IV

Mesh Formulation of the Yield Line Method by Mathematical Programming

La formulation en mailles de la méthode des lignes de rupture par la programmation mathématique

Die Netzformulierung der Bruchlinienmethode mit Hilfe der mathematischen Programmierung

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SUMMARY
The yield line method has been formulated as a mathematical programming problem using a mesh description of a finite element network. The fundamental structural relations have been transformed to an equivalent primal-dual pair of mesh linear programs using the Kuhn-Tucker theory. These programs, along with the primal-dual pair of nodal linear programs derived previously, provide a choice of four programs available for computation. A comparison has been made of the relative computational effort required for these programs when using a simplex-based computer code.

RESUME
La formulation en mailles développée pour la méthode des éléments finis a été appliquée à la méthode des lignes de rupture. On a obtenu une paire „primal-dual" de programmes linéaires équivalente aux relations structurales qui gouvernent la dite description, en utilisant la théorie de Kuhn-Tucker. Ces programmes offrent, avec la paire „primal-dual" de programmes linéaires déjà dérivée pour la description nodale, un éventail de quatre programmes de calcul. On a finalement comparé les difficultés de calcul inhérentes à chacun de ces programmes en utilisant l'algorithme du Simplex.

ZUSAMMENFASSUNG
Die Bruchlinienmethode wurde mit Hilfe einer Netzbeschreibung eines finiten Element-Netzes dargestellt. Die grundlegenden strukturellen Beziehungen wurden unter Verwendung der Kuhn-Tucker Theorie zu einem entsprechenden „primal-dualen" Paar von linearen Programmen transformiert. Diese, und das früher hergeleitete „primal-duale" Paar von linearen Programmen für die Knotenbeschreibung, liefern eine Auswahl von vier für die Berechnung verfügbaren Programmen. Schliesslich wurde der für jedes dieser Programme erforderliche relative Rechenaufwand bei Verwendung des Simplex Algorithmus verglichen.

1. INTRODUCTION

The plastic limit analysis and synthesis of structural *frames* may be formulated conveniently as linear programs (LPs) using either a mesh [1] or nodal [2] description of the structure to formulate the fundamental static and kinematic relations. Each description leads to a primal-dual pair of LPs and thus, for the numerical computation, a choice must be made between four possible programs. Whilst the nodal description is most commonly used, it has been shown [3] [4] recently that the required computational effort with respect to simplex-based algorithms is greatly reduced when the mesh description is employed.

A particularly simple and convenient form of manually-computed plastic limit analysis is embodied in the yield line method (YLM) [5] [6]. This method may be automated to the plastic limit analysis [7] and synthesis [8] of r.c. slabs through FEs and linear programming. The LPs formulated in this way have the same algebraic structure as those obtained previously for frames using the *nodal* description. Since the programs obtained from the *mesh* description for frames had computational advantages, it would appear logical to seek a corresponding mesh formulation for the slab problem and to see if these advantages carry over to this different class of problem.

The nodal description commences with a statement of the (nodal) fundamental kinematic relations and then seeks the corresponding static relations such that an appropriate criterion of consistency is satisfied. The criterion adopted [7] is that of static-kinematic duality (SKD) [1]. The mesh description to be presented herein commences with a statement of the (mesh) fundamental static relations and then derives the corresponding kinematic relations such that SKD is maintained.

2. STATICS

The normal bending moments ($\underset{\sim}{m}$) at the FE sides are considered as the superposition of a particular solution ($\underset{\sim}{m}_o$) which equilibrates the loading and a complementary solution ($\underset{\sim}{m}_c$) which consists of a linear combination of independent self-equilibrating moment fields.

$$\underset{\sim}{m} = \underset{\sim}{m}_o + \underset{\sim}{m}_c \qquad (1)$$

The particular solution can be readily obtained if, as indicated on Fig. 1, the slab is split into cantilevers.

