element interfaces; finally some complications arise for multiply connected domains.

An other, generally applicable approach to obtain stress assumptions satisfying a priori, at least approximately, equilibrium conditions would be to use linear elastic analysis to find both the inhomgeneous stress state $\{\bar{\sigma}\}$ and the homogeneous ones building the columns of the matrix $[\Psi]$. These can be obtained by specifying as load cases any number of different initial strain distributions resulting in an equal number of homogeneous (but not necessarily linearly independent) stress states.

6. ON PLASTIC OPTIMUM DESIGN

If some kinds of relation between the c_{kq}-coefficients representing the resistance of the structure at a checkpoint q for a stress direction $\{f_k\}$ (see Fig. 4) and a "merit"-function M $(\ldots.,c_{kq},\ldots.)$ can be mathematically established, an optimum design problem leading to an optimal distribution of the resistance coefficients c_{kq} for a prescribed design load $\{p\}$ can be formulated. Using, for simplicity, stress assumptions satisfying a priori equilibrium (i.e. Eq. (32) with $\lambda = 1$), the following mathematical program for the unknown c_{kq}- and Σ-coefficients is found:

$$M(\ldots.,c_{kq},\ldots.) \rightarrow \text{optimum},$$
$$\{0\} \leq -[f]^T\{\bar{\sigma}_q\} - [f]^T[\Psi_q]\{\Sigma\} + \{c_q\} \qquad (q = 1 \text{ to } Q). \tag{34}$$

An obvious difficulty of this approach lies in the choice of the merit function M. An other difficulty arises when several different loading cases govern the design of the structure as different sets of Σ-coefficients defining an "optimal" homogeneous stress state for each of the loading cases considered would have to be determined.

This last difficulty can be avoided when the inhomogeneous stress distributions $\{\sigma_n\}$ for each of the N loading cases considered (n = 1 to N) can be found by linear elastic analysis. This is only possible if the c_{kq}'s , i.e. the plastic resistance distribution within the structure, can be assumed not to have any influence on the elastic stress distribution (e.g. this is possible when looking for an optimal reinforcement distribution in a given concrete structure). From Eq. (28) the yield conditions for k = 1 to K and q = 1 to Q can then be formulated as follows:

$$0 \leq -\max_n(\{f_k\}^T\{\bar{\sigma}_{nq}\}) - \{f_k\}^T\{\Psi_q\}\{\Sigma\} + c_{kq}, \tag{35}$$

where at each checkpoint q and for each stress direction $\{f_k\}$ only the most unfavourable load case n is explicitely checked ($\{\bar{\sigma}_{nq}\}$ represents the elastic stresses due to the n'th load case at a checkpoint q), while for all loading cases together a single "optimal" homogeneous stress distribution defined by the stress parameter vector $\{\Sigma\}$ is introduced. According to the so-called shake-down theorem of the plasticity theory, this procedure will result in the design of a structure capable of stabilising for any conceivable load cycle, i.e. a structure which will behave perfectly elastically after plastic flow has occurred in the first load cycles. But the real advantage of this procedure, when applicable, is that the optimum design problem will be much simplified when several loading cases have to be considered which is, of course, almost always the case.

7. OVERVIEW AND CONCLUSIONS

In the field of plasticity most procedures suggested to date are based on an elasto-plastic approach, the initial stress method, with or without stiffness modification, being certainly the most generally applicable one. In different well-known general purpose finite element computer programs this kind of analysis is implemented. The main advantage of the elasto-plastic approach is that it can provide all needed informations on structural behaviour from working conditions until collapse. Other non-linear effects due to large deformations, crack propagation, creep, contact problems, friction, in fact, at least in principle, to any kinds of material behaviour that can be mathematically described can be taken into account by step-by-step iterative methods. An other important field of application is non-linear dynamic analysis by time-step integration of the dynamic equations.

However, the difficulties involved in an elasto-plastic analysis when applied to real life problems should not be underestimated. The computational effort needed will generally be high as reiterate solutions of large systems of linear equations will be necessary as well as reiterate evaluations of internal forces and stiffness matrices for each element by numerical integration procedures. Modeling problems might also arise as it is often necessary, in order to reduce computing time, to approximate reality by simple models, i.e. by coarse finite element meshes. This requires from the user of the computer program a very clear understanding of the way the program internally works and of the approximations involved. Finally, the interpretation of results and their relation to the actual design of the structure may also present some difficulties.

Rigid-plastic limit load analysis has received, so far, less attention than elasto-plastic analysis. This is probably due to the limited scopes that can be pursued by such an approach, as no information on working stresses or on displacements before collapse can be obtained. For real life problems an additional linear elastic analysis will therefore in most cases be necessary. An other difficulty arising from the rigid-plastic approach is caused by the great computational effort generally needed for solving the large linear programs involved. It is felt that more research work is needed for finding faster solution methods taking advantage of the peculiar nature of the problem. If this succeeds, however, rigid-plastic limit analysis, possibly combined in the same computer program with linear elastic analysis, could well become a widely used tool for everyday's structural engineering, being certainly easier to apply to real life problems than elasto-plastic analysis.

Rigid-plastic optimum design, and actually any kinds of direct optimum design procedures has found very few applications in civil engineering. In fact the prevailing attidude today is that the design of a structure cannot be done in a completely automatic way, but always requires a close interaction between the designer and the computer, which is more a problem of man - machine communication than of the theoretical approach used for the design. In some cases, however, the most important being probably the problem of finding a minimum weight reinforcement distribution for a given concrete structure, plastic optimum design methods can be useful to find the best solution among a narrow choice specified by the designer working in an interactive computer aided design environment.

REFERENCES

[1] O.C. Zienkiewicz: "The Finite Element Method in Engineering Science", McGraw-Hill, 1971.

[2] J.T. Oden: "Finite Elements of Non-linear Continua", McGraw-Hill, 1972.

[3] J. Robinson: "Integrated Theory of Finite Element Methods", John Wiley and Sons, 1973.

[4] E. Anderheggen: "Starr-plastische Traglastberechnungen mittels der Methode der Finiten Elemente", Swiss Federal Institute of Technology Zürich, Institute of Structural Engineering, Report No 32, 1971, Birkhäuser Verlag Basel and Stuttgart.

Collapse Load Analysis of Engineering Structures by Using New Discrete Element Models

Calcul à la ruine de structures, à l'aide de modèles nouveaux d'éléments discrets

Berechnung der Traglast von Baukonstruktionen mit neuen diskreten Elementen

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SUMMARY
Combining advantages of the concept of limit analysis and standard load incremental procedure in existing finite element method, a method of collapse load analysis is introduced in this paper. It should be mentioned that this method is not a method of rigid plastic analysis, but effects of elasticity, finite deformation or instability can be taken into account.
Motivation of development of this method and its theoretical basis will be explained first and justification of the present method is illustrated by several numerical examples including collapse load analysis of concrete slabs.

RESUME
Le rapport présente une méthode de calcul à la ruine combinant les avantages de la théorie des charges limites et du procédé de l'augmentation progressive des charges, tels qu'ils existent dans des méthodes actuelles par éléments finis. Cette méthode n'est pas une méthode de calcul rigide-plastique, car elle tient compte des effets de l'élasticité, de déformations limitées et d'instabilité. Les raisons du développement de cette méthode et ses bases théoriques sont présentées. Plusieurs exemples numériques, parmi lesquels le calcul à la ruine de dalles en béton armé, illustrent et prouvent la valeur de la méthode développée ici.

ZUSAMMENFASSUNG
Eine Methode zur Traglastberechnung wird vorgestellt, welche Vorteile der Traglastverfahren einerseits und der bei der Anwendung der Methode der finiten Elemente üblichen Verfahren der schrittweisen Laststeigerung andererseits kombiniert. Es handelt sich nicht um eine starr-plastische Berechnungsmethode, denn das elastische Verfahren, endliche Verformungen oder Instabilitäten können ebenfalls berücksichtigt werden. Die Entwicklung der Methode wird begründet, und ihre theoretischen Grundlagen werden dargestellt. Anhand mehrerer numerischer Beispiele, unter anderem zur Berechnung der Traglast von Stahlbetonplatten, wird die Anwendung der Methode erläutert.

ABSTRACT

Combining advantages of the concept of limit analysis and standard load incremental procedure in existing finite element method, a method of collapse load analysis is introduced in this paper. It should be mentioned that this method is not a method of rigid plastic analysis, but effects of elasticity, finite deformation or instability can be takn into account.
Motivation of development of this method and its theoretical basis will be explained first and justification of the present method is illustrated by several numerical examples including collapse load analysis of concrete slabs.

