Reinforced concrete corbels - some exact solutions

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Reinforced Concrete Corbels — Some Exact Solutions

Consoles en béton armé — quelques solutions complètes

Stahlbetonkonsolen — einige vollständige Lösungen

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SUMMARY
The paper presents some equations for the load carrying capacity of reinforced concrete corbels. The solutions are exact solutions based on the classical theory of plasticity.

RESUME
La résistance ultime des consoles en béton armé est examinée en appliquant la théorie classique de la plasticité. Quelques solutions complètes sont obtenues.

ZUSAMMENFASSUNG
Die Tragfähigkeit von Stahlbetonkonsolen wird mit Hilfe der klassischen Plastizitätstheorie untersucht. Einige vollständige Lösungen werden angegeben.

1. INTRODUCTION

In this paper some lower and upper bounds for the load carrying capacity of reinforced concrete corbels will be presented.

The solutions are based on the assumptions of zero tensile strength in concrete and plane stresse field. Concrete and reinforcement are idealized to rigid-perfectly plastic materials. The bars carry forces in the axial directions only. The yield criterion of concrete is the well known square yield criterion, see fig. 1. From the work on shear problems carryed out in Denmark, it is known that the strength of concrete cannot be expected to be the normal uniaxial compressive strength f_c, see f.ex. [1]. The same condition is well known from normal bending calculations. To take this into account we introduce the effective compressive strength νf_c, where $\nu \leq 1$.

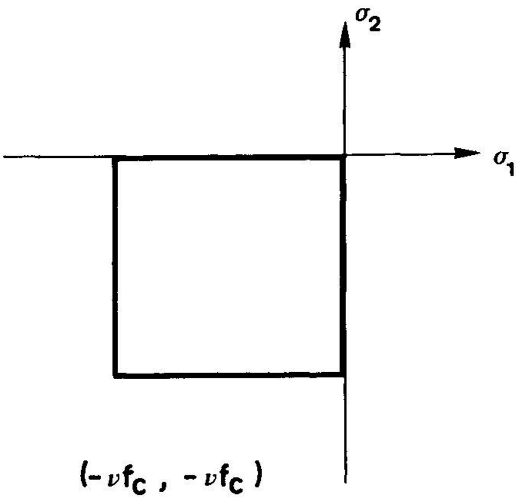

Fig. 1. Square yield criterion for concrete.

2. HORIZONTAL CONCENTRATED REINFORCEMENT

2.1 Lower bound solutions

Consider the stress destribution shown in fig. 2. The force T in the reinforcement is transferred to the concrete at the length EF. In the regions ABC and DEF there are hydrostatic compression. In the region BCDF there is unaxial compression. The hydrostatic stress is equal to the unaxial stress, and both are equal to the concrete strength νf_c.

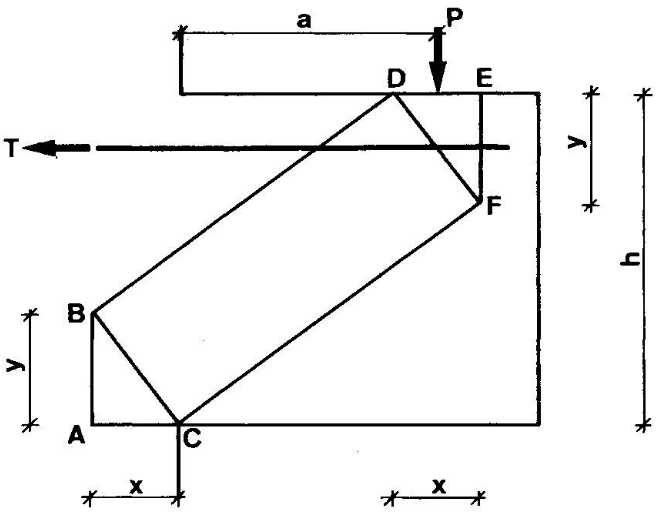

Fig. 2. Stress destribution.

