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Optimality Problem in Artificial Intelligence

Problème d'optimisation de l'intelligence artificielle

Optimierungsproblem in der künstlichen Intelligenz-Technologie

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SUMMARY

The combinatorial searching, an essence of Artificial Intelligence Technology, plays an important role in structural decision problems including the present failure load analysis of frame systems. Both generation of a combined failure mode and test of a failure load factor should be completed over all combinations between elementary failure modes with a result of explosive increase of burden on computer which can be, herein, improved effectively by means of a heuristic rule of the similarity index. Thus, the present method becomes a powerful tool when the mode approach is applied to reliability analysis or design.

RÉSUMÉ

La recherche combinatoire est une partie essentielle de la technologie de l'intelligence artificielle et joue un rôle important dans les problèmes impliquant des choix déterminants dans les ouvrages, par exemple dans la détermination de la charge de rupture des cadres. Il faudrait prendre en compte toutes les combinaisons possibles de mécanismes élémentaires, aussi bien dans la génération d'un mécanisme de rupture combinée que dans l'essai d'un facteur de charge ultime; ceci entraînerait un accroissement de type explosif du volume des calculs à l'ordinateur. Une règle heuristique de l'indice de similitude est plus efficace, grâce auquel la méthode des mécanismes devient plus performante dans l'analyse de fiabilité et dans le dimensionnement.

ZUSAMMENFASSUNG

Wesentlicher Bestandteil in der Technologie der künstlichen Intelligenz ist das kombinatorische Suchen. Es spielt auch eine wesentliche Rolle bei Entscheidungen im konstruktiven Ingenieurbau, etwa bei der Bestimmung der Grenztragfähigkeit von Rahmensystemen. Sowohl bei der Generierung eines kombinierten Versagensmechanismus als auch beim Test eines Grenzlastfaktors sollten alle denkbaren Kombinationen der Elementarmechanismen einbezogen werden, wodurch der Rechenaufwand explosionsartig grösser würde. Als effizientes Mittel hat sich eine heuristische Regel des Aehnlichkeitsindex erwiesen, mit der die Mechanismusmethode in der Zuverlässigkeitsanalyse und Bemessung sehr leistungsstark wird.

1. INTRODUCTION

The development of AI technology was triggered by Dartmouth Summer Conference in 1956 where the role of computer in the future was emphatically discussed. Subsequently, the important idea was realized such as GPS, LISP language and the frame theory. Furthermore, expert systems were developed such as Dendral at Stanford, MYCIN and Prospector which could get their successful position because of dealing with specified subjects though under enormous consumption of man power. The fifth generation project in Japan has aimed at high performance of inference of knowledge base. Through the development of AI technology the effective searching technique in an enormous database theoretically and practically was obviously a main subject to approach. Particularly, when the database consists of combination of elements, its space becomes growing so exponentially that search for optimality becomes significantly laborious and frequently almost impossible because of its nondifferentiability(Polak[1987]). Such combinatorial searching subject as the traveling salesman problem(Kernighan[1973]) can be found not only in application field of AI technology but many structural engineering analyses including the present failure load analysis. Many searching techniques for the combinatorial optimality are developed(Padberg[1987], Lin[1973], Johnson[1989]), among which the branch-and-bound method that is closely related to the dynamic programming is a generalized technique(Ibaraki[1991]). A large class of structural engineering design problems is also transcribed into the form of a nondifferentiable optimization problem with inequality constraints involving maximum function. When dealt with such nondifferentiable optimal problems(discrete optimization), the exhaustive enumeration including the generate-and-test procedure should be inevitably required. This is partly due to a lack of information of extrapolation on a searching space which is explosively enormous practically. It is laborious to describe algorithm of exhaustive enumeration in procedural language such as FORTRAN. On the contrary, declarative languages including Prolog(Clocksin[1984]) can handle it directly. Some important properties of Prolog are backtracking and nondeterminism to search for prescribed goal. The transitivity and inheritance inferences extend the searching efficiency so to large extent that combinatorial problems are more practically approached(Corkill[1983], Fennell[1977]). Thus, regarding nondifferentiable combinatorial optimality problem to accomplish an effective searching for an appropriate goal is equivalent to establishment of the pruning-futile-alternative technique including the branch-and-bound method, the heuristic approach and the qualitative reasoning that can realize sub-dimensionalization of searching space with a result of its rapid shrinkage. Frequently they are applied interactively. Unfortunately such pruning technique depends heavily upon particularity of the problem. Thus, an attempt of its generalization tends to lose sharpness of their efficiency as a result. Herein, the heuristics implies in wide sense a pruning technique to reduce the amount of generate-and-test drastically. Furthermore, the fact is that the rigorous goal cannot be necessarily attained even by the laborious generate-and-test method unless certain problem-oriented pruning technique is applied or unless the problem is relaxed into searching feasible goals. In general, the branch-and-bound method that belongs to exhaustive enumeration methods is applied with the aid of effective algorithms such as depth-first, best-bound and heuristic algorithm. Since the problem-oriented technique or the heuristic algorithm that can prune futile alternatives depends largely upon particularity of the problem and hence incidental human flair, the systematic development of heuristic algorithm becomes almost impossible. Recent conspicuous approaches such as the simulated annealing and neural network technique are mooted with considerable success(Hopfield[1985]).