1. INTRODUCTION

1.1 Plastic Analysis and its Limitation

Consider any structure or solid subjected to external load (statical or dynamical). As long as the external load is small, it may deform elastically and the induced stresses and strains are so small that their distribution can be determined by well established theory of elasticity.
With the increase of external loads, however, strain distribution may reach the stage where no longer it is small and finite strain distribution will set up locally or partially in the deformed body under consideration, and then the structure may be subjected to large deformation or may buckle in case of ductile materials.
Upon further increase of the load, deformed structures may start to yield and develop so called plastic hinges, hinge lines (or slip lines) or slip surfaces, and plastic zone will grow and spread out and finally cracks may initiate from some overstressed region.
At the final stage of loading or the ultimate load, a certain link mechanism which may usually consist of plastic hinges or plastic hinge lines or slip surfaces will be formed in the structure.
Under such condition the structure may loose stability and it will start to move freely just like a linked rigid bodies and it will be *collapsed*.
According to the theory of plasticity, it can be proved that a collapse load solution can be uniquely determined if the solution satisfies the following three conditions:

(i) equilibrium condition
(ii) plasticity condition
(iii) mechanism condition

In general it is extremely difficult to obtain such a solution analytically and two different procedure of obtaining approximate solutions have been proposed by basing on the well-known upper bound and lower bound theorems in the theory of limit analysis which were originally proposed by Prager, Drucker and many others. These approximate solution procedures have provided a very

powerful tool for collapse load analysis of plane frames, simple plate and shell structures and resulted in development of plastic analysis and design which are now accepted in routine structural design of those engineering structures mentioned.
Practical application of plastic analysis, however, has been limited to collapse load analysis of plane frames and furthermore influences of stability, crack initiation can not be taken into account. Especially dynamic collapse load analysis is still under state of arts and because of this reason, its application to structural dynamics has been quite limited.

1.2 Finite Element Method and Current Status of Development

Advent of the finite element method has changed this situation completely. By using the standard load incremental procedure it is possible to trace step by step the equilibrium state of structures and elasto-plastic stress distribution corresponding to the load condition at the time prescribed and sequence of formation of collapse mechanism.
At the present moment it is not too difficult in principle to obtain the collapse load solution of any complex structure under a certain loading condition by using standard computer programs. Very serious drawback of the finite element method, however, is computing time and cost, especially in nonlinear analysis.
In order to solve such a difficult problems active work has been done all over the world. Actual problems, however, are still far beyond control of any existing method.

1.3 Concept of the Rigid Body-Spring Element

The present author believed that development of a new discrete models might be only a possible way to solve this problem. He tried to find a new physical model rather than mathematical in which essential feature of deformable bodies is retained without introducing highly complex mathematical manipulation. In general when structures or solids reach their ultimate state of loading, they may be yielded, collapsed and crushed into pieces. At the limiting state each part or piece of the structures may move like rigid bodies. Based on such experimental evidences, the following *Rigid Body- Spring* model has been conceived. Consider the bending problem of a beam under lateral loads as shown in Fig. 1.
Within elastic range, deformation is distributed throughout the beam, but once plastic deformation starts either at the point of load application or at the beam ends, strain energy will be absorbed in the narrow portion of a beam where plastic deformation takes place and at the ultimate stage of loading a number of the so-called *plastic hinges* will be formed so that the beam structure will collapse into a link mechanism. This mechanism consists of rigid bars and plastic hinges. In case of bending problems of concrete slabs as shown in Fig. 1, similar experimental evidence will be observed. That is, within the range of elastic bending, deformation is distributed over the whole plate area, however, at the final stage of loading the plate will collapse under a certain mechanism which consists of rigid plate segments, and plastic hinge lines connecting those plate segments.
The so-called *slip line theory* is also well known in also plane stress as well as plane strain problems in the theory of plasticity (See slso Fig. 1). According to this theory, it is assumed that two dimensional solids will move under a certain mechanism which consists of two dimensional rigid segments and slip lines connecting those segments, and along which relative sliding of two neighboring segments will occur. In the following section theoretical basis of these new models will be described.

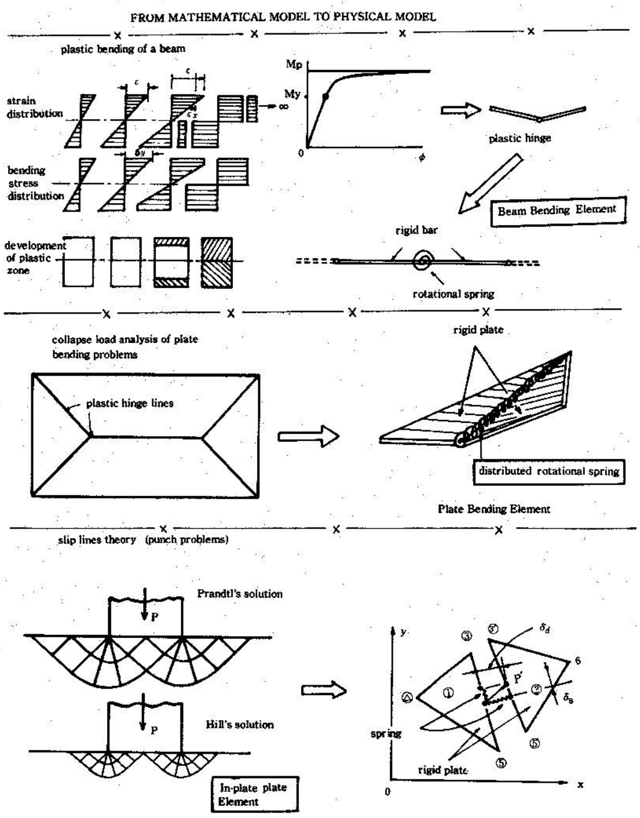
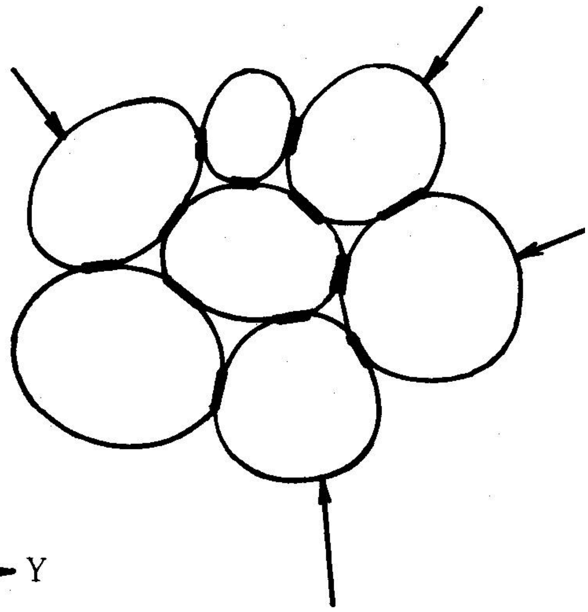


Fig. 1 Concept of the Rigid Body-Spring Model

2. THEORETICAL BASIS OF RIGID BODY-SPRING MODELS

2.1 Physical Basis of Rigid Body-Spring Models

Consider a set of three dimensional rigid bodies of arbitary shape as shown in Fig. 2. They are assumed to be in equilibrium with external loads, and reaction forces are produced by the spring system which is distributed over the contact surface of two adjacent bodies. For further development of new element models, it will be assumed that the contact area is known and fixed*.



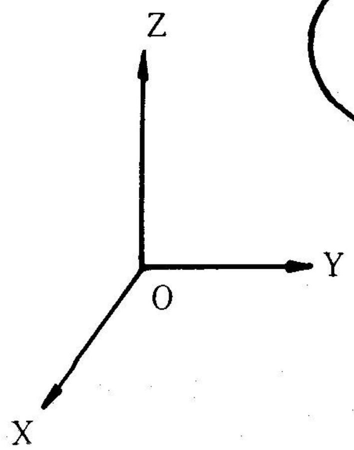


Fig. 2

* It should be mentioned here that in actual contact problem, the contact surface are not known a priori, and therefore it can be determined only in iterative way.

Taking such two rigid bodies under contact, infinitesimal deformation of the spring system is considered. (Fig. 3). Displacement $\mathbf{u}$ of an arbitrary point in a rigid body can be given by the following vectorial equation:

$$\mathbf{u} = \mathbf{u}_G + \boldsymbol{\Theta} \times (\mathbf{r} - \mathbf{r}_G) \qquad (1)$$

where $\mathbf{u}_G$ is displacement vector of the centroid, $\boldsymbol{\Theta}$ is the infinitesimal rotation vector and $(\mathbf{r} - \mathbf{r}_G)$ is a position vector of arbitrary point with respect to the centroid before deformation.

$$\mathbf{u}_G = (u_g, v_g, w_g) \qquad \boldsymbol{\Theta} = (\theta, \phi, \chi) \qquad (2)$$

Denoting the displacement vectors of arbitrary point P(x,y,z) in body (I) and (II) by $\mathbf{u}'$, $\mathbf{u}''$, respectively, they are given by the following equations:

$$\begin{aligned} \mathbf{u}' &= \mathbf{u}_1 + \boldsymbol{\Theta}_1 \times (\mathbf{r} - \mathbf{r}_1) \\ \mathbf{u}'' &= \mathbf{u}_2 + \boldsymbol{\Theta}_2 \times (\mathbf{r} - \mathbf{r}_2) \end{aligned} \qquad (3)$$