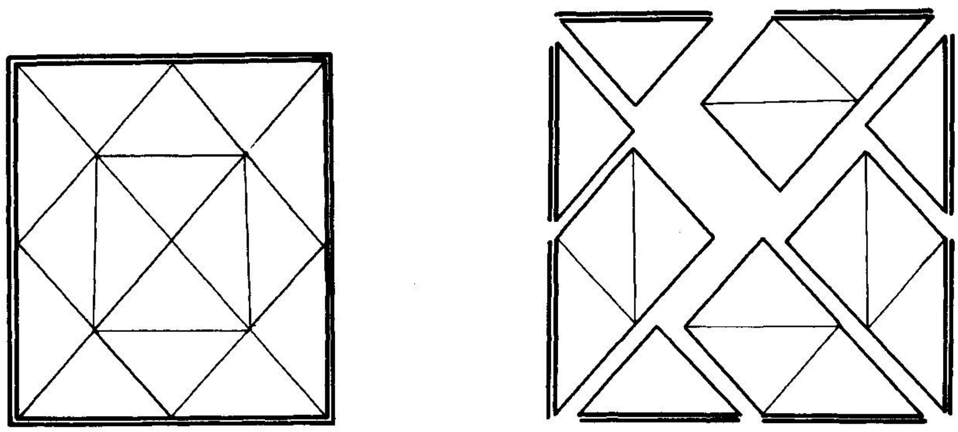
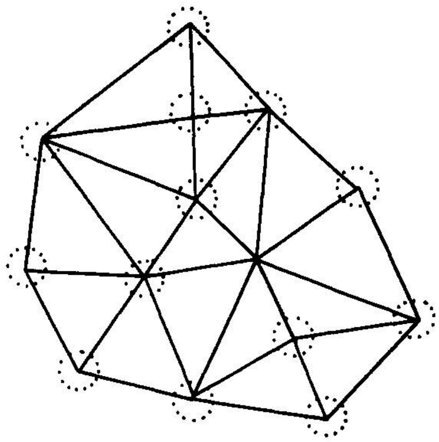


Fig. 1: Cantilever Slabs

Clearly, such a procedure implies edge fixity. However, if other boundary conditions pertain, then they can be incorporated as outlined elsewhere [9].

The complementary solution ($\underset{\sim}{m}_c$) will be based on a set of linearly independent meshes. The simplest such set will be obtained from the FEs which are incident on each of the nodes except one. Such a set is shown in Fig. 2 and these meshes are analogous to the regional meshes of frame theory [4].

Fig. 2: Linearly independent set of finite element meshes



For example, the complementary solution total normal bending moments at the FE sides of the mesh represented in Fig. 3 may be expressed in terms of two static parameters (p_1 and p_2) in the following form

$$\begin{bmatrix} m_{s1} \\ m_{s2} \\ m_{s3} \\ m_{s4} \\ m_{s5} \end{bmatrix} = \begin{bmatrix} \sin\alpha_1 & \cos\alpha_1 \\ \sin\alpha_2 & \cos\alpha_2 \\ \sin\alpha_3 & \cos\alpha_3 \\ \sin\alpha_4 & \cos\alpha_4 \\ \sin\alpha_5 & \cos\alpha_5 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \tag{2}$$

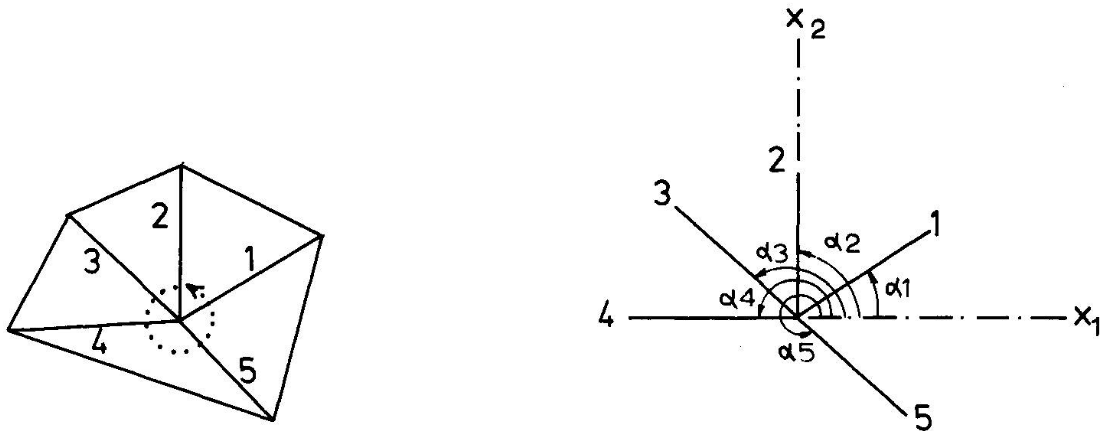


Fig. 3: Example of finite element mesh

Now, if relations (2) are established for all such regional meshes, then they can be assembled in the following compact form

$$\underset{\sim}{m}_c = \underset{\sim}{B}\,\underset{\sim}{p} \tag{3}$$

and the parameters $\underset{\sim}{p}$ will be termed the *mesh actions*. Thus, the mesh equilibrium equations (1) become

$$\boxed{\underset{\sim}{m} = \underset{\sim}{m}_o + \underset{\sim}{B}\,\underset{\sim}{p}} \qquad (4)$$