Failure load analysis of structural systems from kinematically admissible field belongs to a typical combinatorial searching problem. Watwood[1979] proposes the generation of elementary mechanisms and their linear combinations of frames by the linear programming technique. Gorman[1981] presents an automatic method to generate the failure mode equations for all possible failure modes. Systematic generation of failure modes is important, because to add further constraints such as minimum weight criteria and reliability threshold(Henley[5]) more realistic description of structural design can be attained(Ditlevsen[1984], Melchers[1985]). The present study deals with the failure load analysis of rigid-plastic frames by the upper bound theorem which shows a combinatorial problem. When a kinematically admissible mode is assumed, the virtual work equation provides the corresponding failure load factor, γ_j. After generation of elementary failure modes from kinematically admissible displacement fields by topology and geometry(Onodera[1967]) their linear combinations produce successively the remaining failure modes whose predominance should be tested. Thus, the present optimal problem can be described by minimization of the objective function or the virtual work equation of possible failure modes.

2. ESSENCE OF AI TECHNOLOGY - COMBINATORIAL OPTIMALITY

Practical implementation of discrete optimality requires any of enumeration approaches such as the dynamic programming technique(DP), the branch-and-bound method and the exhaustive enumeration. DP has limitation to combination problem to some extent(Miyamura[1992]). The branch-and-bound method is applicable to widely diversified problems. It is accepted as a method to transform the combinatorial problem, which is difficult to solve directly by recursive decomposing, into partial problems until a set of more simplified problems. These partial problems have less large number of parameters while the number of problems to solve demand large amount of computing time due to explosive increase of combination. The branch-and-bound method can be summarized as follows: First, initialization of both the tentative value of evaluation function equal to infinitive, $z = \infty$, and the active partial problem the original problem, P_o, to solve. Second, searching for a new active partial problem, P_i, or ending for no more active partial problem. Third, testing P_i and a new z is obtained for a upper bound of feasible solution, $\{x\}$. Lastly, branching of descendent partial problems to add the active space and searching.

Thus, the branch-and-bound method tells that when fail occurs by test for any solution generated from an active partial problem, then further branch operation is not required with a result of decrease of combinatorial generate-and-test. Regarding searching for a new partial problem, P_i, this can be attained by the following two criteria: First, when the optimal solution is obtained from a partial problem, P_i, it is not necessary to deploy further branch operation. Second, if a partial problem cannot provide optimal solution of the original problem, it is not necessary to extend further branch operation.

These two criteria to halt further branch operation is the bound operation that thus can terminate the partial problem, P_i. Practical implementation of the bound operation can be made by either the lower bound test based upon relationship between optimal solutions from relaxed problems and admissible solutions or the dominance test based upon binary relationship of the evaluation function and constraint between two partial problems, P_k and P_l. The conventional exhaustive enumeration or blind searching corresponds to the case that the evaluation fuction, $f(\{x\})$ can be calculated after completion of branch operations and a set of feasible solutions are obtained from parameter vector including the optimal solution.

The present failure load analysis relates closely to the combinatorial optimality problem in the sense that a minimum load factor should be searched between possible kinematically admissible fields or failure modes including linear combination modes of elementary modes. Conventional LP(linear programming method) requires combination of k elementary modes to determine mode weighting coefficient, C_i, to optimize an evaluation function, where $C_i \neq 0$ for N active modes and $C_i = 0$ for non-active modes. Thus, combinatorial searching is accomplished for any N modes from k modes, and a memory size of combination defined by numbers of both elementary mode and member become practically enormous. On the contrary, the branch-and-bound method does not necessarily require a large memory size for searching optimality, when effective rules, frequently from heuristic knowledge, can bound non-active searching trees or descent futile alternatives. Herein, two bounding rules or heuristics are applied: generation of complete failure mode with one degree-of-freedom by the recursive expression of combination and pruning by the similarity index, S_{ij}, that can estimate similarity of plastic hinge distribution between two failure modes. Between the present generate-and-test technique and the conventional branch-and-bound method there is a significant difference: the evaluation function from virtual work equation cannot guarantee monotony. This suggests necessity of exhaustive enumeration of a larger combination space. However, any even higher order combination requires at least to possess a common plastic hinge between combined modes. This becomes less possible for the higher order combinations, which is empirically recognized from numerical simulations.