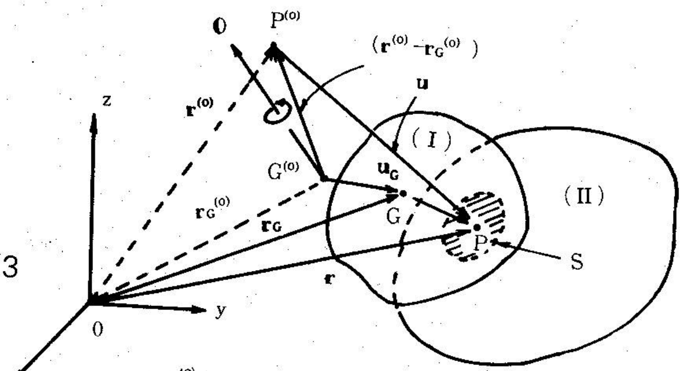


Fig. 3

More precisely,

$$\left.\begin{aligned} u' &= u_1 + (z - z_1)\phi_1 - (y - y_1)\chi_1 \\ v' &= v_1 + (x - x_1)\chi_1 - (z - z_1)\theta_1 \\ w' &= w_1 + (y - y_1)\theta_1 - (x - x_1)\phi_1 \end{aligned}\right\} \qquad (4\text{-a})$$

$$\left.\begin{aligned} u'' &= u_2 + (z - z_2)\phi_2 - (y - y_2)\chi_2 \\ v'' &= v_2 + (x - x_2)\chi_2 - (z - z_2)\theta_2 \\ w'' &= w_2 + (y - y_2)\theta_2 - (x - x_2)\phi_2 \end{aligned}\right\} \qquad (4\text{-b})$$

Therefore denoting the point P after displacement in bodies (I) and (II) by P' and P" respectively, the relative displacement vector of the point P can be defined as follows:

$$\overrightarrow{P'P''} = \mathbf{u}'' - \mathbf{u}' \qquad (5)$$

Denoting the unit normal drawn outward to the contact surface at the point P by $\mathbf{n}$, (See Fig. 4) the normal displacement δ_d to the surface S can be given as follows:

$$\delta_d = (\overrightarrow{P'P''}, \mathbf{n}) = l(u'' - u') + m(v''-v') + n(w''-w') \qquad (6\text{-a})$$

where

$$\mathbf{n} = (l, m, n) \qquad (6\text{-b})$$

Similarly the displacement component δ_s in the tangential plane to the surface can be given by the following equation:

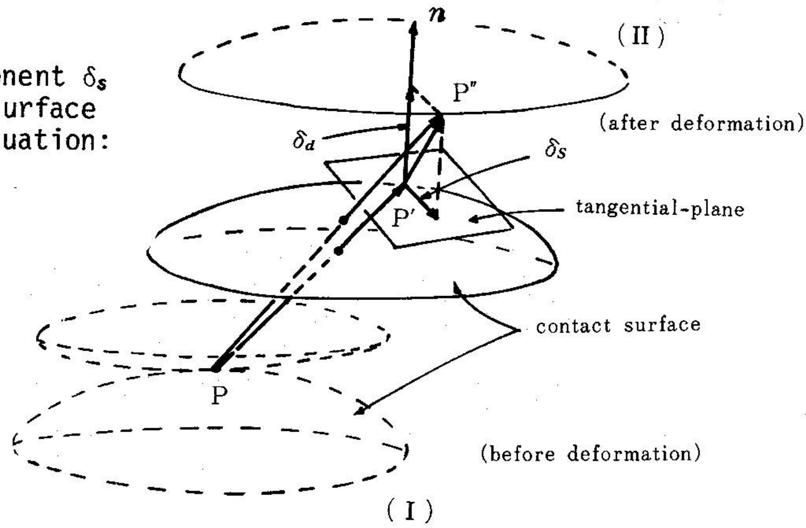


Fig. 4

$$
\begin{aligned}
\delta_s^2 &= \left| \boldsymbol{n} \times \overrightarrow{P'P''} \right|^2 \\
&= \{m(w'' - w') - n(v'' - v')\}^2 + \{n(u'' - u') - l(w'' - w')\}^2 \\
&\quad + \{l(v'' - v') - m(u'' - u')\}^2
\end{aligned} \tag{7}
$$

Basing on the above preliminaries, strain energy due to the relative displacements (δ_d, δ_s) of the spring system distributed over the contact surface S can be given by the following equation:

$$
V = \frac{1}{2} \iint_S (k_d \, \delta_d^2 + k_s \delta_s^2) \, dS \tag{8}
$$

Substituting eqs.(6) and (7) into eq.(8) the following equation can be easily obtained:

$$
\begin{aligned}
V &= \frac{1}{2} \iint_S [k_s (\delta_x^2 + \delta_y^2 + \delta_z^2) + (k_d - k_s)(l^2 \delta_x^2 + m^2 \delta_y^2 + n^2 \delta_z^2 \\
&\quad + 2 \, l m \, \delta_x \delta_y + 2 \, m n \, \delta_y \delta_z + 2 \, n l \, \delta_z \delta_x)] \, dS \\
&= \frac{1}{2} \iint \boldsymbol{\delta}^T \bar{\mathbf{D}} \, \boldsymbol{\delta} \, dS
\end{aligned} \tag{9-a}
$$

where

$$
\begin{aligned}
\delta^T &= \lfloor \delta_x , \delta_y , \delta_z \rfloor \\
\delta_x &= u'' - u', \quad \delta_y = v'' - v', \quad \delta_z = w'' - w'
\end{aligned} \tag{9-b}
$$

$$
[\bar{D}] = \begin{bmatrix}
k_d l^2 + k_s (1 - l^2) & (k_d - k_s) l m & (k_d - k_s) l n \\
(k_d - k_s) l m & k_d m^2 + k_s (1 - m^2) & (k_d - k_s) m n \\
(k_d - k_s) l n & (k_d - k_s) m n & k_d n^2 + k_s (1 - n^2)
\end{bmatrix} \tag{10}
$$

Displacement vector $\boldsymbol{\delta}$ can be also expressed by the following matrix equation:

$$
\boldsymbol{\delta} = \mathbf{B} \, \mathbf{d} \tag{11}
$$

where

$$
[\mathbf{B}] = \begin{bmatrix}
-1 & 0 & 0 & 0 & -(z-z_1) & (y-y_1) & 1 & 0 & 0 & 0 & (z-z_2) & -(y-y_2) \\
0 & -1 & 0 & -(z-z_1) & 0 & -(x-x_1) & 0 & 1 & 0 & (z-z_2) & 0 & (x-x_2) \\
0 & 0 & -1 & -(y-y_1) & (x-x_1) & 0 & 0 & 0 & 1 & (y-y_2) & -(x-x_2) & 0
\end{bmatrix} \tag{12}
$$

$$
\mathbf{d}^T = \lfloor u_1, v_1, w_1, \theta_1, \phi_1, \chi_1 ; \; u_2, v_2, w_2, \theta_2, \phi_2, \chi_2 \rfloor \tag{13}
$$

Substituting eq.(11) into eq.(9-a), the following expression will be finally obtained:

$$
V = \frac{1}{2} \mathbf{d}^T \mathbf{k} \, \mathbf{d} \tag{14-a}
$$

where

$$
\mathbf{k} = \iint_S \mathbf{B}^T \bar{\mathbf{D}} \, \mathbf{B} \, dS \tag{14-b}
$$

where S is area of the contact boundary surface on two adjacent elements. The complete form of the stiffness matrix [K] is already obtained in the author's previous papers.

Applying Castigliano's theorem to eq.(14-a), the following stiffness equation can be derived:

$$\mathbf{R} = \frac{\partial V}{\partial \mathbf{d}} = \mathbf{k}\,\mathbf{d} \tag{15-a}$$

where $\mathbf{k}$ is a (12 x 12) symmetric matrix given by the following equation:

$$\mathbf{k} = {}_{12}[\ k_{ij}\ ]^{12} \tag{15-b}$$

and $\mathbf{R}$ is nodal reaction vector defined by the following equation:

$$\mathbf{R}^T = \lfloor X_1, Y_1, Z_1, L_1, M_1, N_1;\ X_2, Y_2, Z_2, L_2, M_2, N_2 \rfloor \tag{15-c}$$

Spring constants k_d and k_s can be determined systematically by using the finite difference expression for strain components as follows:
On the contact surface S shown in Fig. 3 , normal and tangential stresses σ_n, τ_{ns} satisfy the following equations:

$$\sigma_n = E'\varepsilon_n^* \ , \qquad \tau_{ns} = G\,\gamma_{ns} \tag{16}$$

Strain components ε_n and γ_{ns} are approximated by the following finite defference expressions

$$\varepsilon_n = \delta_V/h \quad , \qquad \gamma_{ns} = \delta_H/h \tag{17}$$

where $h = h_1 + h_2$ is the projection of the vector G_1G_2 on $\mathbf{n}$.
On the other hand, the following relations are obtained from the definition of spring constants

$$\sigma_n = k_d\delta_V \quad , \qquad \tau_{ns} = k_s\gamma_{ns} \tag{18}$$

Therefore comparing eqs. (16) and (18), the following formulae can be derived

$$k_d = E'/h \quad , \qquad k_s = G/h \tag{19}$$

The stiffness equation defined by eq.(15) must be obtained for each contact surface if a given rigid body (I) has a number contact surfaces with other rigid bodies including the body (II), and for equilibrium of a given total system of rigid bodies, they should be summed up and the final form of the stiffness equation can be given by the following standard form of the finite element method.

$$\mathbf{K}\,\mathbf{U} = \bar{\mathbf{F}} \tag{20}$$

where

$$\mathbf{K} = \Sigma\,\mathbf{k}, \quad \mathbf{U} = \Sigma\,\mathbf{d}\ , \quad \bar{\mathbf{F}} = \Sigma\,\mathbf{f} \tag{21}$$

$\bar{\mathbf{F}}$ is a given external load vector.
However, care must be exercised in constructon of eq.(20), because in this method the centroid of each rigid body is selected as the node and therefore superposition of stiffness matrices are somewhat different from that of the standard finite element method.