From right-angled triangles in fig. 2 we get

$$(a + \frac{x}{2})^2+(h-y)^2 = (a-\frac{x}{2})^2+ h^2 - (x^2 + y^2) \tag{1}$$

From this equation we get

$$x = -a + \sqrt{a^2 + 2y(h-y)} \tag{2}$$

The lower bound solution is thus

$$P = x\,\nu\,\sigma_c \qquad (3)$$

Using $\tau = {}^{P}/h$ we get

$$\frac{\tau}{\sigma_c} = -\nu\frac{a}{h} + \sqrt{(\nu\frac{a}{h})^2 + 2\nu\frac{y}{h}(\nu - \nu\frac{y}{h})} \qquad (4)$$

Maximum for (4) is found for ${}^{y}/h = \frac{1}{2}$. However, the tensile force T is equal to the compression force at the region EF, i.e.

$$T = y\,\nu\,f_c \leq A\,f_y \qquad (5)$$

Here is A the area of the reinforcement and f_y is the yield strength. We introduce the degree of reinforcement $\Phi = \frac{A\,f_y}{h\,f_c}$. From (5) we then get $y = \Phi\,{}^{h}/\nu$ but with maximum ${}^{y}/h = \frac{1}{2}$.

Inserting in (4) we get

$$\frac{\tau}{\sigma_c} = \begin{cases} -\nu\frac{a}{h} + \sqrt{(\nu\frac{a}{h})^2 + 2\,\Phi(\nu - \Phi)} & \Phi \leq \frac{\nu}{2} \qquad (6a) \\ -\nu\frac{a}{h} + \nu\sqrt{(\frac{a}{h}) + \frac{1}{2}} & \Phi \geq \frac{\nu}{2} \qquad (6b) \end{cases}$$

These equations are lower bound solutions if the reinforcement is placed with an effective height $h_e = h - \frac{1}{2}y$ or

$$h_e = \begin{cases} h(1 - \frac{1}{2}\frac{\Phi}{\nu}) & \Phi \leq \frac{\nu}{2} \qquad (7a) \\ \frac{3}{4}h & \Phi \geq \frac{\nu}{2} \qquad (7b) \end{cases}$$

If the effective height is less than given in equation 7 the stress destribution in fig. 3 can be used.

The stress destribution is in fact the same as in fig. 2, except the region GHDE. In this region we have uniaxial stress equal to the concrete strength νf_c.

The lower bound solutions are found in the same way as (6), and we get

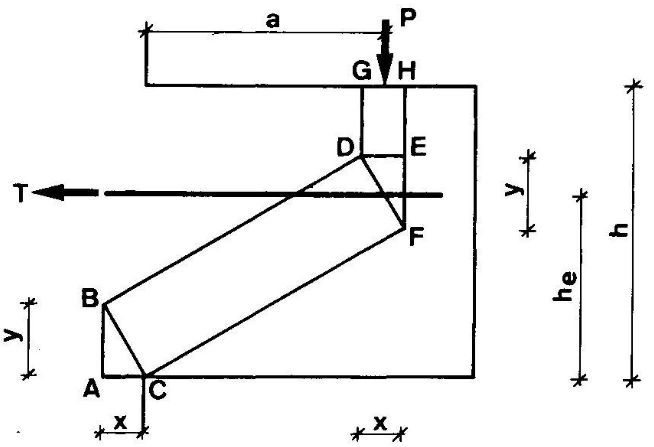

Fig. 3. Stress destribution

$$\frac{\tau}{\sigma_c} = \begin{cases} -\nu\frac{a}{h} + \sqrt{(\nu\frac{a}{h})^2 + \Phi(2\nu\frac{h_e}{h} - \Phi)} & \Phi \leq \nu\frac{h_e}{h} \qquad (8a) \\ -\nu\frac{a}{h} + \nu\sqrt{(\frac{a}{h})^2 + (\frac{h_e}{h})^2} & \Phi \geq \nu\frac{h_e}{h} \qquad (8b) \end{cases}$$

These lower bound solutions are valid when h_e is less than given in (7). If h_e is greater we cannot use this consideration because the tensile force in the reinforcement now is determined by the distance $h - h_e$. At the limit $h = h_e$ the lower bound in this way will be zero.

If h_e is greater than given in (7) we can use the stress destribution in fig. 4. In this case the tensile force is transferred to the concrete as a shear stress $\tau = T/z$. In the region ABC there is hydrostatic compression and in region BCDF there is uniaxial compression. All stresses are equal to the concrete strength νf_c.

