Constitutive equations for concrete

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Equations de base du béton

Stoffgleichungen für Beton

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SUMMARY
A summary of the current state-of-the-art in the mathematical modeling of the mechanical behaviour of concrete is presented. A general discussion of some experimental facts is followed by a detailed description of the five basic types of models: 1. uniaxial and equivalent uniaxial models; 2. linear elastic and fracture models; 3. nonlinear elastic and variable moduli models; 4. elastic-perfectly plastic and fracture models; and 5. elastic-strain hardening plastic and fracture models. Their relative merits and limitations are discussed and some of the interrelationship between various models are demonstrated. Directions of further research are indicated.

RESUME
Le rapport présente l'état actuel des connaissances sur les modèles mathématiques exprimant le comportement mécanique du béton. Quelques remarques générales sont faites à partir de quelques faits expérimentaux. Une description détaillée des cinq types fondamentaux de modèle est donnée:
1. modèle uniaxial et quasi-uniaxial; 2. modèle linéaire élastique et de rupture; 3. modèle non-linéaire élastique; modèle élastique-parfaitement plastique et de rupture; 5. modèle linéaire élastique-écrouissable et de rupture. Leur valeur relative et leurs limites sont discutées et quelques relations entre les différents modèles sont données. Des directions sont indiquées pour des recherches futures.

ZUSAMMENFASSUNG
Es wird ein Überblick gegeben über die heutigen Kenntnisse an mathematischen Modellen, die das mechanische Verhalten von Beton beschreiben. Nach einer Erörterung einiger Aspekte des in Versuchen beobachteten Verhaltens werden fünf Arten von Modellen beschrieben: 1. einachsiale und quasieinachsiale Modelle; 2. linear elastisches Verhalten bis zum Bruch; 3. nichtlinear elastisches Verhalten; 4. elastisch-ideal plastisches Verhalten bis zum Bruch; 5. elastisch-verfestigend plastisches Verhalten bis zum Bruch. Die Vorzüge und Grenzen der einzelnen Modelle werden einander gegenübergestellt und einige Beziehungen zwischen den verschiedenen Modellen werden dargestellt. Richtungen für die weitere Forschungstätigkeit werden angedeutet.

1. INTRODUCTION

A complete *progressive failure analysis* of reinforced concrete structures under static and seismic loading conditions requires the consideration of loading and ground motion input, generalized material behavior, and analytical procedure. The terms "loading" and "ground motion" refer to the specific loadings and motions that should be considered in design and analysis of reinforced concrete structures. This includes the magnitude and direction of loading, the duration and frequency content of the motion, and the acceleration, velocity and displacement parameters. This is beyond the scope of the present paper.

The term "generalized material behavior" refers to multi-dimensional stress-strain relations which adequately describe the basic characteristics of reinforced concrete materials subjected to monotonical and cyclic loading. These are called *constitutive equations*. These are the most fundamental relations required for any analysis of reinforced concrete structures. This paper sets out those which have been most widely and successfully used in analytical and numerical approaches to reinforced concrete problems. To discuss the mathematical modeling.of non-linear reinforced concrete behavior, three areas must be examined: the behavior of concrete, the response of steel reinforcement and the bond-slip phenomenon between steel and concrete.

Since steel reinforcement is comparatively thin, it is generally assumed capable of transmitting axial force only and thus, a uniaxial stress-strain relationship is sufficient for general use. The most commonly used plasticity model for steel reinforcement is the linearly elastic-perfectly plastic type which ignores the Bauschinger effect but allows elastic unloading. As for concrete, however, a knowledge of multiaxial stress-strain behavior is required. This is still far from complete, although a large variety of models have been proposed in recent years. Herein, a critical review of these constitutive equations for concrete is given in the forthcoming. For the present, we limit our discussion to the cases of short-time loading for which the creep effects may be neglected. Although the analysis and design of reinforced concrete structures require not only each relationship between stresses and strains of steel and concrete but also the bond-slip relation between steel and concrete, only the constitutive relations for plain concrete will be reviewed and evaluated here. Once the stress-strain relation of each material is available and a bond-slip relation is assumed, steel reinforcements can then be placed in proper positions in concrete elements and constitutive equations for the composite response of reinforced concrete element can be readily formulated. The mechanics of bond-slip phenomenon between steel and concrete will not be considered here. In most practical applications, a perfect bond is generally assumed.

The term "analytical procedure" refers to the mathematical and numerical aspects of calculation used to obtain solutions. In recent years there has been a growing interest in the application of the *finite element* procedure to the analysis of reinforced concrete problems. The present state of development of computer programs and finite-element method are limited to linear, non-linear, or plasticity material models for two-dimensional analysis of reinforced concrete structures under static and seismic loading conditions. The relative merits and limitations of different constitutive models for concrete will be reviewed here with particular emphasis on their use inthe numerical analysis of reinforced concrete structures.

2. EXPERIMENTAL DATA

We begin by examining some typical experimental data for concrete under uniaxial, biaxial and triaxial states of stress. These data, which are essential in the generalized development of mathematical modeling of concrete, serve the following two major purposes: (1) to give a guidance on the proper type of material behavior to be developed in the mathematical modeling; and (2) to provide data for the determination of the various material constants which appear in the mathematical models.
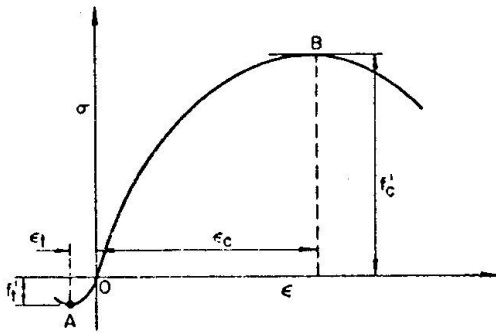
Fig. 1 Typical Stress-Strain Curve
        for Concrete Under Uniaxial
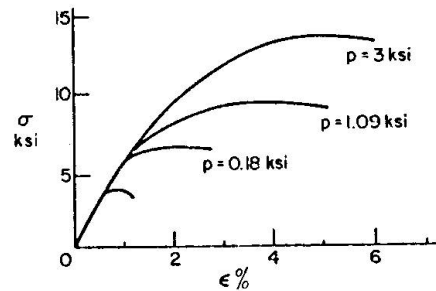   Tension and Compression

Fig. 2 Typical Stress-Strain Curve for
        Concrete Under Compression and
        Lateral Pressure (1 ksi =
        6.89 NM/m^2)(Richart et al. 1928,
        Ref. 1)

Under ordinary experience, concrete is brittle in tension and limitedly deformable in compression. A typical uniaxial stress-strain curve for concrete is shown in Fig. 1. The maximum tensile stress f_t' is about 8-12 times *less* than the maximum compressive stress f_c'. Similar proportions occur between corresponding strains ε_t and ε_c; whereas ε_c takes on values of about 0.25%, ε_t does not usually exceed 0.015%. The crushing of concrete is usually followed by an unstable portion of strain softening with a crushing strain ε_u of about 0.35%. The brittleness of the tensile failure does not necessarily mean that the behavior is linearly elastic up to the fracture, as shown in Fig. 1. A characteristic of tensile type of failure is that the failure itself is abrupt with the development of one major crack, while in the case of compressive type of failure, more major cracks develop. Crack formation at failure occurs in the direction perpendicular to the direction of the greatest principal strain or stress.

Ordinary experience simply is not adequate to predict the behavior of concrete under triaxial compression state of stresses. The stress-strain curves of Fig. 2 illustrate the change in behavior of a concrete as it deforms under increasingly higher confined pressure. Hydrostatic pressure is seen largely to increase both maximum stress and maximum strain during compression, and the unstable strain softening portion gradually vanishes for increasing pressures.

Considerable experimental data are available regarding the strength, deformational characteristics and microcracking behavior of concrete subjected to biaxial state of stresses. Fig. 3 shows typical experimental stress-strain curves for concrete under biaxial compression (Fig. 3a), combined tension and compression (Fig. 3b) and biaxial tension (Fig. 3c).