In case where the body (I) is supported by other bodies through its whole boundary surface S, i.e.

$$S = S_1 + S_2 + \dots\dots + S_m$$

this model is idealization of three dimensional elastic continuum as shown in Fig. 5 in which the shape of each element can be chosen arbitrary.

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* $E' = \dfrac{(1-\nu)E}{(1+\nu)(1-2\nu)}$

The method outlined so far will be called hereafter as the Rigid Bodies-Spring Method (RBSM) or Stiffness Lumping Method (SLM). Using this method stress analysis of deformable bodies under contact will be possible in iterative way, typical application of which is analysis of the rockfill dam.

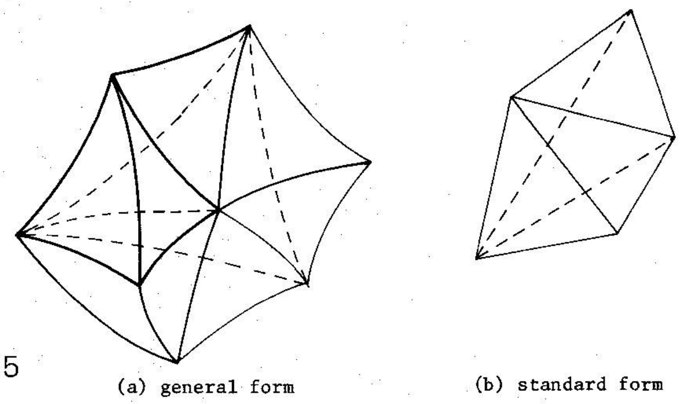


Fig. 5

2.2 Variational Basis of RBSM

RBSM model is originally proposed by basing on physical consideration and therefore it is desperately needed to establish the mathematical basis for those elements although a series of simple bending and vibration analyses were conducted. In the following section the mathematical basis of the present elements will be briefely described. For the sake of simplicity, consider a set of triangular elements in the plane stress problem as shown in Fig. 6.

Each triangular element is assumed that they are conected by a set of boundary elements $\overline{AB}$, $\overline{BC}$, $\overline{CA}$.
Infinitesimal displacement field for each element is also given by the following equation:

$$\begin{Bmatrix} U \\ V \end{Bmatrix} = \begin{bmatrix} 1 & 0 & -(y-y_G) \\ 0 & 1 & (x-x_G) \end{bmatrix} \begin{Bmatrix} u_G \\ v_G \\ \chi \end{Bmatrix} + \begin{bmatrix} x-x_0 & 0 & \frac{1}{2}(y-y_G) \\ 0 & y-y_0 & \frac{1}{2}(x-x_G) \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad (22)$$

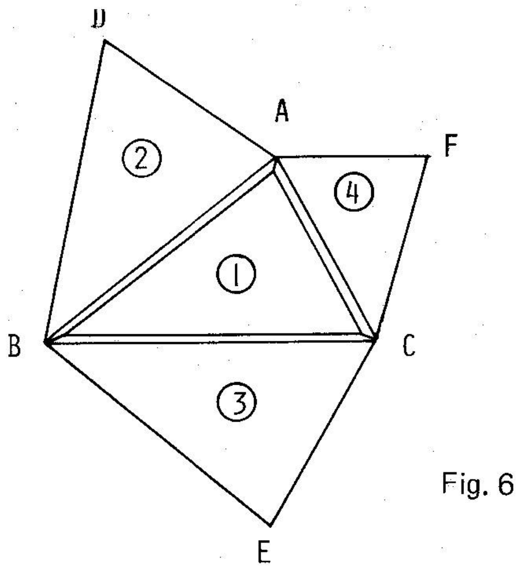


Fig. 6

Fig. 6 A Set of Triangular Plate elements for the pla stress problem

Or in compact form:

$$\mathbf{u}(X) = \mathbf{A}(X)\mathbf{d} + \mathbf{B}(X)\boldsymbol{\varepsilon} \qquad (23)$$

where

$$\mathbf{u}(X)^T = \lfloor U(x,y), \ V(x,y) \rfloor$$

$$\mathbf{A}(X) = \begin{bmatrix} 1 & 0 & -(y-y_G) \\ 0 & 1 & (x-x\ ) \end{bmatrix}$$

$$\mathbf{B}(X) = \begin{bmatrix} x-x_G & 0 & \frac{1}{2}(y-y_G) \\ 0 & y-y_G & \frac{1}{2}(x-x_G) \end{bmatrix}$$

and

$$\mathbf{d}^T = \lfloor u_G, \ v_G, \ \chi \rfloor \qquad \boldsymbol{\varepsilon}^T = \lfloor \varepsilon_x, \ \varepsilon_y, \gamma_{xy} \rfloor \qquad (24)$$

Eqs. (22) or (23) implies that the linear displacement field consists of two independent parameters, i.e. the rigid body movement of the centroid $\mathbf{d}$ and uniform strain distribution $\boldsymbol{\varepsilon}$ in the said element.

Since nodal parameters of each element can be assumed independently and continuity condition of displacements along the interface boundary should be imposed by means of Lagrangian multiplier in the variational formulation of total potential energy.
There are several methods of formulation which are called hybrid displacement method and they are discussed clearly in the texts of Professor Washizu, Zienkiewicz, Gallagher and many other's [2],[4],[5].
Here an approach originally proposed by Ping Tong [4] is adopted.
Consider the functional π_{PH} given by the following equation:

$$\pi_{PH} = \sum_{n} \Big(\iint_{S_n} \frac{1}{2} \boldsymbol{\varepsilon}^T D \boldsymbol{\varepsilon}\, dS - \iint_{S_n} \bar{b}^T \cdot u\, dS - \int_{C\sigma_n} \bar{T} u\, dS - \int_{C_n} (n\boldsymbol{\sigma})^T (u - u_b)\, dS \qquad (25)$$

$\sum_{n}$ implies summation of the total n elements.
The first term of the right hand side of eq.(25) is the strain energy to be stored in the element, the second and the third represent potential energy of external body force $\bar{b}$ and boundary load $\bar{T}$. The last term is the additional potential to be imposed on the displacement field to secure their continuity along the interface boundary lines in which $\boldsymbol{\sigma}$ is the stress matrix and it can be expressed by

$$\boldsymbol{\sigma} = D \boldsymbol{\varepsilon} \qquad (26)$$

where D is the stress-strain matrix, and u_b represents the displacement vector of the boundary elements.
Therefore the functional π_{PH} is a function of d , $\boldsymbol{\varepsilon}$ of all elements as well as d_b , $\boldsymbol{\varepsilon}_b$ of all boundary elements. As already mentioned before nodal parameters d and $\boldsymbol{\varepsilon}$ of each element is assumed independently, variations are taken first with respect to d and $\boldsymbol{\varepsilon}$ of a typical element and the following matrix equations can be derived:

(i) with respect to variation δd

$$E_d^T \boldsymbol{\varepsilon} + \bar{Q}_d = 0 \qquad (27)$$

where

$$\left.\begin{aligned} E_d^T &= \int_{C_n} A(X)^T\, n\, D\; dS \\ \bar{Q}_d &= \iint_{S_n} A(X)^T\, \bar{b}\; dS + \int_{C\sigma_n} A(X)^T\, \bar{T}\; dS \end{aligned}\right\} \qquad (28)$$

(ii) with respect to variation $\delta \boldsymbol{\varepsilon}$

$$E_d^T d - S D \boldsymbol{\varepsilon} + \bar{Q}_{\varepsilon} - G^T q = 0 \qquad (29)$$

where

$$\left.\begin{aligned} &S_0 = \iint_{S_n} dS\ ,\quad S_1 = \int_{C_n} B(X)^T n\; dS\ ,\quad S = S_0 - 2S_1 \\ &\bar{Q}_{\varepsilon} = \iint_{S_n} B(X)^T\, \bar{b}\; dS + \int_{C_n} A(X)^T\, \bar{T}\; dS \\ &G = \lfloor \int_{C_n} D\, n^T A(X)\; dS,\ \int_{C_n} D\, n^T B(X)\; dS \rfloor = \lfloor E_d\ ,\ S_1^T D \rfloor \\ &q = \lfloor d_b\ ,\ \boldsymbol{\varepsilon}_b \rfloor \end{aligned}\right\} \qquad (30)$$