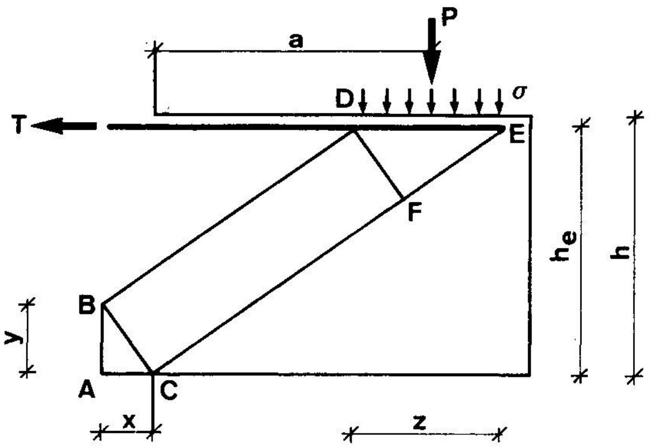

Fig. 4. Stress destribution.

From right-angled triangles in fig. 4 we get

$$(a - \frac{x}{2})^2 + (h_e - \frac{y}{2})^2 - x^2 - y^2 = (a + \frac{x}{2})^2 + (h_e - \frac{y}{2})^2 \qquad (9)$$

From (9) we find

$$x = -a + \sqrt{a^2 + y(2h_e - y)} \qquad (10)$$

Also from geometrical considerations we get

$$z = \frac{h_e\sqrt{a^2 + y(2h_e - y)} + y\,a - h_e\,a}{h_e - \frac{y}{2}} \qquad (11)$$

Now we regard the equlibirum of the triangle DEF. The moment equation gives

$$\sigma = \nu\sigma_c \frac{x^2 + y^2}{z^2} \qquad (12)$$

And then the load P turns out to be

$$P = \sigma z = \nu\sigma_c(-a + \sqrt{a^2 + y(2h_e - y)}\,) \qquad (13)$$

This force is equal to the vertical force on AC.

Maximum for (13) is found for $y = h_e$. However, horizontal projektion gives

$$T = y\,\nu\,\sigma_c = \tau\,z \leq A f_y \qquad (14)$$

This means that $y = \Phi\,h/\nu$, but with maximum $y = h_e$. Inserting in (13) we get the same lower bound solutions as in (8), which then i a general lower bound solution. The stress destribution in fig. 4 can of course be used instead of the stress destribution in fig. 2 and fig. 3. However, the way of transferring the tensile force to the concrete is quite different.

2.2 Upper bound solutions

The failure mechanism is shown in fig. 5. Part II rotates the angle α about A. The yield line AB has pure compression and AC is a tensile crack. The tensile crack reaches the upper side of the corbel at an arbitrary place inside the load. The work equation consist of 3 contributions W_E, W_{IR}, W_{IC} from the external force, the reinforcement and the concrete.

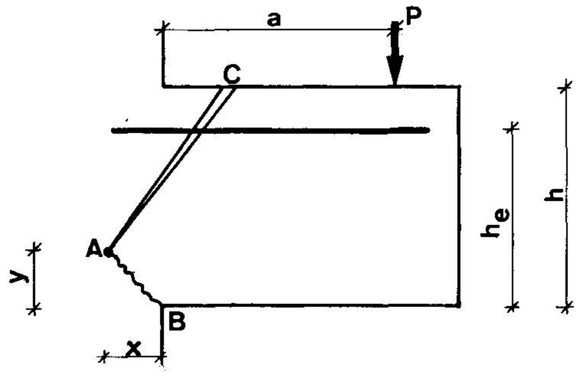

Fig. 5. Failure mechanism.

The contributions are

$$W_E = P(a + x)\alpha \tag{15}$$

$$W_{IR} = A f_y (h_e - y)\alpha \tag{16}$$

$$W_{IC} = \tfrac{1}{2} \nu f_c (x^2 + y^2)\alpha \tag{17}$$

The work equation $W_E = W_{IR} + W_{IC}$ then yields

$$\frac{P}{f_c} = \frac{\frac{1}{2}\nu(x^2 + y^2) + \Phi h(h_e - y)}{a + x} \tag{18}$$

where $\Phi = (A f_y)/(h f_c)$

Minimizing with respect to x and y we get

$$y = \frac{\Phi}{\nu} h \tag{19}$$

$$x = -a + \sqrt{a^2 + \frac{\Phi h}{\nu}(2 h_e - \frac{\Phi h}{\nu})} \tag{20}$$