First, it is seen that the maximum compressive strength increases for the biaxial compression state. A maximum strength increase of approximately 25% is achieved

at a stress ratio of σ_2/σ_1= 0.5 and is reduced to about 16% at an equal biaxial compression state $(\sigma_2/\sigma_1 = 1)$. Under biaxial compression-tension, the compressive strength decreases almost linearly as the applied tensile stress is increased. Under biaxial tension, the strength is almost the same as that of uniaxial tensile strength (see Fig. 4).
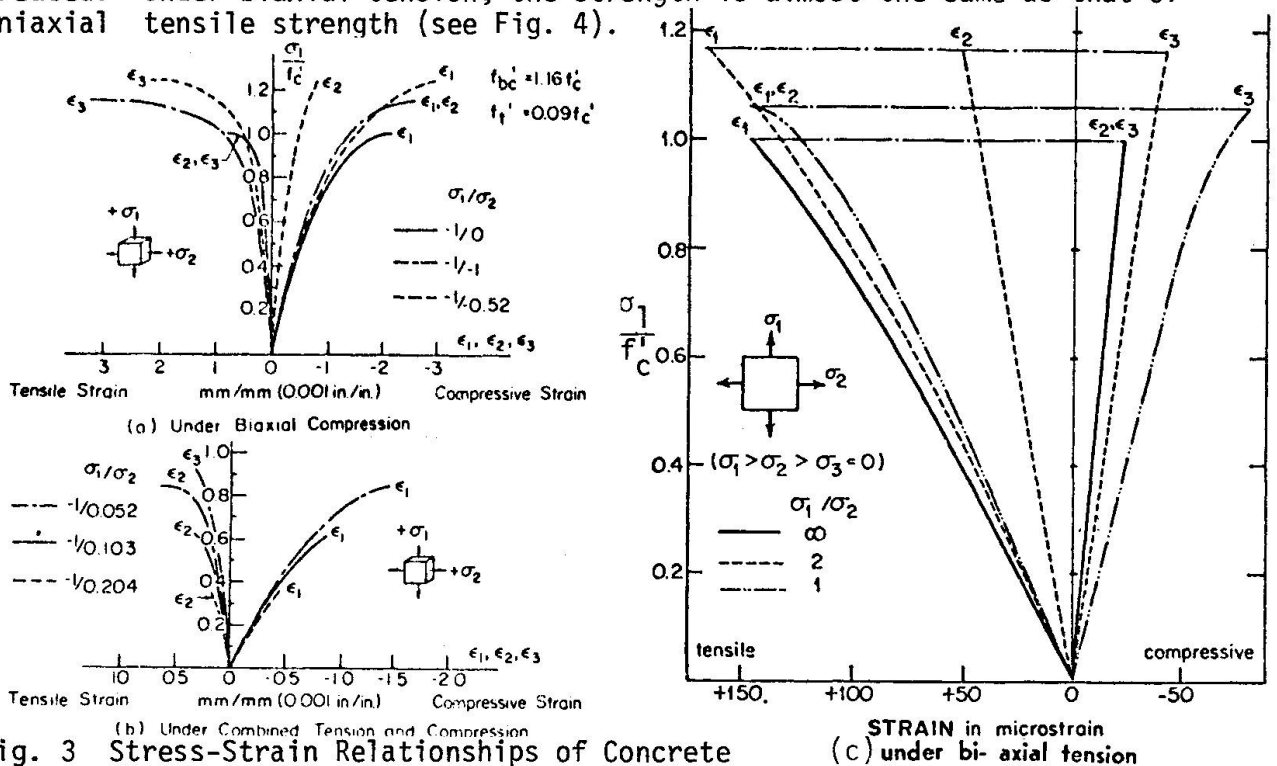

Fig. 3   Stress-Strain Relationships of Concrete

(c) under bi-axial tension

Second, concrete "ductility" under biaxial state of stresses has different values depending on the nature of stress states: compressive type or tensile type. For uniaxial and biaxial compression type (Fig. 3a), the average value of the maximum strain is about 2500 microstrain, and the average value of the maximum tensile strain varies approximately from 1000 to 3000 microstrain. The tensile ductility is greater under biaxial compression state than in uniaxial compression (Fig. 3a). In biaxial compression-tension (Fig. 3b), the magnitude at failure of both the principal compressive strain and the principal tensile strain decrease as the tensile stress increases. In uniaxial, axial and biaxial tension (Fig. 3c), the average value of the maximum principal tensile strain is about 150 microstrain. Note that the existence of a descending branch under biaxial stress states has not generally been observed. However, by using a constant rate of straining, Nelissen [3] has been able to achieve the descending portions of stress strain curves in biaxial loading test.

Third, failure of concrete occurs by tensile splitting with the fractured surface orthogonal to the direction of the maximum tensile strain. Tensile strains are found to be of crucial importance in the failure criterion and failure mechanism of concrete.

Fourth, as the failure point is approached, an increase in volume occurs as the compressive stress continues to increase, as shown in Fig. 5. This inelastic volume increase is called "dilatancy", and is usually attributed to progressive growth of major microcracks of concrete.

There are several approaches for defining this complicated stress-strain behavior of concrete under various stress states. They can be divided in three main groups: (1) representation of given stress-strain curves by using curve-fitting methods, interpolation or mathematical functions, (2) linear and nonlinear

elasticity theories, and (3) perfect and work-hardening plasticity theories. We shall describe some of the main constitutive models used in the analysis of reinforced concrete structures.
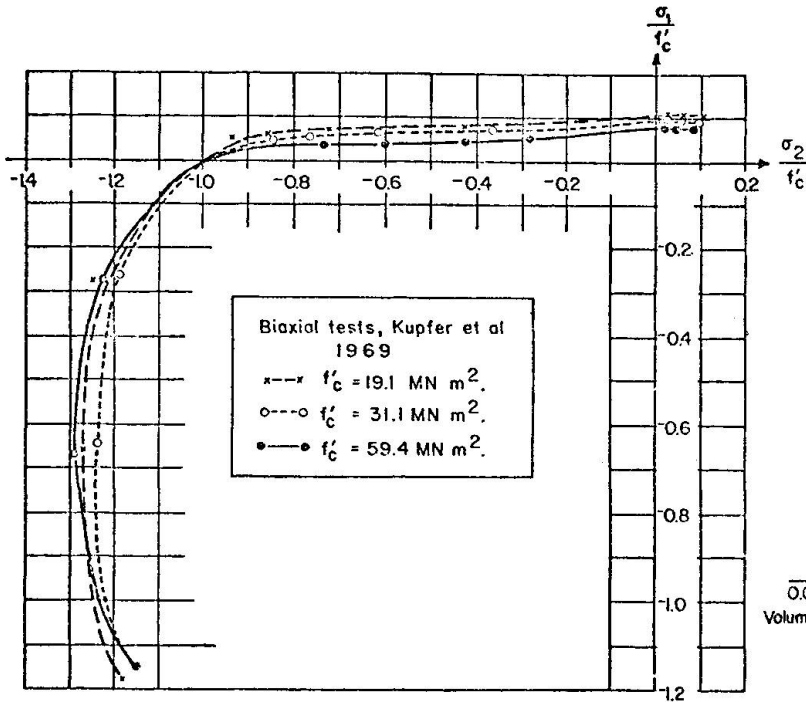

Fig. 4 Biaxial Strength Envelope of Concrete.
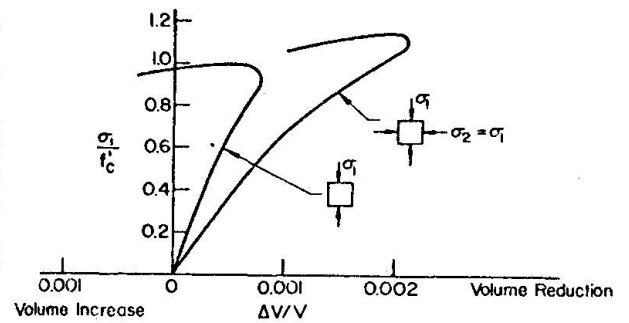
Fig. 5 Typical Stress-Strain Curve for Concrete Volume Change Under Biaxial Compression.

3. UNIAXIAL AND EQUIVALENT UNIAXIAL MODELS

The widely used function for simulation of stress-strain curves of concrete under uniaxial as well as biaxial state of stresses has the form [5]

$$\sigma = \frac{a\varepsilon}{1 + \left(\frac{a\varepsilon_p}{\sigma_p} - 2\right)\left(\frac{\varepsilon}{\varepsilon_p}\right) + \left(\frac{\varepsilon}{\varepsilon_p}\right)^2} \tag{1}$$

where σ, ε = stress and strain in the principal stress direction,
   σ_p, ε_p = experimentally determined values of maximum principal stress and corresponding strain, and
   a = experimentally determined coefficient which represents the initial tangent modulus.

Equation (1) has a horizontal tangent modulus at the point of peak stress and corresponding strain $(\sigma_p, \varepsilon_p)$. For a uniaxial stress state, the points of peak stress are either (f'_c, ε_c) or (f'_t, ε_t) as shown in Fig. 1 and the uniaxial initial elastic modulus is $a = E$.

For biaxial stress states, the maximum stress point, σ_p, is the value which can be determined from the biaxial strength envelop such as the one shown in Fig. 4, and the corresponding value of maximum strain, ε_p, in the major direction may be fixed at about 2500 micro-compressive-strain for uniaxial and biaxial compression state (Fig. 3a), or at about 150 micro-tensile-strain for uniaxial and

biaxial tension state (Fig. 3c).   Since under biaxial compression-tension
state, the compressive strength, σ_p, decreases almost linearly as the tensile
stress is increased, the corresponding decrease in compressive strain, ε_p,
can be estimated by proportioning its value to the tensile stress increase
(Fig. 3b). Note that the value of ε_p in the minor direction will vary. Various
curve-fitting expressions for the biaxial ultimate strength envelope and corres-
ponding major and minor strains can be found in References [4-7].