Combining together eqs. (27) and (29), the following matrix equations can be derived:

$$\begin{bmatrix} 0 & E_d \\ E_d^T & -S D \end{bmatrix} \begin{Bmatrix} d \\ \boldsymbol{\varepsilon} \end{Bmatrix} = \begin{Bmatrix} -\bar{Q}_d \\ G^T q - \bar{Q}_{\varepsilon} \end{Bmatrix} \qquad (31)$$

Let's define the inverse matrix of the left hand side of eq.(31) by the following equation:

$$\begin{bmatrix} 0 & \mathbf{E}_d \\ \mathbf{E}_d^T & -S\,\mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \Phi_{dd} & \Phi_{d\varepsilon} \\ \Phi_{\varepsilon d} & \Phi_{\varepsilon\varepsilon} \end{bmatrix} \tag{32}$$

Then nodal parameters ($\mathbf{d}$, $\boldsymbol{\varepsilon}$) can be expressed as follows:

$$\left.\begin{aligned} \mathbf{d} &= -\Phi_{dd}\bar{Q}_d - \Phi_{d\varepsilon}\bar{Q}_\varepsilon + \Phi_{d\varepsilon}G^T q \\ \boldsymbol{\varepsilon} &= -\Phi_{\varepsilon d}\bar{Q}_d - \Phi_{\varepsilon\varepsilon}\bar{Q}_\varepsilon - \Phi_{\varepsilon\varepsilon}G^T q \end{aligned}\right\} \tag{33}$$

Using eq. (33), element nodal parmenters ($\mathbf{d}, \boldsymbol{\varepsilon}$) can be eliminated from the functional Π_{pH} given by eq.(25).
After some calculations, Π_{pH} can be given in the following form:

$$\Pi_{pH}(\mathbf{d}_b, \boldsymbol{\varepsilon}_b) = -\sum_{n}\left(\frac{1}{2}q^T k q - \bar{Q}\,q + C_n\right)$$

Needless to say, minimization of $\Pi_{pH}(q)$ with respect to q will yield the standard equilibrium equation of a given structure in the finite element method. Summarizing the method proposed, unknown stresses or strains in the elements can be obtained by using the principle of the minimum potential energy under a given boundary displacement $\mathbf{d}_b$ on the element interfaces.
More precisely the element stiffness matrix can be expressed in terms of boundary displacements $\mathbf{d}_b$ and strain components $\boldsymbol{\varepsilon}_b$ as shown in Fig. 7.
It may be the most reasonable approach to derive a new discrete model in which the boundary interface can be regarded as a slip line when the corresponding boundary element is plastically yielded.
There are several variations of this element model, some of which are given by Fig. 8.
As a matter of fact, the following conclusions can be drawn from careful comparative study of the RBSM and Model II:

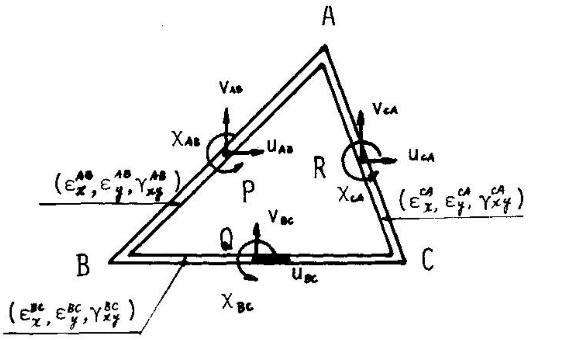


Fig. 7 A New Discrete Element with Boundary Displacements $\mathbf{d}_b$ and Strain Components $\boldsymbol{\varepsilon}_b$

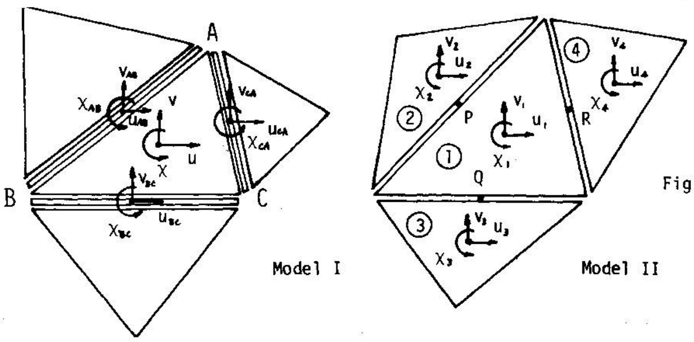


Fig. 8 Two Possible Variations of a New Discrete Element

(i) By two types of spring system whose intensities k_d and k_s , material properties of isotropic solids can be completely represented.

(ii) Convergency of elastic solutions is often considerably influenced by the mesh division. This is a serious disadvantage of the original rigid body-spring element.
Poor convergency of this element may be attributed to lacking of some cross coupling terms among elements②,③and④ in the stiffness matrix (See Fig. 8).

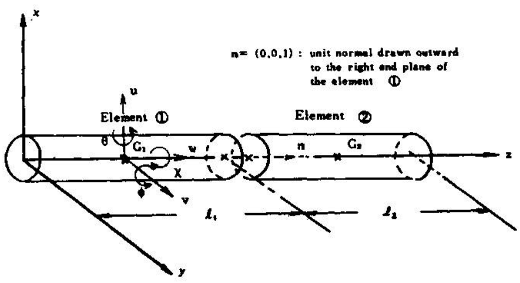
(iii) It can be expected that the Model II of Fig. 8 might give appropriate base for convergency study of the RBSM.

2.3 Stiffness Matrices of Beam and Plate Elements

A series of element matrices are now under development for practical application of the present method. In any element total number of degrees of freedom never exceeds 6 because it is assumed to be rigid.
In case of a beam element, deformation consists of axial, bending (about two principal axes) and torsional deformation, and in bending problem effect of shear deformation can be easily taken into account.
In case of plate and shell problems, membrane stiffness as well as bending stiffness can be defined by this (6 x 6) stiffness matrix.
Consideration of the shear deformation can be also made.
In what follows stiffness matrices of a straight beam element of constant cross section and a flat triangular plate element will be given.

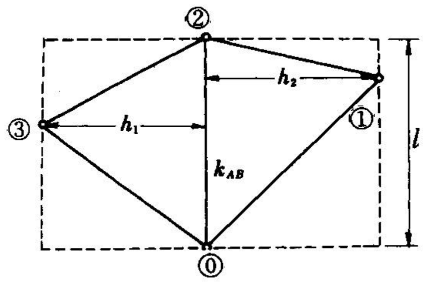
GENERAL STIFFNESS MATRIX $[k_{ij}^{12}]$ OF THREE DIMENSIONAL RIGID BODIES-SPRING ELEMENT

	u_1	v_1	w_1	θ_1	ϕ_1	χ_1	u_2	v_2	w_2	θ_2	ϕ_2	χ_2
X_1	K_1	0	0	0	$\frac{l_1}{2}K_1$	$-K_4$	$-K_1$	0	0	0	$\frac{l_2}{2}K_1$	K_4
Y_1		K_1	0	$-\frac{l_1}{2}K_1$	0	K_3	0	$-K_1$	0	$-\frac{l_2}{2}K_1$	0	$-K_3$
Z_1			K_2	K_4	$-K_6$	0	0	0	$-K_2$	$-K_7$	K_6	0
L_1	S.Y.M			$K_{16}+\frac{l_1^2}{4}K_1$	$-K_{18}$	$-\frac{l_1}{2}K_3$	0	$\frac{l_1}{2}K_1$	$-K_7$	$-K_{16}+\frac{l_1 l_2}{4}K_1$	K_{18}	$\frac{l_1}{2}K_3$
M_1					$K_{15}+\frac{l_1^2}{4}K_1$	$-\frac{l_1}{2}K_4$	$-\frac{l_1}{2}K_1$	0	K_6	K_{18}	$-K_{15}+\frac{l_1 l_2}{4}K_1$	$\frac{l_1}{2}K_4$
N_1						K_9+K_{10}	K_4	$-K_3$	0	$-\frac{l_2}{2}K_3$	$-\frac{l_2}{2}K_4$	$-K_9-K_{10}$
X_2	$K_1=\int k_n ds, K_2=\int k_s ds$						K_1	0	0	0	$-\frac{l_2}{2}K_1$	$-K_4$
Y_2	$K_3=\int k_s x ds, K_4=\int k_s y ds$							K_1	0	$\frac{l_2}{2}K_1$	0	K_3
Z_2	$K_6=\int k_n x ds, K_7=\int k_n y ds$								K_2	K_7	$-K_6$	0
L_2	$K_9=\int k_s x^2 ds, K_{10}=\int k_s y^2 ds$									$K_{16}+\frac{l_2^2}{4}K_1$	$-K_{18}$	$\frac{l_2}{2}K_3$
M_2	$K_{15}=\int k_n x^2 ds, K_{16}=\int k_n y^2 ds, K_{18}=\int k_n xy ds$										$K_{15}+\frac{l_2^2}{4}K_1$	$\frac{l_2}{2}K_4$
N_2												K_9+K_{10}