3. KINEMATICS

The conditions of compatibility for every mesh must ensure that the modal angular deformation rates ($\underset{\sim}{\dot\theta}$) across yielding FE sides correspond to continuity of vertical displacements. For example, for the FE sides of the mesh represented in Fig. 3,

$$\begin{bmatrix} \sin\alpha_1 & \sin\alpha_2 & \sin\alpha_3 & \sin\alpha_4 & \sin\alpha_5 \\ \cos\alpha_1 & \cos\alpha_2 & \cos\alpha_3 & \cos\alpha_4 & \cos\alpha_5 \end{bmatrix} \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3 \\ \dot\theta_4 \\ \dot\theta_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (5)$$

Now, if these compatibility conditions are imposed on every mesh of a FE system, then they can be stated in the following compact form

$$\boxed{\underset{\sim}{B}^T\,\underset{\sim}{\dot\theta} = \underset{\sim}{0}} \qquad (6)$$

The contragredient relation connecting equations (3) and (6) is a manifestation of their consistency with respect to SKD.

Once again, the special treatment of various boundary conditions is discussed more fully elsewhere [9].

4. CONSTITUTIVE RELATIONS

The YLM employs a simple yield criterion

$$\underset{\sim}{N}^T\,\underset{\sim}{m} - \underset{\sim}{m}_* \leqslant \underset{\sim}{0} \qquad (7)$$

where $\underset{\sim}{m}_*$ is the vector of the magnitudes of plastic moments of resistance and the normality matrix $\underset{\sim}{N}$ is given by

$$\underset{\sim}{N} = \left[\underset{\sim}{I} \;\vdots\; -\underset{\sim}{I}\right]$$

where $\underset{\sim}{I}$ is the identity matrix.

It is convenient to express the plastic modal deformations ($\underset{\sim}{\dot\theta}$) in terms of non-negative components

$$\dot\theta_i = x_i^+ - x_i^- \qquad \text{where} \quad x_i^+ \geqslant 0 \;, \quad x_i^- \geqslant 0$$

Thus

$$\underset{\sim}{\dot{\theta}} = \underset{\sim}{N}\,\underset{\sim}{x} \tag{8}$$

where

$$\underset{\sim}{x} \equiv \begin{bmatrix} \underset{\sim}{x}^{+} \\ \hline \underset{\sim}{x}^{-} \end{bmatrix} \geqslant \underset{\sim}{0}$$

and $\underset{\sim}{N}$ is the normality matrix as previously described. Relations (8) constitute the flow rule for the considered problem.

The parity rule linking the static and kinematic variables can be expressed in the following complementary way.

$$\underset{\sim}{x}^{T} \left[\underset{\sim}{N}^{T}\,\underset{\sim}{m} - \underset{\sim}{m}_{*}\right] = 0 \tag{9}$$

5. FUNDAMENTAL STRUCTURAL RELATIONS

The particular solution bending moments $(\underset{\sim}{m}_{o})$ may be expressed as the sum of those due to dead load $(\underset{\sim}{m}_{do})$ and those due to live load $(\underset{\sim}{m}_{\ell o})$.

$$\underset{\sim}{m}_{o} = \underset{\sim}{m}_{do} + \underset{\sim}{m}_{\ell o} \tag{10}$$

If the $\underset{\sim}{m}_{\ell o}$ vector is expressed in terms of a single load parameter (λ),

$$\underset{\sim}{m}_{o} = \underset{\sim}{m}_{do} + \underset{\sim}{r}_{o}\lambda \tag{11}$$

where $\underset{\sim}{r}_{o}$ is the vector of live load particular solution bending moments per unit value of the load parameter (λ).

Substituting from (11) into (4), the equilibrium equations become

$$\underset{\sim}{m} = \underset{\sim}{m}_{do} + \underset{\sim}{r}_{o}\lambda + \underset{\sim}{B}\,\underset{\sim}{p} \tag{12}$$

and the statical admissibility conditions are obtained by substituting from (12) into the yield conditions (7),

$$\underset{\sim}{A}^{T}\,\underset{\sim}{y} - \underset{\sim}{c} + \underset{\sim}{t} = \underset{\sim}{0} \tag{13}$$

where

$$\underset{\sim}{A} \equiv \begin{bmatrix} \underset{\sim}{r}_{o}^{T}\,\underset{\sim}{N} \\ \hline \underset{\sim}{B}^{T}\,\underset{\sim}{N} \end{bmatrix} \qquad \underset{\sim}{y} \equiv \begin{bmatrix} \lambda \\ \hline \underset{\sim}{p} \end{bmatrix} \qquad \underset{\sim}{c} = \underset{\sim}{m}_{*} - \underset{\sim}{N}^{T}\,\underset{\sim}{m}_{do}$$

and $\underset{\sim}{t}$ are non-negative slack variables.