The biaxial compressive stress-strain curves of Fig. 3(a) show increasing *initial*
stiffnesses for increasing values of lateral compression and this is caused main-
ly by the effect of Poisson's ratio. Thus the strain measured in the same direc-
tion as the stress includes the contribution from lateral direction. Similarly,
Poisson's ratio has a decreasing effect on the *initial* stiffnesses of the biaxial
tensile stress-strain curves (Fig. 3c). For a linearly elastic, isotropic
material, the biaxial stress-strain relation can be expressed as

$$\sigma = \frac{E\varepsilon}{1 - \nu\alpha} \tag{2}$$

where α = ratio of the principal stress in the orthogonal direction to the
            principal stress in the direction considered,
        E = initial tangent modulus in uniaxial loading, and
        ν = Poisson's ratio in uniaxial loading.

As an approximation, the *effective initial modulus*, $E/(1-\nu\alpha)$, due solely to the
Poisson's effect can be used as the initial modulus, $a = E/(1-\nu\alpha)$, in Eq. (1)
which describes the nonlinear biaxial stress-strain relationship of concrete.
Since this biaxial stress-strain equation passes through the point of peak stress
and strain $(\sigma_p, \varepsilon_p)$, which accounts for mainly the microcrack confinement effect
in the presence of biaxial stress, the tangent modulus which is the slope at any
point of the biaxial stress-strain curve, $E_t = d\sigma/d\varepsilon$, include both the microcrack
confinement effect as well as the Poisson's effect.

The basic concept of this model is to treat the biaxial stress-strain behavior
of concrete as an *equivalent uniaxial* relation. According to this approach,
the strain increment in each principal direction is evaluated solely by the
principal stress increment in the same direction and the corresponding tangent
stiffness, which is a function of the prinxipal stress ratio, α, accounts for
all the biaxial effects. In this, Poisson's ratio is assumed to be a constant
which is near 0.2. According to experimental evidence, this is a fairly rea-
sonable approximation up to about 80% of peak stress, but after this point it
progressively deviates (see Fig. 5)

The main advantages of this model are that it is simple and that the required
data are either readily obtainable from uniaxial tests on the concrete or readily
available from various biaxial tests reported in literature. The model is mainly
applicable to planar problems such as beams, panels and thin shells where the
stress is predominantly biaxial. However, it is immediately apparent from Fig.
5 that there is an abrupt volume increase near peak stress under biaxial compres-
sion and from Fig. 2 that there is a marked influence of hydrostatic pressure on
the behavior of concrete under triaxial stress states. These behaviors can not
be accounted for by the present equivalent uniaxial approach. Thus the model
described will have little validity in three-dimensional situations.

4. LINEAR ELASTIC-FRACTURE MODELS

One of the most important characteristics of concrete is its low tensile strength which results in tensile cracking at very low stress compared to compressive stresses. The tensile cracking reduces the stiffness of the concrete and is usually the major contributor to the nonlinear behavior of some reinforced concrete structures such as panels and shells where the stress is predominantly biaxial tension-compression type. For these structures, accurate modeling of cracking behavior of concrete is undoubtedly the most important factor and linear elastic-fracture models have been developed and used by many investigators to study the nonlinear response of reinforced concrete beams, panels and shells [see, for example, Refs. 8 and 9].

In the following, a linear elastic-fracture model under general stress states is developed in three parts: elasticity, fracture criteria and fractured concrete.

4.1 Linear Elasticity

The basic relations between stress and strain, upon which uncracked and cracked concrete are based, are those of *linear elasticity* which is described completely by two elastic constants, Poisson's ratio, ν, and Young's modulus, E. The elastic limit envelope in general stress space is the same as that of the fracture envelope.

For the uncracked isotropic concrete, the constitutive relationships are well known, in the usual notation [10]

$$p = \frac{\sigma_{kk}}{3} = \frac{E}{3(1 - 2\nu)} \, \varepsilon_{kk} = K \, \varepsilon_{kk} \tag{3}$$

$$s_{ij} = \frac{E}{1 + \nu} \, e_{ij} = 2G \, e_{ij} \tag{4}$$

where $K = E/3(1-2\nu)$ and is called the *bulk modulus* and $G = E/2(1+\nu)$ and is the *shear modulus*. Volume change $\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \varepsilon_x + \varepsilon_y + \varepsilon_z$ is produced by the mean normal stress $p = \sigma_{kk}/3 = (\sigma_{11} + \sigma_{22} + \sigma_{33})/3 = (\sigma_x + \sigma_y + \sigma_z)/3$; distortion or shear deformation $e_{ij} = \varepsilon_{ij} - \varepsilon_{kk}\delta_{ij}/3$ is produced by the shear stress or the stress deviation $s_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$. Each is independent of the other. A generalization of Eqs. (3) and (4) for defining isotropic nonlinear stress-strain behavior of concrete will be given later when the nonlinear elastic and variable moduli models are presented.

For the cracked concrete, the stress-strain relations are still linear elastic but the material stiffness or the tangent elasticity matrix is modified to reflect the fact that the stresses in the direction normal to the crack are zero. Overall this implies that crack propagation is solved by succession of transitions from one instantaneous elastic stiffness to another. However, the entire amount of stresses existing in the crack plane before cracking must be released suddenly and thrown back into the structure for redistribution. This redistribution may introduce additional cracking and results in a considerable complication in numerical analysis. Stress-strain relationships of fractured concrete including the sudden release of fracuring stresses will be given later when the general criteria for defining fracture of concrete is presented in what follows.

4.2 Fracture Criteria

When the state of stress reaches a certain critical value, concrete will fail by fracturing. Fracture of concrete can occur in two different types: (1) the "cracking" type of fracture occurs when the principal stresses are either in the tension-tension state or tension-compression state and their value exceeds the limit value; and (2) the "crushing" type of fracture occurs when the principal stresses are in the compression-compression state and their value exceeds the limit value. When concrete cracks the material only loses its tensile strength normal to the crack direction but retains its strength parallel to the crack direction. On the other hand, when concrete crushes, the element loses its strength completely.

To determine the fracture of concrete under multi-axial state of stresses or strains, a *fracture criterion* which specifies the limit value is needed. Most of the existing fracture (or failure) criteria for brittle materials are written in terms of stresses, which, in many cases, are not adequate to predict the failure characteristics of concrete material. In the following, a dual representation of fracture criterion expressed in terms of both stresses and strains is described. The dual criterion has the following forms:

(1) **stress criterion**

$$f(\sigma_{ij}) = 0 \qquad (5)$$

The simplest type of fracture criterion in compression states is to assume a linear relation between octahedral shear stress $\tau_{oct} = \sqrt{2J_2/3}$ and octahedral normal stress $\sigma_{oct} = I_1/3 = p$,

$$\tau_{oct} = a + b\,\sigma_{oct} \qquad (6)$$

or alternatively,

$$\sqrt{J_2} = k - \alpha p \qquad (7)$$

where a and b or k and α are material constants to be determined from experimental data and

$$I_1 = \sigma_{ii} = 3p$$
= first stress invariant which corresponds to the mean normal stress component of the stress state   (8)

$$J_2 = \frac{1}{2} s_{ij}s_{ij} = \frac{1}{2}\sigma_{ij}\sigma_{ij} - \frac{1}{6}\sigma_{ii}^2$$
= second invariant of deviatoric stresses   (9)

If α is zero, Eq. 7 reduces to the von Mises yield condition for metal. The criterion of Eq. 7 is called the *extended von Mises criterion*. Drucker and Prager have shown that the function (7) reduces to the well-known Coulomb criterion in the case of plane strain condition if [11]:

$$\alpha = \frac{\tan\phi}{\sqrt{(9 + 12\tan^2\phi)}} \quad , \quad k = \frac{3c}{\sqrt{(9 + 12\tan^2\phi)}} \qquad (10)$$

where the angle ϕ is known as the angle of internal friction and c is the cohesion of a concrete. The constants c and ϕ can be looked upon simple as parameters which characterize the total resistance of the concrete to shear.

On the basis of biaxial experimental data, a fracture function for concrete in compression states in the form of

$$3J_2 + f_c' \, I_1 + I_1^2/5 = f_c'^2/9 \tag{11}$$

has been proposed by Buyukozturk [12].