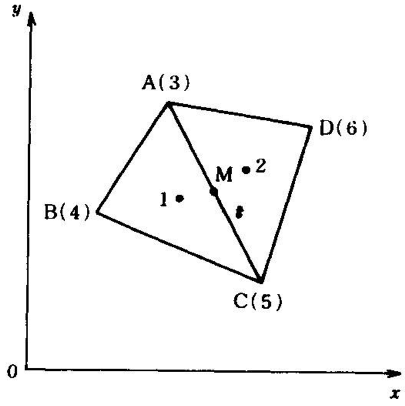


Stiffness matrix of a new plate bending element $\left(\times \frac{k_{AB}}{\Delta_{10}\Delta_{20}}\right)$

	W_0	W_1	W_2	W_3
Z_0	$\frac{\Delta_{20}}{\Delta_{10}}(y_{12}^2+x_{12}^2)+\frac{\Delta_{10}}{\Delta_{20}}(y_{23}^2+x_{23}^2)$ $-2(y_{12}y_{23}+x_{12}x_{23})$	$\frac{\Delta_{20}}{\Delta_{10}}(y_{12}y_{20}+x_{12}x_{20})$ $-(y_{23}y_{20}+x_{23}x_{20})$	$\frac{\Delta_{20}}{\Delta_{10}}(y_{12}y_{01}+x_{12}x_{01})$ $+\frac{\Delta_{10}}{\Delta_{20}}(y_{23}y_{30}+x_{23}x_{30})$ $-(y_{12}y_{30}+y_{23}y_{01}+x_{12}x_{30}+x_{23}x_{01})$	$\frac{\Delta_{10}}{\Delta_{20}}(y_{23}y_{02}+x_{23}x_{02})$ $-(y_{12}y_{02}+x_{12}x_{02})$
Z_1		$\frac{\Delta_{20}}{\Delta_{10}}(y_{20}^2+x_{20}^2)$	$\frac{\Delta_{20}}{\Delta_{10}}(y_{20}y_{01}+x_{20}x_{01})$ $-(y_{20}y_{30}+x_{20}x_{30})$	$y_{20}^2+x_{20}^2$
Z_2			$\frac{\Delta_{20}}{\Delta_{10}}(y_{01}^2+x_{01}^2)+\frac{\Delta_{10}}{\Delta_{20}}$ $(y_{30}^2+x_{30}^2)-2(y_{01}y_{30}+x_{01}x_{30})$	$\frac{\Delta_{10}}{\Delta_{20}}(y_{30}y_{02}+x_{30}x_{02})$ $-(y_{01}y_{02}+x_{01}x_{02})$
Z_3	SYM.			$\frac{\Delta_{10}}{\Delta_{20}}(y_{30}^2+x_{30}^2)$



	u_1	v_1	θ_1	u_2	v_2	θ_2
X_1	$k_d y_{53}^2+k_s x_{53}^2$		$2\Delta_{11}=x_{53}(x_{31}+x_{41})+y_{53}(y_{31}+y_{41})$ $2\Delta_{12}=x_{53}(y_{32}+y_{62})-y_{53}(x_{32}+x_{62})$ $2\Delta_{21}=-x_{53}(y_{31}+y_{41})+y_{53}(x_{31}+x_{41})$ $2\Delta_{22}=-x_{53}(x_{32}+x_{62})-y_{53}(y_{32}+y_{62})$			
Y_1	$-(k_d-k_s)x_{53}y_{53}$	$k_d x_{53}^2+k_s y_{53}^2$				
M_1	$k_d y_{53}\Delta_{11}-k_s x_{53}\Delta_{21}$	$-(k_d x_{53}\Delta_{11}+k_s y_{53}\Delta_{21})$	$k_d\Delta_{11}^2+k_s\Delta_{21}^2+k_r l_{35}^2$	SYM.		
X_2	$-(k_d y_{53}^2+k_s x_{53}^2)$	$(k_d-k_s)x_{53}y_{53}$	$-(k_d y_{53}\Delta_{11}-k_s x_{53}\Delta_{21})$	$k_d y_{53}^2+k_s x_{53}^2$		
Y_2	$(k_d-k_s)x_{53}y_{53}$	$-(k_d x_{53}^2+k_s y_{53}^2)$	$k_d x_{53}\Delta_{11}+k_s y_{53}\Delta_{21}$	$-(k_d-k_s)x_{53}y_{53}$	$k_d x_{53}^2+k_s y_{53}^2$	
M_2	$k_d y_{53}\Delta_{22}-k_s x_{53}\Delta_{12}$	$-(k_d x_{53}\Delta_{22}+k_s y_{53}\Delta_{12})$	$k_d\Delta_{11}\Delta_{22}+k_s\Delta_{21}\Delta_{12}-k_r l_{35}^2$	$-(k_d y_{53}\Delta_{22}-k_s x_{53}\Delta_{12})$	$k_d x_{53}\Delta_{22}+k_s y_{53}\Delta_{12}$	$k_d\Delta_{22}^2+k_s\Delta_{12}^2+k_r l_{35}^2$



3. THEORETICAL BASIS OF NONLINEAR ANALYSIS [7],[8],[9],[13]

In general nonlinear structural problems are coupled problems of large deformation, inelasticity and crack, and they may be solved by using the incremental procedure. In what follows, essentials of solution procedure of nonlinear structural problems will be given.

3.1 Geometrical nonlinear problem

In case of finite displacement, assumption of the infinitesimal angular displacement is no longer valid and eq.(1) should be replaced by the following equation:

$$\mathbf{u}' = \mathbf{u}_G + (\mathbf{T} - \mathbf{I})(\mathbf{r} - \mathbf{r}_G) \tag{35}$$

$\mathbf{T}$ is a coordinate transformation matrix of local coordinates attached to the centroid between before and after deformation as follows:

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{36}$$

or

$$\mathbf{r}' = \mathbf{T}\,\mathbf{r}$$

and $\mathbf{I}$ is an unit matrix.
An unit normal $\mathbf{n}$ drawn outward at P of the element (I) before deformation may be subjected to the similar transformation as follows:

$$\mathbf{n}' = \mathbf{T}\,\mathbf{n} \tag{37}$$

where

$$\mathbf{n} = (l, m, n), \quad \mathbf{n}' = (l', m', n')$$

Consequently components (δ_d , δ_s) of the relative displacement $\overrightarrow{P'P''}$ will be given by the following equation:

$$\left.\begin{aligned} \delta_d &= (\overrightarrow{P'P''}, \mathbf{n}') = l'(u_{II} - u_I) + m'(v_{II} - v_I) + n'(w_{II} - w_I) \\ \delta_s^2 &= (\overrightarrow{P'P''} \times \mathbf{n}')^2 = \{m'(w_{II} - w_I) - n'(v_{II} - v_I)\}^2 \\ &\quad + \{n'(u_{II} - u_I) - l'(w_{II} - w_I)\}^2 + \{l'(v_{II} - v_I) - m'(u_{II} - u_I)\}^2 \end{aligned}\right\} \tag{38}$$

Knowing the strain energy V, and applying the principle of virtual work statical equilibrium equation can be derived where effect of finite rotation of elements is considered. From this equation the following standard incremental form of stiffness equation can be derived after some calculation.

$$(\mathbf{K} + \mathbf{K}_0 + \mathbf{K}_G)\,\mathbf{d}^* = \bar{\mathbf{F}}^* - \mathbf{F}_r \tag{39}$$

where $\mathbf{K}_0$ is the initial strain matrix, $\mathbf{K}_G$ * the geometrical stiffness matrix, $\mathbf{d}^*$, $\bar{\mathbf{F}}^*$ are increments of the displacement and exteral loads respectively, and $\mathbf{F}_r$ is an unbalance force due to manipulation error in previous stage of loading. Detail of the derivation is given in the previous papers of the author.