The plastic collapse deformation rates $(\underset{\sim}{\dot{\theta}})$ for a single-degree-of-freedom mode are fixed only up to a single parameter whose magnitude remains arbitrary. It will therefore be necessary to introduce some form of scaling so that the problem will have a finite kinematic solution. A convenient scaling is

$$\underset{\sim}{r}_{o}^{T}\,\underset{\sim}{N}\,\underset{\sim}{x} = 1 \tag{14}$$

The compatibility relations (6) and the flow rule (8) lead to

$$\underset{\sim}{B}^T \underset{\sim}{N} \underset{\sim}{x} = \underset{\sim}{0} \qquad (15)$$

The kinematic admissibility conditions are obtained from equations (14) and (15)

$$\begin{bmatrix} \underset{\sim}{r}_o^T \underset{\sim}{N} \\ \hline \underset{\sim}{B}^T \underset{\sim}{N} \end{bmatrix} \underset{\sim}{x} = \begin{bmatrix} 1 \\ \hline \underset{\sim}{0} \end{bmatrix}$$

or, more compactly,

$$\underset{\sim}{A} \underset{\sim}{x} = \underset{\sim}{b} \qquad (16)$$

Thus, the full set of fundamental structural relations in mesh form becomes

Statical Admissibility	$\underset{\sim}{A}^T \underset{\sim}{y} - \underset{\sim}{c} + \underset{\sim}{t} = \underset{\sim}{0}$	
Kinematical Admissibility	$\underset{\sim}{A} \underset{\sim}{x} = \underset{\sim}{b}$	(17)
Parity	$\underset{\sim}{x}^T \underset{\sim}{t} = 0$	
Sense restrictions	$\underset{\sim}{x} \geqslant \underset{\sim}{0} \qquad \underset{\sim}{t} \geqslant \underset{\sim}{0}$	

6. LINEAR PROGRAMS

The relations (17) constitute a linear complementarity problem (LCP). If they are regarded as Kuhn-Tucker conditions [10] then, from Kuhn-Tucker equivalence, their solution is also the solution of the following mesh primal-dual LPs of the YLM

Min $z = \underset{\sim}{c}^T \underset{\sim}{x}$ $\underset{\sim}{A} \underset{\sim}{x} = \underset{\sim}{b}$ $\underset{\sim}{x} \geqslant \underset{\sim}{0}$	Max $w = \underset{\sim}{b}^T \underset{\sim}{y}$ $\underset{\sim}{A}^T \underset{\sim}{y} \leqslant \underset{\sim}{c}$	(18) (19)
Mesh Primal LP	Mesh Dual LP	

7. SOLUTION BOUNDS

From the duality theory [11] of LP, it follows that the optimal values of the two objective functions coincide and are equal to the collapse load (λ_c) for the YLM-FE model.

$$z_* = w_* = \lambda_c \qquad (20)$$

Since the necessary compatibility requirements are satisfied, the plastic collapse load parameter (λ_c) for the YLM-FE model is an *upper bound* to the collapse load parameter (λ_c^c) for the continuous model.

$$\lambda_c \geqslant \lambda_c^c \qquad (21)$$

Clearly, if the FE boundaries contain the yield lines of the true collapse mode of the continuous model, then the strict equality applies in relation (21).

The upper bound nature of the YLM-FE model also applies when a nodal description [7] is adopted. However, it has been shown elsewhere [9][12][13] that an FE formulation using approximating field functions can be devised such that the collapse load parameter is a lower bound on that of the continuous model.

8. COMPUTATIONAL EFFORT

If n_c is the number of constraints and n_v is the number of variables in the standard form of an LP, then the computational effort involved in a simplex-based computer code varies as $(n_c^3\ n_v)$. It can be shown that the primal (unsafe) LP always involves less computation than the dual (safe) LP, irrespective of the description (nodal or mesh) used. The choice therefore lies between the nodal primal LP and the mesh primal LP for a YLM-FE model. The comparison between these two programs with respect to computational effort depends, to some extent, on the boudnary conditions. However, as the number of FEs increases and the FE network tends to an infinitely fine one, then the influence of boundary conditions becomes less important and the nodal primal LP tends to require 450% of the computational effort of the corresponding mesh LP.

Another important consideration is the complexity of data preparation and organisation prior to entering the simplex-based code. Here the position with regard to the mesh description is, as yet, less satisfactory. However, this was also considered to be a disadvantage with respect to the mesh LPs for frames, but recent developments have largely overcome the problems and further research should improve the position with regard to slabs.

9. CONCLUSIONS

The mesh description, which has proved to be particularly convenient with respect to frames, can readily be adapted to a YLM-FE model of a reinforced concrete slab. The mesh primal-dual LPs for the slab problem have the same algebraic form as those for frames. Whilst the data preparation may require more attention, the computational effort required for the solution of the mesh primal LP is generally considerably less than that for the corresponding nodal primal LP.

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