In tension-compression and tension-tension states, the maximum stress (or strain) criterion is generally adopted in most applications

In a recent development, a fracture function for concrete of the following form has been proposed in [13]:

compression-compression domain $(\sqrt{J_2} \leq - \dfrac{1}{\sqrt{3}} I_1 \text{ and } I_1 \leq 0)$

$$f(\sigma_{ij}) = J_2 + \frac{A_u^c}{3} \, I_1 = \tau_u^{2c} \tag{12}$$

tension-compression or tension-tension domain $(\sqrt{J_2} > - \dfrac{1}{\sqrt{3}} I_1 \text{ or, } I_1 > 0)$

$$f(\sigma_{ij}) = J_2 - \frac{1}{6} I_1^2 + \frac{A_u^T}{3} \, I_1 = \tau_u^{2T} \tag{13}$$

where A_u and τ_u are material constants

$$A_u^C = \frac{(f_{bc}'/f_c') - 1}{2(f_{bc}'/f_c') - 1} \, f_c' \qquad , \qquad A_u^T = \frac{1 - (f_t'/f_c')}{2} \, f_c'$$

$$\tau_u^{2c} = \frac{2(f_{bc}'/f_c') - (f_{bc}'/f_c')^2}{3[2(f_{bc}'/f_c') - 1]} \, f_c'^2 \quad , \quad \tau_u^{2T} = \frac{(f_t'/f_c')}{6} \, f_c'^2 \tag{14}$$

to be determined from the uniaxial tensile strength f_t', uniaxial compressive strength f_c', and equal biaxial compression strength f_{bc}'.

Fig. 6 shows the fracture surface of Eqs. (12) and (13) in the general $(I_1, \sqrt{J_2})$ stress space together with the initial discontinuous surface and sub-sequent loading surfaces. These initial and subsequent surfaces are obtained by simply scaling the fracture surface down to different sizes. For biaxial tests, initial discontinuity is seen to occur at about 75% of the fracture strength in the uniaxial and biaxial compression tests (Fig. 3a); and at about 60% of the fracture strength in tests involving direct tension (Fig. 3c). These surfaces are required in the later part of the paper where the theory of work-hardening plasticity is applied to construct the incremental stress-strain relationships for the elastic-plastic behavior of concrete.

(2) strain criterion

$$g(\varepsilon_{ij}) = 0 \tag{15}$$

The simplest type of fracture criterion in terms of strains in compression states is to assume that it has the same form as that of stress criterion. For instance, in the case of Equation (12) the corresponding strain criterion in

compression states with the maximum tensile strain criterion as cut-off in tension states has the forms

$$g(\varepsilon_{ij}) = J_2(\varepsilon_{ij}) + \frac{A_u^C}{3} \left(\frac{\varepsilon_u}{f_c'}\right) \quad I_1(\varepsilon_{ij}) = (\tau_u^C)^2 \left(\frac{\varepsilon_u}{f_c'}\right)^2 \qquad (16)$$

or maximum of the principal tensile strains $= \varepsilon_t$ \hfill (17)

in which

$$I_1(\varepsilon_{ij}) = \varepsilon_{ii} \qquad (18)$$

$$J_2(\varepsilon_{ij}) = \frac{1}{2} \varepsilon_{ij}\varepsilon_{ij} - \frac{1}{6} \varepsilon_{ii}^2, \text{ and} \qquad (19)$$

A_u^C is defined in Eq. (14) and ε_u and ε_t specify the maximum ductilities of concrete under uniaxial compressive and tensile loading conditions, respectively. The fracture surface defined by Eqs. (16) and (17) in the biaxial principal strain space is shown in Fig. 7.
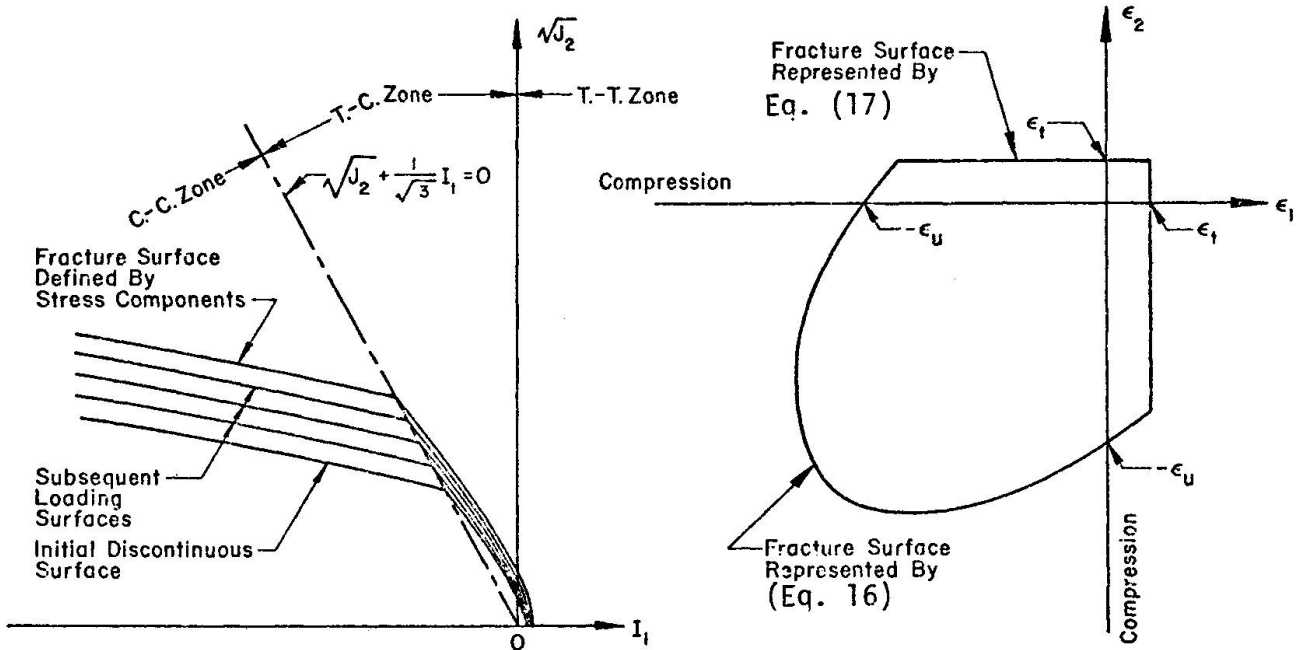
Fig. 6   Loading Surfaces of Concrete in $(I_1, \sqrt{J_2})$ Space.

Fig. 7   Fracture Surface Defined by Strain Components in the Biaxial Strain Space.

When the stress state in the concrete satisfies either the stress criterion, Eq. (5), or the strain criterion, Eq. (15), fracture of concrete is assumed to occur.

4.3   Stress-Strain Relations of Fractured Concrete

In the following, we discuss the stress-strain relations of a fractured concrete, using a proper physical model which simulates the kinematics of a fractured concrete: crushing type and cracking type.

The term *"crushing"* is used to indicate the complete rupture and disintegration of the material under compression type of stress states. After crushing, the current stresses drop suddenly to zero and the concrete is assumed to lose its resistance completely against further deformation. The term *"cracking"* is used to indicate a partial collapse of the material across the plane of cracking under tensile type of stress states. An infinite number of parallel fissures are assumed to occur in the direction normal to the offending principal tensile stress or strain. Once a crack has formed, the tensile stress across the crack drops suddently to zero and the resistance of the material normal to the crack direction is reduced to zero in this direction against further deformation. However, material parallel to the crack is assumed to carry stress according to the uniaxial or biaxial conditions prevailing parallel to the crack. The conditions for further cracks or opening and closing of the existing cracks will be described later when the matrix constitutive relationships for fractured concrete are discussed.

The tensile type of stress states (including tension-compression type) and multi-axial compressive type of stress states can be defined in the following manner:

(1) **In terms of stress invariants**

For example, in the case of Eqs. (12) and (13), it can be shown that when a stress state satisfies the condition

$$\sqrt{J_2} \leq - \frac{1}{\sqrt{3}} I_1 \quad \text{and} \quad I_1 \leq 0 \tag{20}$$

The stress state is of compression type and crushing type of fracture is assumed to occur. Otherwise, it is of tensile type and cracking type of fracture is assumed to occur.

(2) **In terms of principal stresses σ_1, σ_2 and σ_3:**

If all the three principal stresses are compressive (negative) or zero, the stress state is of compressive type and "crushing" type of fracture is assumed to occur. Otherwise, the stress state is of tensile type and "cracking" type of fracture is assumed to occur.