3.2 Material Nonlinearity Problems

For simplicity, displacement of a given body is assumed to be infinitesimal, and therefore the problem will be reduced to integration of the following stiffness equation based on the well-established incremental procedure.

$$K\,d^* = \bar{F}^* - \bar{F} \tag{40}$$

For integration of eq.(40), yield or failure criterion of a given material should be introduced.
For this purpose the elastic strain energy density of the spring system V_0 is considered, and it is given by the following formula.

$$V_0 = \frac{1}{2}\left(k_n \delta_n^2 + k_s \delta_s^2\right) = \frac{1}{2}\left(\frac{\sigma_n^2}{k_n} + \frac{\tau_{ns}^2}{k_s}\right) \tag{41}$$

It can be concluded from eq.(41) that if the maximum strain energy criterion is adopted, the material may fail if $V_0 = \sigma_y^2/2E$. According to this theory it will be seen that yielding will occur if $\tau_{ns} = \sigma_y/\sqrt{2(1+\nu)}$, while brittle failure will initiate if $\sigma_n = \sqrt{(1+\nu)(1-2\nu)}\sigma_y/(1-\nu)$.
As alternative failure criteria the maximum shearing stress theory may be adopted for ductile materials, while the maximum stress theory can be considered for brittle materials.
To avoid unnecessary confusion in further development, it is assumed that material is ductile and ideal plastic.
Solution of eq.(40) based on this assumption will give generalized solution of limit analysis which is well established in framed structures. A series of such solutions have been given in previous papers of the authors.
The present method of analysis on the material nonlinear problem can be generalized by replacing the spring system connecting rigid elements by the spring-dashpot system as shown in the Fig. 9
Using such rigid bodies-spring-dashpot system, static and dynamic analysis of viscoelastic-plastic problem under thermal loading may be possible.

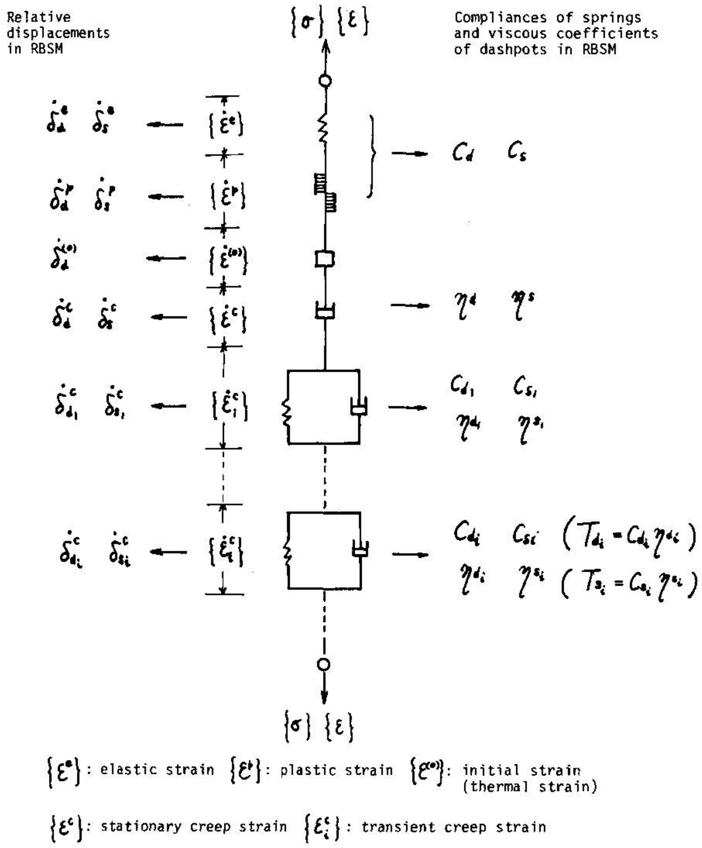


Fig. 9 Mechanical Model of Viscoelastic-Plastic Materials

3.3 Consideration of Crack Initiation and Growth in the Present Anaysis

In analysis of the material nonlinear problems described in the last paragraph yield criterion is applied pointwisely on the contact boundary surface. Therefore in component calculation of the stiffness matrix for each contact surface, appropriate scheme of numerical integration should be adopted.
More precisely, for example, k_{11} of the general 3D stiffness matrix is given by

$$k_{11} = \iint \left\{ k_n \ell^2 + k_s(1-\ell^2)\right\} dS$$

And therefore if the boundary surface is curved or k_n , k_s depend on stresses and strain, calculation of k_{11} should be made, for example, by using Gauss' integration scheme, Using such integration scheme it is possible to pursue gradual development of plastic hinge lines, slip lines or slip surfaces on the contact boundary, and the ultimate load can be calculated. In real structures, however, it is usual to consider that initiation and propagation of crack may reduce substantially the ultimate load. At the present moment, the criterion for crack initiation and propagation is not well established and therefore the

following simple criterion is adopted for the time being.
Crack intiation and propagation may take place when the shearing strain γ exceeds γ_b which may be equivalent to the concept of *COD*. It is not difficult to incorporate this criterion with the yield criterion in analysis of material nonlinear problems. As a matter of fact, crack analysis of two dimensional notched plates were conducted by the present authors and reasonable results were obtained. Effect of large scale yielding, however, was not considered in this analysis and therefore more refined analysis will be planned in near future by taking into account of such an effect.

4. SOME EXAMPLES OF COLLAPSE LOAD ANALYSIS [6],[10],[11],[13]

To show validity of the present new elements, a series of numerical anaysis has been conducted and most of the results obtained were reported in the conference proceedings or engineering journals.
Therefore only some new results will be shown here without explanation.

(I)
- (a) collapse analysis of square concrete slabs.
- (b) two dimensional punch problem
- (c) three dimensional elasto-plastic anaysis of a through crack problem.

(II) (d) shake down analysis of a simply supported square plate subjected to variable transverse loads.

(III) (e) collapse analysis of cylindrical shell roofs simply-supported on four edges and subjected to external radial pressure.

(IV) (f) dynamic collapse of automobile front structures.

5. CONCLUSION

Outline is briefly explained on a new discrete method of analysis which has been proposed by the present author. This method may be suitable for analysis of highly nonlinear problems where plasticity, large deformation and crack growth are coupled. Therefore broad application may be expected in future to analysis and design of the reinforced concrete structures where punching shear crack growth, creep etc. are important design prarameters. The followings are conclusion so far obtained from a series of numerical analysis.

(i) Stiffness of a given body is lumped on the contact sufaces of neighbouring rigid elements and yielding or failure is assumed to occur only on these contact surfaces. Consequently the analysis of material nonlinear problems becomes much simpler than that of conventional finite element method.

(ii) Concept of node superposition in the conventional finite element analysis is completely discarded in the present analysis and *slip* due to plastic deformation or frictional force on the contact surface can be easily represented in this method.

(iii) Since the lower order shape function is employed for element stiffness formulation, computing time for stiffness calculation will be considerably reduced to compare with the conventional finite element method.

(iv) Variational formularion of the present method is now under way. It is expected in near future to give rational basis for this discrete analysis.

(v) Although it can be concluded by a series of test analyses that the present method may be very powerful for the collapse load analysis, accumlation of results of numerical analysis of more realistic structures should be necessary for verification of the method.

ACKNOWLEDGEMENTS

The author would like to express his sincere appreciation to Dr. Kazuo Kondou, Hiroshima University and Yutaka Toi, a graduate student at University of Tokyo for their sacrificing effort shown in the course of this study. He also wishes to express his thanks to Miss Sueko Suzuki and Mr. Tetsuo Aso for their help in typing manuscript and drawing of figures and tables.

REFERENCES

1. DRUCKER, D.C., PRAGER, W., and GREENBERG, H.J.: Extended Limit Design Theorems for Continuous Media. Quarterly of Applied Mathematics, Vol.9, No.4, pp 381-389, 1952
2. WASHIZU, K.: Variational Methods in Elasticity and Plasticity. Pergamon Press, 1968
3. ASME: Limit Analysis Using Finite Elements, 1976
4. ZIENKIEWICZ, O.C.: The Finite Element Method. Third Edition, McGraw-Hill, 1977
5. GALLAGHER, R.H.: Finite Element Analysis: Fundamentals. Prentice-Hall, 1975
6. KAWAI, T.: New Discrete Models and Their Application to Seismic Response Analysis of Structues. to be Published in International Journal of Nuclear Engineering and Design.
7. KAWAI, T.: A New Discrete Analysis of Nonlinear Solid Mechanics Problems Involving Stability, Plasticity and Crack. Symposium on Applications of Computer Methods in Engineering, Los Angeles California, August 23-26, 1977
8. KAWAI T.: New Discrete Structural Models and Generalization of the Method of Limit Analysis. International Conference on Finite Elements in Nonlinear Solid and Structural Mechanics, Geilo, Norway, August 29-September 1, 1977
9. KAWAI. T.: A New Discrete Model for Analysis of Solid Mechanics Problems. International Conference on Numerical Methods in Fracture Mechanics, Swansea, U.K., January, 9-13, 1978
10. KAWAI, T, and TOI. Y.: Crush Analysis of Engineering Structures. HOPE International JSME Symposium, Tokyo JAPAN, October 30 - November 5, 1977
11. KAWAI. T, and KONDOU, K.: Shakedown Analysis of Engineering Structures by Using New Discrete Elements. HOPE International JSME Symposium, Tokyo JAPAN October 30 - November 5, 1977
12. KAWAI. T.: New Element Models in Discrete Structural Analysis. Journal of the Society of Naval Architects of Japan, Vol. 141, May, 1977
13. TOI, Y. and KAWAI. T.: A New Discrete Analysis on Dynamic Collapse of Structures. Journal of the Society of Naval Architects of Japan, Vol. 143, 1978