Inserting in (18) we get the minimum

$$\frac{\tau}{\sigma_c} = -\nu \frac{a}{h} + \sqrt{(\nu \frac{a}{h})^2 + \Phi(2\nu \frac{h_e}{h} - \Phi)} \tag{21a}$$

The upper limit for the validity of (21a) is the contribution 0 from the reinforcement, that is $h_e = y$ or $\Phi = (h_e/h)$. If the degree of reinforcement is greater we get

$$\frac{\tau}{\sigma_c} = -\nu\frac{a}{h} + \nu\sqrt{(\frac{a}{h})^2 + (\frac{h_e}{h})^2} \qquad \Phi \geq \nu\frac{h_e}{h} \qquad (21b)$$

The upper bound solutions (21) are identical with the lower bound solutions (8). The solutions therefore are exact.

3. INCLINED REINFORCEMENT

3.1 Lower bound solutions

In case of inclined reinforcement. The stress destribution is shown in fig. 6. In the regions IKFD, KLGF and CDEB there are uniaxial compression and in the regions ABC, DFE and FGE there are hydrostatic compression. All stresses are equal to the concrete strength. The side EF is perpendicular to the reinforcement and the reinforcement is passing through the middle of EF.

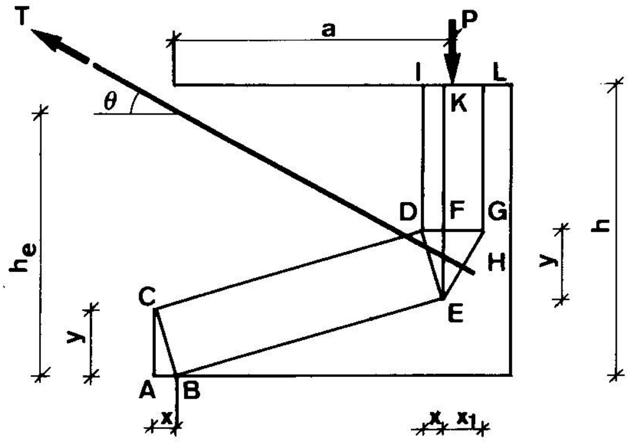

Fig. 6. Stress destribution.

From the geometri we can find two different expressions for the diagonal BD. Then we get

$$(a - \tfrac{1}{2}(x + x_1))^2 + (h_e - (a + \tfrac{1}{2}x)\tan\theta + \tfrac{1}{2}y)^2 - (x^2 + y^2)$$

$$= (a + \tfrac{1}{2}x - \tfrac{1}{2}x_1)^2 + (h_e - (a + \tfrac{1}{2}x)\tan\theta - \tfrac{1}{2}y)^2 \qquad (22)$$

In this we have $x_1 = y\tan\theta$ and from (22) we find

$$x = -a + \sqrt{a^2 + y(2h_e - 2a\tan\theta - y)} \qquad (23)$$

The lower bound solutions is thus

$$P = \nu\,\sigma_c(x + x_1) \qquad (24)$$

and we get

$$\frac{P}{\sigma_c} = \nu\,y\tan\theta + \nu\sqrt{a^2 + y(2h_e - 2a\tan\theta - y)} - \nu a \qquad (25)$$

Maximum for (25) is found for

$$y = h_e - a\tan\theta + \sin\theta\sqrt{h_e^2 + (\frac{a}{\cos\theta})^2 - 2a\,h_e\tan\theta} \qquad (26)$$

However the tensile force must be equal to the compression force on the side EF, which yields

$$y \nu f_c \le A f_y \cos\theta \qquad (27)$$

This means that

$$y = \frac{\Phi}{\nu} h \cos\theta \qquad (28)$$

and with the maximum given by (26).

Inserting in (25), and using $\tau = P/h$, we get the lower bound solutions

$$\frac{\tau}{\sigma_c} = \begin{cases} \Phi \sin\theta - \nu \frac{a}{h} + \sqrt{(\nu\frac{a}{h})^2 + \Phi\cos\theta(2\frac{h_e}{h}\nu - 2\frac{a}{h}\nu\tan\theta - \Phi\cos\theta)} & (29a) \\ \nu\frac{h_e}{h}\tan\theta - \nu\frac{a}{h}(1+\tan^2\theta) + \frac{\nu}{\cos\theta}\sqrt{(\frac{h_e}{h})^2 + (\frac{a}{h\cos\theta})^2 - \frac{2 a h_e}{h^2}\tan\theta} & (29b) \end{cases}$$

(29a) is valid when

$$\Phi\cos\theta \le \nu\frac{h_e}{h} - \nu\frac{a}{h}\tan\theta + \frac{\nu}{h}\sin\theta\sqrt{h_e^2 + (\frac{a}{\cos\theta})^2 - 2 a h_e \tan\theta} \qquad (30)$$

3.2 Upper bound solutions

The failure mechanism is the same as in fig. 5, see fig. 7.