Incremental Stress-Strain Relationships

The fracture model shown schematically in Fig. 8 is used here to discuss the required incremental stress-strain relationships of fractured concrete. The slopes of lines 0-1 and 2-3 represent the "material stiffness" before and after the occurence of fracture. The total stresses released are denoted by the stress vector $\{\sigma_o\}$ (line 1-2 in Fig. 8). The released stresses are redistributed to adjacent material of the entire structure. The released stresses are assumed to be generated discontinuously from zero to the specified magnitudes at the instant of fracturing. The incremental stress-strain relationship after fracturing can be represented by the well-known relationship

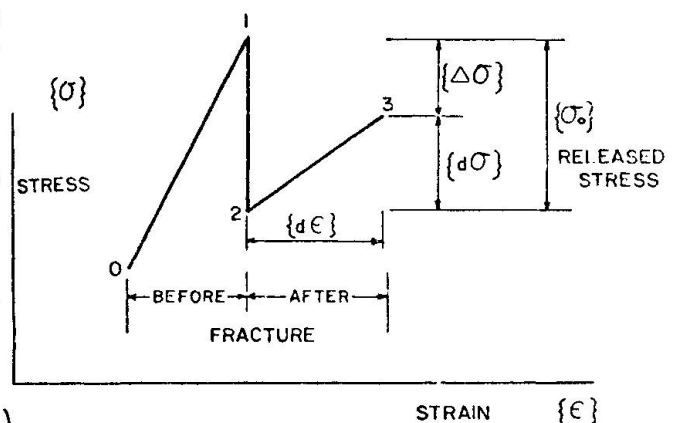$$\{d\sigma\} = [D]_c \{d\epsilon\} \tag{21}$$

Fig. 8   Stress-Strain Model of a Fractured Concrete.

The total stress change $\{\Delta\sigma\}$ in the fractured material during this process can be written formally as

$$\{\Delta\sigma\} = \{d\sigma\} - \{\sigma_0\} = [D]_c\{d\varepsilon\} - \{\sigma_0\} \tag{22}$$

in which

$[D]_c$ = material stiffness matrix after fracturing (either the cracking type or the crushing type),

$\{\sigma_0\}$ = the released stress vector during fracturing

Crushing Type of Fracture

It is usually assumed that at the instant of crushing, all stresses at the point just prior to the crushing are released complete and thereafter the concrete is assumed to lose its resistance completely against any type of further deformation. This implies that the stress point 2 in Fig. 8 drops to zero and the slope of line 2-3 is also zero ($[D]_c = 0$ in Eq. 22 and $\{\sigma_0\}$ = current stress vector at the point just prior to crushing).

Cracking Type of Fracture

A crack is assumed to form in the planes (or surfaces for axisymmetric problems) perpendicular to the maximum principal tensile stress direction if the stress fracture criteria controls (Eq. 5), or perpendicular to the maximum principal tensile strain direction if the strain fracture criteria controls (Eq. 15). In order to avoid the complexities of the problem, further restriction pertaining to the crack formation is usually introduced. Cracks are assumed to form only in the planes perpendicular to the plane for planar problems or only in the axisymmetrical surfaces for axisymmetric problems.

It is further assumed that at the instant of the crack formation, only the normal stress perpendicular to the cracked plane and the shear stress parallel to the cracked direction are released and the other stresses are assumed to remain unchanged. It follows that the stress states of the cracked material are reduced to (1) the uniaxial stress states parallel to the cracked direction for plane stress problems; (2) the biaxial stress states in the cracked and z directions for plane strain problems; or(3) the biaxial stress states in the cracked and circumferential (θ) directions for axi-symmetric problems, respectively.

Assuming the behavior of the sliced material between two adjacent cracked planes is linearly elastic, the incremental stress-strain relationships for the cracked material can be derived. For instance, in the case of plane stress, the stress-strain relationship in the global coordinate system has the form, in the usual notation for stresses and strains,

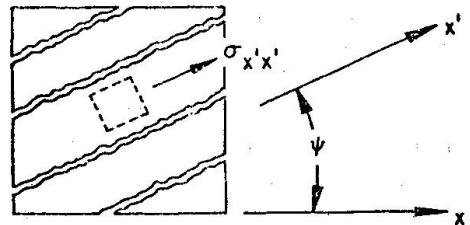**Fig. 9** Pattern of Cracks and Stress Distribution in a Cracked Concrete.

$$\begin{Bmatrix} \Delta\sigma_x \\ \Delta\sigma_y \\ \Delta\tau_{xy} \end{Bmatrix} = \left[E\{b(\Psi)\}\{b(\Psi)\}^T \right] \begin{Bmatrix} d\varepsilon_x \\ d\varepsilon_y \\ d\gamma_{xy} \end{Bmatrix} - \left[[I]-\{b(\Psi)\}\{b'(\Psi)\}^T \right] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \tag{23}$$

where σ_x, σ_y and τ_{xy} are the current stress components at the point just prior to the formation of a crack and

$$\{b(\Psi)\} = \begin{Bmatrix} \cos^2\Psi \\ \sin^2\Psi \\ \sin\Psi \cos\Psi \end{Bmatrix} , \quad \{b'(\Psi)\} = \begin{Bmatrix} \cos^2\Psi \\ \sin^2\Psi \\ 2\sin\Psi \cos\Psi \end{Bmatrix} , \quad [I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (24)$$

in which Ψ = the direction of the cracks as shown in Fig. 9.

For the case of plane strain, the stress-strain relationship (23) still holds except that E be replaced by $E/(1-\nu^2)$. Details of this development together with a plasticity formulation for the nonlinear behavior of concrete for both plane and axisymmetric problems can be found in Ref. 14.

Further Cracking or Crushing of the Cracked Concrete

After the formation of initial cracks, the structure can often deform further without overall collapse. Thus, the possibility of crack closing and opening and the formation of further cracks may arise. These additional transition charac-teristics in the cracked material can be taken into account throughout the analysis. This is described further in what follows.

Formation of the Secondary Cracks

For a cracked concrete, the stress state reduces to the uniaxial one for plane stress problems or to the biaxial ones for plane strain and axisymmetric problems, respectively (see Fig. 9). The dual fracture criteria as defined by Eqs. (5) and (15) are still applicable for such a concrete. Once the reduced stress state satisfies one of the fracture criteria, further fracture (crushing or cracking) is assumed to occur and the direction of the new cracks are allowed to occur perpendicularly to the first cracks. The first cracks and the new cracks are called the *primary* cracks and the *secondary* cracks, respectively.

In the following, the incremental stress-strain relations for the fractured con-crete are summarized:

(1) For the crushed concrete or for the cracked concrete in which more than a single set of cracks are open, the material stiffness is assumed to be zero for further loading and the current state of stress at the point just prior to fracture is assumed to be released completely;

(2) For the cracked concrete in which a single set of cracks are open, the incremental stress-strain relation of the type as given by Eq. (23) can be used;

(3) For the cracked concrete in which all sets of cracks are closed, the concrete is assumed to be completely healed and it behaves as a linear elastic material.

5. NONLINEAR ELASTIC AND VARIABLE MODULI MODELS

The linear relations between the mean response and the deviatoric response of concrete

$$p = K\varepsilon_{kk} \qquad (25)$$

$$s_{ij} = 2Ge_{ij} \tag{26}$$

written in the preceding section are *isotropic* and *reversible*. Clearly, then, a simple extension of these relations with the elastic constants replaced by scalar functions associated with either the stress invariants or strain invariants will have the property of isotropy and reversibility also. Experimental results show that approximately unique relationships existed between hydrostatic stress p and volumetric strain ε_{kk} (or octahedral normal stress-strain $\sigma_{oct} = p$, $\varepsilon_{oct} = \varepsilon_{kk}/3$), and between deviatoric stress and strain (conveniently represented by octahedral shear stress and strain $\tau_{oct} = \sqrt{2J_2/3}$, $\gamma_{oct} = 2\varepsilon_{oct} = \sqrt{8J_2^{(\varepsilon)}/3}$) until fairly close to peak or ultimate stress conditions [15,16]. The unique octahedral normal stress and strain, and octahedral shear stress and strain relations have been determined experimentally by several investigators [7,15,16,17] from which the tangent bulk and shear moduli as nonlinear functions of the stress and/or strain tensor invariants can be derived. Note that these nonlinear relationships are developed mainly from experimental data under multiaxial compressive stress state.

The simplest type of such invariant relationships is to assume the tangent bulk modulus K_t and the tangent shear modulus G_t in Eqs. (25) and (26) to be functions of the first and second stress invariants I_1 and J_2 respectively, i.e.

$$K_t = K_t(I_1) = K_t(\sigma_{ii}) \tag{27}$$

$$G_t = G_t(J_2) = G_t(\tfrac{1}{2} s_{ij}s_{ij}) \tag{28}$$

These relationships can then be introduced into Eqs. (25) and (26) in terms of stress rate and strain rate (incremental) relations

$$\dot{s}_{ij} = 2G_t\dot{e}_{ij} \tag{29}$$

$$\dot{p} = K_t\dot{\varepsilon}_{kk} \tag{30}$$

values of E_t and ν_t can be obtained by standard expressions from linear elasticity. Once these current tangent moduli are known, the current tangent elasticity matrix, with any modification due to cracking as described in the preceding section, can be determined. Thus the nonlinear deformation response of concrete up to fracture can now be simulated *incrementally* as a piece-wise linear elastic-fracture model, using the updated or current values of the tangent moduli.