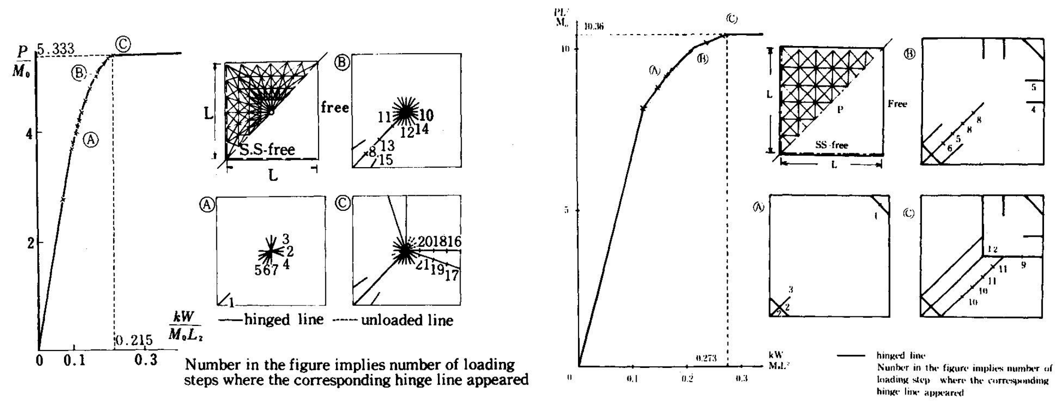
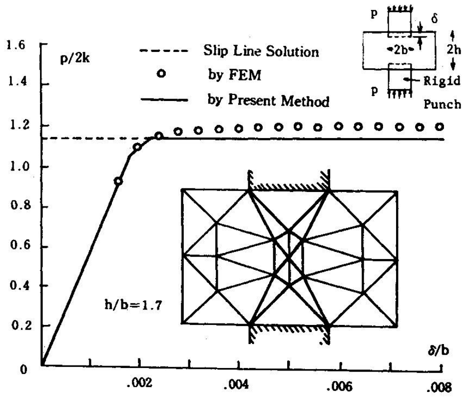
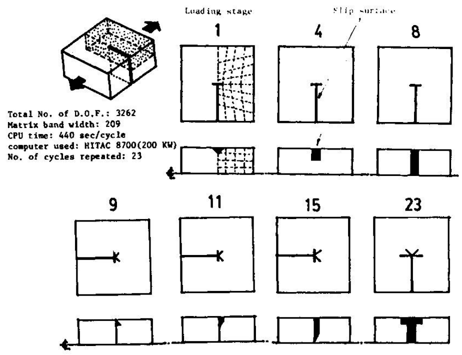


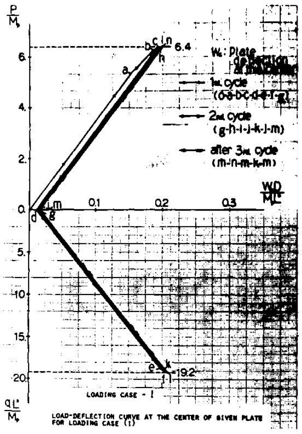
PLATE BENDING COLLAPSE



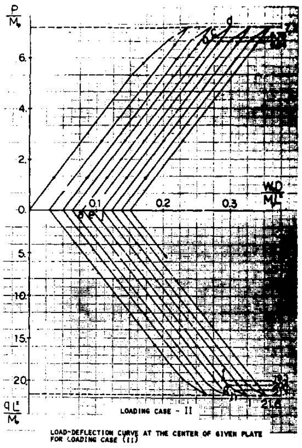
20 PUNCH PROBLEM



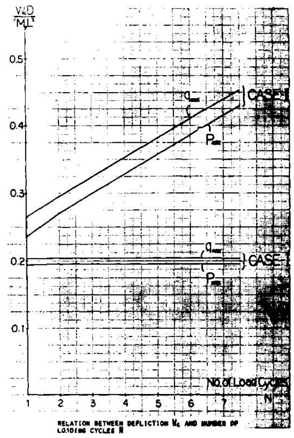
3D CRACK ANALYSIS



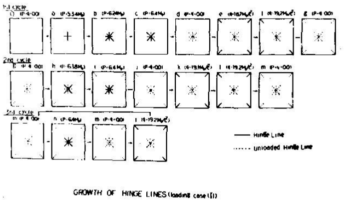
LOAD-DEFLECTION CURVE AT THE CENTER OF GIVEN PLATE FOR LOADING CASE (I)



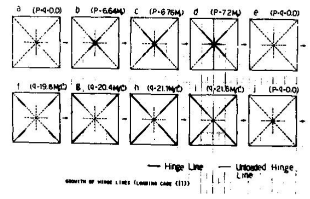
LOAD-DEFLECTION CURVE AT THE CENTER OF GIVEN PLATE FOR LOADING CASE (II)



RELATION BETWEEN DEFLICTION W_c AND NUMBER OF LOADING CYCLES N

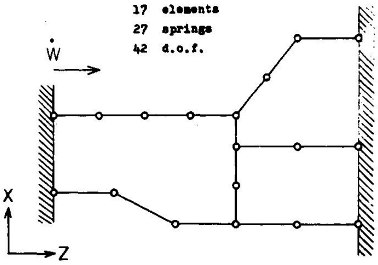


GROWTH OF HINGE LINES (loading case (I))

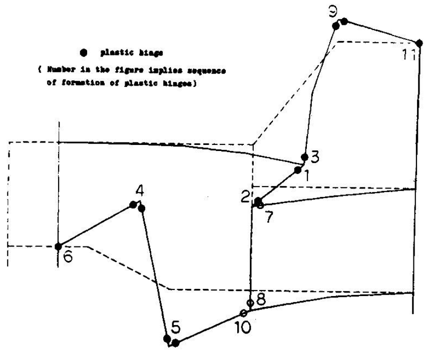


GROWTH OF HINGE LINES (LOADING CASE (II))

SHAKE DOWN ANALYSIS
OF A SIMPLY SUPPORTED
SQUARE PLATE

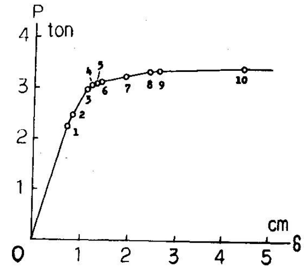


FINITE ELEMENT MODEL OF AUTOMOBILE BODY STRUCTURE



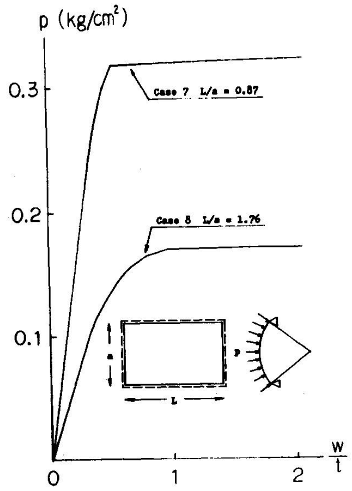
(c) final deflected shape

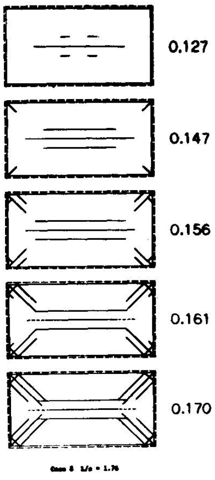
DYNAMIC COLLAPSE OF AUTOMOBILE BODY STRUCTURE

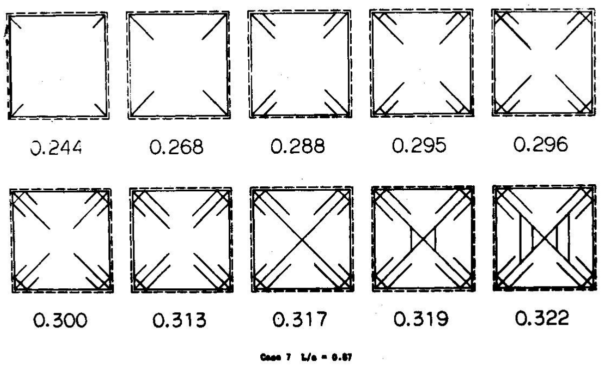


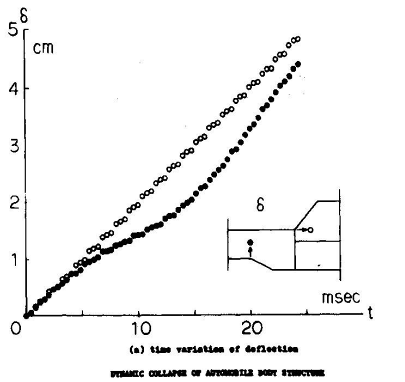
(b) load-deflection curve

STATIC COLLAPSE OF AUTOMOBILE BODY STRUCTURE
(Number in the figure implies number of loading steps where the corresponding hinge appeared.)









CRASH ANALYSIS OF AN AUTOMOBILE FRONT BODY

COLLAPSE ANALYSIS OF A SHELL ROOF STRUCTURE