The contributions to the work equation from concrete and from the external load are given by (17) and (16). The contribution from the reinforcement is determined by the distance from A perpendicular to the reinforcement. This distance is $(h_e - y)\cos\theta + x\sin\theta$, and the work equation then is

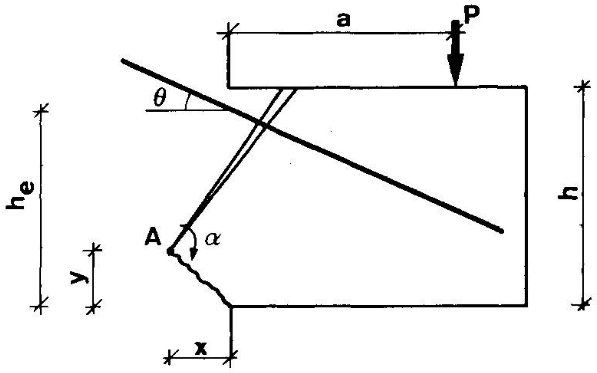

Fig. 7. Failure mechanism.

$$\frac{P}{\sigma_c} = \frac{\frac{1}{2}\nu(x^2+y^2) + \Phi h((h_e - y)\cos\theta + x\sin\theta)}{a + x} \qquad (31)$$

Minimum is found for

$$y = \frac{\Phi}{\nu} h\cos\theta \qquad (32)$$

$$x = -a + \sqrt{a^2 + \frac{\Phi}{\nu} h\cos\theta\,(2 h_e - 2 a\tan\theta - \frac{\Phi}{\nu} h\cos\theta)} \qquad (33)$$

Inserting in (31) we find an upper bound solution equal to (29a). The upper limit for the validity of this equation is the contribution 0 from the reinforcement, that is

$$(h_e - y)\cos\theta + x\sin\theta = 0 \qquad (34)$$

Inserting (33) in (31) we get (26), and then we get the same limit as in (30). If (30) is not fulfilled, the upper bound solution is (29b).

4. CONCLUSION

The developed upper bound solutions are identical with lower bound solutions. Therefore the solutions (8) and (29) are exact solutions.

In the same way as here, the load carrying capacity of corbels with destributed reinforcement and with combined horizontal and inclined reinforcement can be found. Fig. 8 shows the stress destribution with horizontal and inclined reinforcement.

It is noteworthy that the failure mechanism in all cases is a sort of bending-shear failure. Only in a very special case, a sliding failure gives the right upper bound solution. The sliding failure mechanism corresponds to the stress destribution in fig. 2, where the yield line will appear between C and D. The load carrying capacity for corbels presented in [1] is therefore only exact when (7) is fulfilled.

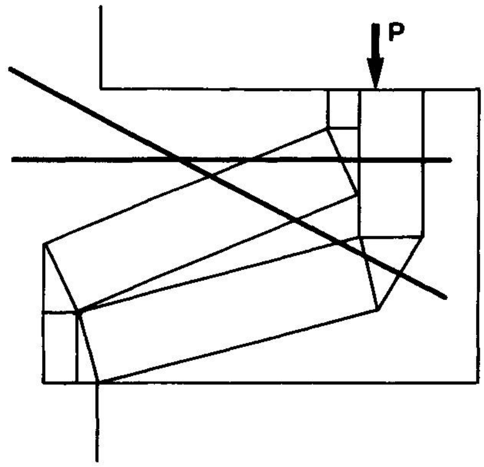

Fig. 8. Stress destribution.

The results presented here can of course be transferred to the shear capacity of beams without shear reinforcement.

Just now we carry out a comparesion of the equations with tests.

5. REFERENCES

[1] Nielsen, M.P., M.W. Bræstrup, B.C. Jensen, & F. Bach: Concrete Plasticity, Special Publication, Danish Society of Structural Science and Engineering, Copenhagen 1978, pp 129.