Note that there is a neat and logical separation between the mean response and the deviatoric or shear response of concrete, which is exhibited very clearly by Eqs. (29) and (30). However, there will be interaction between the two through the change in the magnitude of the scalar function G_t and K_t with variation in $I_1 = \sigma_{kk}$ and $J_2 = \tfrac{1}{2} s_{ij}s_{ij}$ if the scalar functions are generalized to be functions of both I_1 and J_2 as $G_t(I_1,J_2)$ and $K_t(I_1,J_2)$. This implies that volume change ε_{kk} will not depend solely on the mean normal stress p. Similarly, distortion or shear deformation, e_{ij} will not depend solely on the stress deviation or shear stress, s_{ij}. They will depend on each other, and interact through the variation of both bulk and shear moduli, if these moduli are assumed to be functions of both I_1 and J_2.

Equations (25) and (26) are nonlinear stress-strain relations for isotropic, elastic concrete which reduce to the linear forms when G and K are constants.

They represent elastic (reversible) behavior because the state of strain is determined uniquely by the current state of stress without regard to the history of the loading. This is obviously not the case for concrete at relatively high stress levels. These relationships also assume that the principal axes of strain increment always coincide with the principal axes of stress increment (Eqs. 29 and 30). This is true only at low stress levels. At high stresses, especially near failure, however, the strain increment axes in concrete probably coincide more closely with the principal axes of stress (and not of stress increment). Knowledge regarding this aspect of concrete deformation behavior under triaxial stress states is still lacking and mostly unknown. Further, the biaxial compression data show clearly that concrete dilates near and after failure or peak stresses and considerable volume expansion occurs as the compressive stresses increase. This *"dilatancy"* is not predicted by the present nonlinear elastic model. Some of these limitations on nonlinear elastic modeling of concrete can be overcome by the introduction of the so-called *"variable moduli"* models.

Here, as in nonlinear elastic models, Eqs. (29) and (30), both the bulk and shear moduli are taken as nonlinear functions of the stress and/or strain tensor invariants. However, unlike the nonlinear elastic models, different functions are used in initial loading, unloading, and reloading to reflect the irreversible characteristics of inelastic deformations of concrete. A model of this type can fit to a rather complete set of laboratory data for a particular concrete material. At the present time, however, this approach has not been attempted in the field of structural concrete due to the lack of available experimental data on concrete stress-strain behavior subjected to biaxial and triaxial stress reversal and cyclic loading.

The other problem associated with this type of formulation is that the model may not satisfy all rigorous theoretical requirements for all stress histories. To illustrate, for example, when the stresses are near *neutral loading* in shear, the variable moduli model fails to satisfy the continuity condition along such stress paths, and the resulting numerical solutions in practical applications may become questionable. This was found to be the case in a ground shock computation in soil mechanics [18]. Such a concern on continuity condition also can occur in the analysis of reinforced concrete problems.

All these limitations can be overcome if an elastic-plastic stress-strain relationship is employed. If the difference between the nonlinear elastic or variablemoduli and more rigorous plasticity models to be described in what follows is not s gnificant when applied to reinforced concrete structures under certain loading conditions, then the present approach will be very attractive. This is because the present modeling, unlike the plasticity modeling, contains no explicit yield condition and hence it is computationally simple. It is particularly well suited for finite element code calculations where local stiffness is required. The variable moduli model, which is similar to the *deformation theory* of *plasticity*, is capable of fitting repeated hysteretic data in cyclic loading. At present, not enough data are available for such applications in reinforced concrete structures. In the following, we discuss two plasticity models: an elastic-perfectly plastic-fracture model and an elastic-plastic-strain-hardening-fracture model.

6. ELASTIC-PERFECTLY PLASTIC-FRACTURE MODELS

It is known that under triaxial compression, concrete can flow like a ductile material on the yield or failure surface before reaching its crushing strains. The assumption made in Sec. 4 that concrete crushes completely once the fracture surface is reached is rather rough, but a fair first approximation. To account for this limited plastic flow ability of concrete before crushing, a perfectly plastic model can be introducted.

A typical elastic-perfectly plastic and fracture model is generally assumed to set linearly elastically below yield or fracture surface, and is perfectly plastic on the surface until crushing or cracking strains are reached. Thus under uniaxial conditions, the compressive stress-strain curve for this model would exhibit a short horizontal stress plateau at the peak stress value, f_c' and this plateau terminates when the crushing strain of $-\varepsilon_u$ is reached (Fig. 10). More generally, a limited tensile ductility can also be assumed for concrete under tensile type of loading.
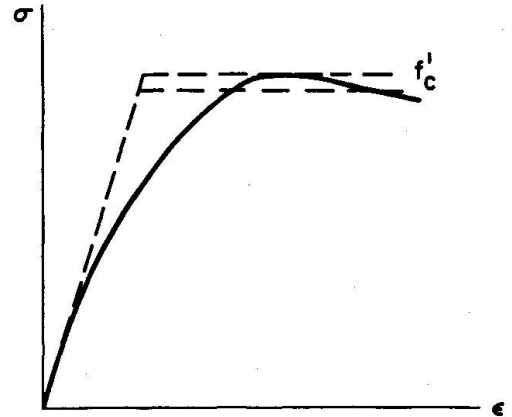
Fig. 10 Stress-Strain Curve for Concrete and Elastic-Perfectly Plastic Idealizations.

6.1 Incremental Stress-Strain Relationships for the Elastic-Plastic Behavior

The complete stress-strain relationships are developed in three parts: (1) before yield; (2) during plastic flow; and (3) after fracture. The linear elastic stress-strain relationships before yield and after fracture have been developed and described in Sec. 4. Only the plastic stress-strain relationship during the plastic flow needs to be added here. To achieve this one must first define (1) the condition of yield; and (2) the strain criterion for fracture. With these boundaries determined, the plastic stress-strain relationship in an incremental form can then be established.

The fracture criterion written in Sec. 4 in terms of stress invariants can be taken as the perfectly plastic yield surface. A considerable amount of numerical work has been done using (1) von Mises criterion, (2) extended von Mises criterion (or Drucker-Prager criterion); and (3) Coulomb or modified Coulomb criterion. With the exception of the von Mises criterion which is used widely in metal plasticity, all yield criteria developed for concrete incorporate a dependence of yield point stress on the mean normal stress $3p = I_1$ in addition to the dependence on the invariant of the "averaged" maximum shear stress J_2.

To construct the stress-strain relation in the plastic range, the *normality* of the plastic deformation rate vector to the yield surface or the so-called *flow rule* is commonly employed. The dependence of yield function on the mean normal stress and the concept of flow rule lead, in general, to a plastic volume increase under pressure. This *"dilatancy"* near failure is usually observed in concrete and rock type of materials. It is of interest to note that this phenomenon is generally not observed in soil type of medium except possibly in densely packed dry sands. Consequently, recent development in soil plasticity attempts to keep this effect as close to zero as possible. Here, as in rock materials, we attempt to allow for dilatancy. The historical development of various types of plasticity models used in soil mechanics as well as in rocklike materials, such as concrete, along with some numerical results using linearly elastic-perfectly plastic models, is given in a recent book by Chen [10].

The normality condition of the plastic deformation vector $d\varepsilon_{ij}^p$ to the yield surface $f(\sigma_{ij})$ can be expressed mathematically in the form:

$$d\varepsilon_{ij}^p = \lambda \frac{\partial f}{\partial \sigma_{ij}} \qquad (31)$$

in which $\lambda > 0$ is a scalar proportionality factor. Once a yield function is determined, the plastic stress-strain relationship can be derived in a rather straight-forward manner by performing the operations indicated in the above equation.

Recently, the post-fracture behavior of concrete under biaxial compression states has been reported [16]. The normality flow rule used for plastic flow of concrete before fracture is not strictly observed in the case of *fractured* concrete. This seems to indicate that Eq. (31) is not proper for the development of a stress-strain relationship of a fractured concrete. Instead, this relationship should be developed on the basis of a proper physical model which simulates closely the kinematics of a fractured concrete: crushing type and cracking type. This is described in Sec. 4.

Due to the lack of available experimental data on concrete ultimate deformation capacity under multiaxial stress states, the strain criterion required for the fracture of concrete under compression states is usually developed by simply converting the yield criterion in stresses directly into strains. This is accomplished, for example, by simply replacing f_c' with ε_u, τ_{oct} or J_2 with the octahedral shearing strain ε_{oct} or $J_2^{(\varepsilon)}$, and p with the hydrostatic strain ε_{kk}. The maximum tensile strain criterion is often used as the cut-off plane for this compression surface. The resulting fracture envelope in the biaxial strain space has the shape shown in Fig. 7. Note that the magnitude of the maximum tensile strains at failure is generally not a constant in the tension-compression states, but increases with the degree of compression. From biaxial stress tests, it is found that concrete can sustain significantly higher indirect tensile strains than direct tensile strains. A general strain criterion for fracture of concrete is still incomplete and further research on this is urgently needed.

At the present time, little of the "typical" experimental concrete data under cyclic loading conditions are available. As more experimental data are expected to be available in the future, some modifications of the present perfect plasticity models are expected in order to match better with these new cyclic loading data. These modifications can be grouped into the following three types: (1) the generalization of the yield condition to include *strain softening* beyond the peak flow stress; (2) the use of nonlinear relations of the type described in the preceding section for concrete before yield; and (3) the generalization of the flow rule for plastic concrete by introducing the concept of *plastic potential surface* which is not assumed to be identical to that of the yield surface. These modifications will lead to the development of a series of extended and advanced elastic-perfectly plastic concrete models.

6.2 Ductility Assumptions and Limit Analysis

If the tensile strength of concrete is assumed to be zero for dimensions of engineering interest, and not just a small fraction of the compressive strength, then the *limit theorems* of perfect plasticity will hold rigorously for this idealization [10]. They can be applied easily when a tensile crack is introduced in the failure mechanism. The early application of the limit theorems approach to the analysis and design of Voussoir arches is a good example.

Kooharian [19] made the extreme but not unreasonable assumption that concrete is unable to take tension and behaves as a rigid, infinitely strong material in compression. This idealization was selected because it was simple and provided a first approximation to the behavior of a real Voussoir arch.

If concrete is assumed to have zero tensile strength and a finite compressive strength, it may almost be considered as a real material. The assumption of infinite ductility in tension at zero strength is rigorous and conservative for the application of limit analysis, but the assumption of infinite ductility in compression at selected compressive stregnth is more questionable. Brittleness in compression and a falling stress-strain curve after a maximum strength is usually observed, as indicated in Fig. 10. This is not compatible with the limit theorems. If the compressive strain of concrete is small and is not repeated often, then the deformability of concrete in compression prior to an appreciable fall-off of stress may be sufficient to permit the consideration of limit theorems with the concrete idealized as perfectly plastic at a yield stress in compression that approximates the ultimate strength f_c'. However, with the safe assumption that concrete is unable to resist any tension, Chen and Drucker [20] have shown that the bearing capacity is just the unconfined compression strength of the column of material directly under the load.

Obviously, the idealization of zero tensile strength is not true. When a small but significant tensile strength is assumed, and if the Coulomb surface for failure in compression is taken to represent a perfectly plastic yield, the predicted capacity is found to be in good agreement with published test results [20]. These results convincingly show that tensile strength and local extensibility of concrete are essential to the ability of concrete blocks to carry the load.

In view of the fact that: (1) a complete elastic-plastic and fracture analysis of concrete and reinforced concrete structures by finite element method is always complicated; and (2) failure or collapse load is usually the governing condition in so many problems in reinforced concrete structures, the drastic idealization of concrete as a rigid, perfectly plastic material together with a proper failure criterion, such as the modified Coulomb criterion with a small tension cutoff, will lead to the simple and efficient method of limit analysis from which the collapse load can be calculated directly. Such idealizations are found acceptable in many practical situations such as (1) the load-carrying capacity of a construction joint in plain concrete [21]; (2) shear strength of reinforced concrete [22]; (3) load carrying capacity of concrete blocks [10,20,21]; and (4) plastic collapse analysis of reinforced concrete plates and shells [23].

7. ELASTIC-STRAIN HARDENING PLASTIC AND FRACTURE MODELS

The latest step in the development of concrete constitutive models is the use of the strain-hardening theory of plasticity. A yield surface called *loading surface* which combines both perfect plasticity and strain-hardening is postulated and an associated flow rule is used for the plastic concrete before fracture. This approach can be considered as a generalization of all previous models, and at the same time satisfies rigorously the basic principles of continuum mechanics such as the requirements of *uniqueness* of solution and *continuity* of near neutral loading paths. It also allows a good fitting of material property data. This is given in the forthcoming.

According to this approach, the stresses in a structure under operation condition are expected to be in the initial discontinuous range such that the concrete behavior can be expected to be characterized as linear elastic and formation of microcracks can be minimized, i.e., one may expect to avoid fatigue. This

initial discontinuous surface is the limiting surface for elastic behavior. The elastic limit is defined here as the initial discontinuous surface similar to the fracture surface but a certain distance within the fracture surface. Fig. 11 shows a trace of these two extreme surfaces in a two-dimensional principal stress space. When the state of stress lies within the initial discontinuous surface, the material is assumed to be linear elastic and the linear elastic constitutive equations can be applied.

When the material is stressed beyond the initial discontinuous or elastic limit surface, a subsequent new discontinuous surface called the *loading surface* is developed. This new surface replaces the initial discontinuous surface. If the material is unloaded from and reloaded within this subsequent loading surface, no additional irrecoverable deformation will occur until this new surface is reached. If straining is continued beyond this surface, further discontinuity and additional irrecoverable deformation results. In other words, at each stage of the history of loading there exists a loading surface in stress space containing all states of stress which can be reached by elastic changes and any straining beyond this surface is accompanied by the irrecoverable plastic deformation. Beyond the elastic limit, the normality condition or the so-called associated flow rule is assumed to govern the post yielding stress-strain relations for concrete. Once the loading surface is defined, constitutive equations based on the concept of flow rule can be derived.

A crush and crack surface called *strain fracture surface* in terms of strains is also postulated to define the complete collapse for the yielded concrete. The *stress fracture surface* in terms of stresses is defined as the outermost extreme of the loading surface similar to that of perfectly plastic yield surface. Once the stress fracture surface is reached, the concrete begins to flow under constant stresses. Finally, the concrete is assumed to crack or crush depending upon the nature of the stress states, when the strain fracture surface is reached. The stress-strain relationship developed in Sec.4 for a fractured concrete can now be applied.
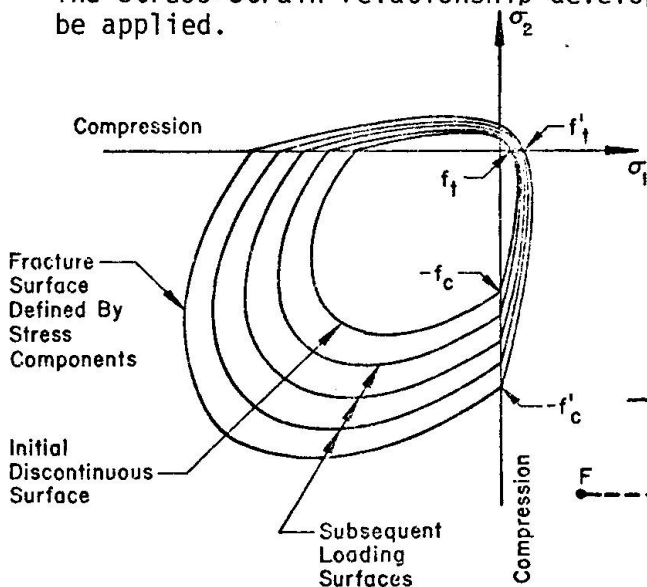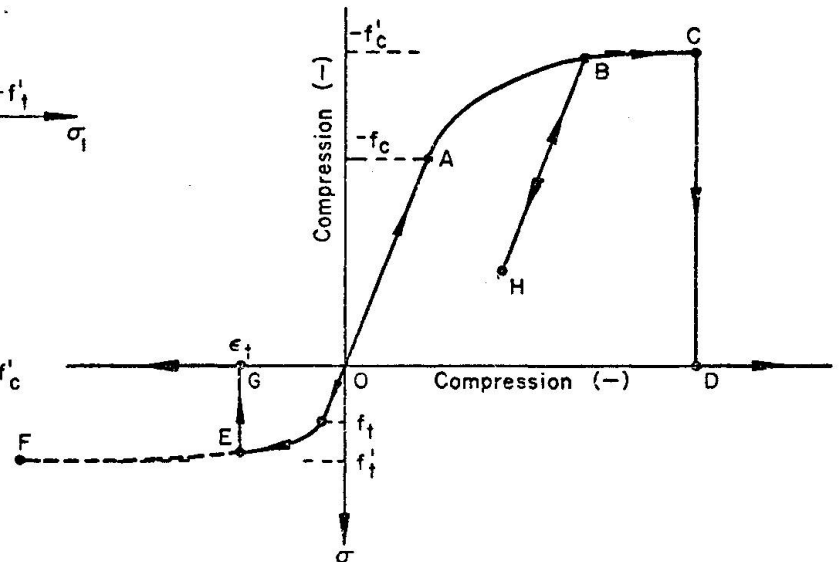

Fig. 11   Loading Surfaces of Concrete   Fig. 12   Idealized Uniaxial Stress-
          in Biaxial Stress Plane.   Strain Curve for Concrete.

7.1 Uniaxial Stress-Strain Curve

The stress-strain characteristics of concrete under general multi-axial stress states described above can best be explained from the uniaxial stress-strain curve shown in Figure 12. The concrete is considered to be a linear elastic strain (or work) hardening plastic, and fracture material in compression as well as in tension. The modulus of elasticity in the elastic range is assumed to be the same for both tensile and compressive stress states. The transition compressive and tensile stresses from elasticity to plasticity are $-f_c$ and f_t respectively. At stress level $-f_c'$, the concrete becomes perfectly plastic until crushing occurs at a compressive strain of $-\varepsilon_u$. The concrete is assumed to have a limited tensile strength of f_t and limited tensile strain of ε_t. When the tensile stress or strain exceeds its limiting value f_t' or ε_t, a crack is assumed to occur in a plane normal to the direction of the offending stress or strain.

Unloading and reloading (line BH in the figure) in the plastic range follow the initial elastic modulus E. The transition stresses from strain hardening plasticity to perfectly plastic flow in compression, which will eventually lead to crushing of concrete, is $-f_c'$ and the transition to brittle fracture in tension which will lead to cracking is f_t'. These limiting stresses are called fracture stresses of concrete.

The complete stress-strain relationships under general stress states are developed in three parts" (1) linear elasticity before yielding; (2) strain hardening plasticity for a post yielded concrete; and (3) linear elasticity after fracture. Items (1) and (3) have been developed in Sec. 4. Item (2) will be discussed in what follows.

7.2 Strain-Hardening

For a plastic concrete, a strain-hardening plasticity model as proposed in Ref. [13] is used here to illustrate the procedure. The loading function adopted in this model has the following form:

Compression-compression domain $(\sqrt{J_2} \leq \frac{1}{\sqrt{3}} I_1$ and $I_1 \leq 0)$

$$f(\sigma_{ij}) = \frac{J_2 + \frac{\beta}{3} I_1}{1 - \frac{\alpha}{3} I_1} = \tau^2 \tag{33}$$

Tension-compression or tension-tension domain $(\sqrt{J_2} > - \frac{1}{\sqrt{3}} I_1$ or $I_1 > 0)$

$$f(\sigma_{ij}) = \frac{J_2 - \frac{1}{6} I_1^2 + \frac{\beta}{3} I_1}{1 - \frac{\alpha}{3} I_1} = \tau^2 \tag{34}$$

where α and β are material constants, and τ is strain-hardening parameter. The loading functions, Eqs. (33) and (34), can be depicted as the loading surfaces shown in Figure 11 in the biaxial principal stress space and in Figure 6 in the $(I_1, \sqrt{J_2})$ space for the more general case of triaxial stress states. For $\tau = \tau_0$ or $\tau = \tau_u$, the loading function represents the initial discontinuous (yield) or fracture surface in the stress space, respectively, whereas $\tau_0 < \tau < \tau_u$ represents a subsequent loading surface. The constants τ_0, τ_u, α and β are to be determined from the following three tests: uniaxial compression f_c', uniaxial tension f_t' and equal biaxial compression f_{bc}' which may be assumed to be 1.16 f_c'.

Further, all the initial yield values for f_c, f_t and f_{bc} can be taken to be 30-60% of the corresponding maximum strength values.

In constructing the relationship between stress and strain increments in the plastic range for general stress states, one must define (a) the current condition for yield; (b) the general form of the stress-strain relationship; and (c) a criterion for work hardening. In the present formation, the current condition for yielding is determined by the maximum value of $\tau = \tau_{max}$ which the concrete has already experienced in its prior loading history. This value of τ_{max} determines the current loading function which controls the condition for yielding.

As for the incremental stress-strain relationships for the elastic-plastic behavior of concrete, we assume that the "normality" condition as commonly used in the incremental theory of metal plasticity still holds for plastic concrete. This normality condition (flow rule) results in the following incremental plastic stress-strain relationship.

$$d\varepsilon_{ij}^{p} = \frac{1}{H'} \cdot \frac{\partial f/\partial\sigma_{ij}}{\sqrt{(\partial f/\partial\sigma_{mn})(\partial f/\partial\sigma_{mn})}} \, df \tag{35}$$

in which

$$H' = \frac{dF}{d\varepsilon_p} = \text{work hardening modulus} \tag{36}$$

where

$$f(\sigma_{ij}) = F(\varepsilon_p) = \text{effective stress and} \tag{37}$$

$$\varepsilon_p = \sqrt{d\varepsilon_{ij}^{p} \, d\varepsilon_{ij}^{p}} = \text{effective strain} \tag{38}$$

Matrix constitutive equations based on the concept of subsequent loading surface and associated flow rule have been derived [24]. A comprehensive review of the plasticity theory as applied to concrete mechanics is given in [10].

The effective stress $F = f(\sigma_{ij})$ is defined here to be able to extrapolate from a simple uniaxial compression test into the multi-dimensional situation. The initial yield occurs when the effective stress, F, equals the initial yield stress, f_c, measured in the uniaxial compression test. With loading continued, the subsequent yielding occurs and plastic strains in the uniaxial direction, ε_1^p, as well as in the lateral directions, $\varepsilon_2^p = \varepsilon_3^p$, are introduced. For each value of effective stress, $F = f$, the corresponding effective strain,

$\varepsilon_p = \sqrt{\varepsilon_1^{p^2} + 2\varepsilon_2^{p^2}}$ can be calculated by simply subtracting the elastic contribution from the total strains in the uniaxial compression test. This results in

$\varepsilon_1^p = \varepsilon_1 - \varepsilon_1^e$ and $\varepsilon_2^p = \varepsilon_2 - \nu\varepsilon_1^e$. The slope of (F,ε_p) curve, H', is known as work hardening modulus. A crushing type of fracture occurs when the effective stress, F, equals the plastic flow stress, f_c' and the uniaxial compressive strain reaches the limit value, ε_u.

To include the microcrack-confinement effect on the work hardening modulus, H', in the multi-dimensional stress situation, the effective stress-strain curve can also be constructed from a biaxial compression test similar to that of uniaxial compression test. Since a biaxial stress-strain expression, expressed in terms of the principal stress ratio, has been developed in Ref. [5], closed-form expressions for the effective stress-strain curve and hence the effective work

hardening modulus, H', under a biaxial stress state may be derived. Similarly, the effective stress-strain curve may also be developed from an almost unique relationship between octahedral stress and strain described in Sec. 5.

Equation (35) which determines the plastic strain increment $d\varepsilon^p_{ij}$, together with the elastic Hooke's law for the elastic strain increment $d\varepsilon^e_{ij}$, give the complete relation between stress increment, $d\sigma_{ij}$, and strain increment

$$d\varepsilon_{ij} = d\varepsilon^e_{ij} + d\varepsilon^p_{ij} \qquad (39)$$

for a strain-hardening plastic concrete.

8. SUMMARY AND CONCLUSIONS

A brief review of various mathematical models for concrete that are currently being used in reinforced concrete analysis have been given in this paper. Based on the results and observations described herein, the following conclusions can be made:

1. The equivalent uniaxial model is the simplest model for fitting to biaxial experimental data. While it approximates many (but not all) features of biaxial experimental tests adequately, it can not match stress-strain data for the triaxial test. The model is particularly appealing to planar type of problems such as beams, panels and shells because (1) the stresses in these structures are predominantly biaxial, and (2) the model has its broad data base.

2. The variable moduli model gives a good fit to the full set of tests (uniaxial, biaxial and triaxial compression data under proportional loading) available. Unique relationships have been established between hydrostatic stress and volumetric strain (or octahedral normal stress and strain), and between deviatoric stress and strain (or octahedral shear stress and strain). From these relationships, tangent bulk and shear moduli can be derived. Thus the nonlinear deformational response of concrete is simulated incrementally as a piecewise linear elastic material with variable moduli. The model is therefore computationally simple and it is particuarly well suited for finite element code calculations where local stiffness is required. The model has also the capability of producing repeated hysteretic effects in cyclic loading. However, experimental data under reversed loading are still very scarce. This does not allow for further development. Variable moduli approach does not model accurately the behavior of concrete at high stress levels (near failure). It may have the continuity problem when stress paths are on or near neutral loading.

3. The strain-hardening plasticity model can be considered as a generalization of all the previous models and it satisfies all the basic principles of continuum mechanics. A set of equations has been presented that provide the general framework for further development. Many of the details have been worked out, but others require further investigation. Some of the interrelationships between linear elasticity, work-hardening plasticity, dual fracture criterion, mechanics of fractured concrete, empirical equation for biaxial stress modulus, and the empirical equations for tangent moduli are demonstrated.

4. Limit analysis which is developed on the basis of perfect plasticity model can be used to calculate the collapse load of some plain and reinforced concrete structures such as bearing capacity of concrete blocks, shear in reinforced concrete beams and slabs. In such a situation, the complicated nonlinear irreversible mechanical behavior of concrete and reinforced concrete can be drastically

idealized as a rigid-perfectly plastic material.

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