3. COMBINED MODE AND HEURISTICS

The present generate-and-test consists of both generation of failure mode and test of failure load factor. Thus, the generate process is classified into two categories: generation of elementary failure modes and their linear combinations. When a rigid-plastic plane frame with m-members, n-nodes and n_r-fixed supports collapses with plastic hinges subject to nodal loading, elementary failure modes, whose number is $(3n - m)$, can be expressed as follows:

$$\{\tau_p\} = [C]\{_I\tau_{pt}\} \tag{1}$$

where $\{\tau_p\}$ means the plastic hinge rotation vector at ends of member, $\{_I\tau_{pt}\}$, the corresponding independent hinge rotation vector of tree members, respectively. The displacements at nodal point can be expressed by means of the path matrix, $[H]$:

$$\begin{aligned} \{D_x\} &= [H][\mu][L][_cH_{mt}]\{\tau_{pt}\} \\ \{D_y\} &= -[H][\lambda][L][_cH_{mt}]\{\tau_{pt}\} \end{aligned} \tag{2}$$

where $\{D_x\}$ and $\{D_y\}$ mean nodal displacement vectors in x- and y-directions, respectively. After generation of elementary failure modes a combined mode is expressed by their linear combination. To implement the generate-and-test effectively it is preferable to describe the combination process in recursive form. When both the hinge rotation vector and nodal displacement vector of the i-th elementary mode is expressed by $\{Y_i\} = \{\{\tau_p\}_i^t, \{D\}_i^t\}$, the combination of any two elementary modes, $\{Y_i\}$ and $\{Y_j\}$, becomes:

$$C_2(\{Y_i\}, \{Y_j\}|\tau_{p,s}) = A_i\{Y_i\} + A_j\{Y_j\} \tag{3}$$

$$A_i\tau_{pi,s} + A_j\tau_{pj,s} = 0 \tag{4}$$

where $\tau_{pi,s}$ and $\tau_{pj,s}$ mean hinge rotations common to both the i- and j-th modes at the critical section, S. The lefthandside of Eq.(3) means the resulting mode combined $\{Y_i\}$ with $\{Y_j\}$, which are not plastic at the critical section. Extending Eq.(3) to combination of the other modes, the following recursive expression can be obtained:

$$\begin{aligned} &C_k(\{Y_k\}, \{Y_{k-1}\}, \ldots, \{Y_1\}|\tau_{p,k-1}, \tau_{p,k-2}, \ldots, \tau_{p,1}) = \\ &\qquad A_I C_{k-1}^I(\{Y_{k-1}\}, \{Y_{k-2}\}, \ldots, \{Y_1\}|\tau_{p,k-2}, \tau_{p,k-3}, \ldots, \tau_{p,1}) + \\ &\qquad A_{II} C_{k-1}^{II}(\{Y_k\}, \{Y_{k-2}\}, \ldots, \{Y_1\}|\tau_{p,k-2}, \tau_{p,k-3}, \ldots, \tau_{p,1}) \end{aligned} \tag{5}$$

$$A_I\tau_{p,k-1}^I + A_{II}\tau_{p,k-1}^{II} = 0 \tag{6}$$

where $\tau_{p,k-1}^I$ and $\tau_{p,k-1}^{II}$ are the hinge rotations at the $(k-1)$-th critical section common to the $(k-1)$-th combined failure modes, $C_{k-1}^I(\cdot)$ and $C_{k-1}^{II}(\cdot)$. Eq.(5) shows that the k-th combined mode can be decomposed into two $(k-1)$-th modes each of which has the same tail of minus-one order but different head. The recursive expression thus generalized ensures both easy composition and decomposition of failure modes by a simple algorithm. Subsequently, the test procedure should be implemented by evaluation of a failure load factor, γ_j, which is given by the following virtual work equation for a generated failure mode:

$$\gamma_j = \sum_{k=1,2,\cdots,n} C_k\{\tau_{pk}\}^t\{M_p\} / \sum_{k=1,2,\cdots,n} C_k\{D_k\}^t\{P\} \rightarrow min \tag{7}$$

where C_k means a weighing coefficient of the k-th mode from n elementary failure modes $(0 \leq C_k \leq 1)$. Summation is implemented for any number of combinations less than that of n elementary modes. $\{\tau_{pk}\}$ means the hinge rotation vector, $\{M_p\}$, the member yielding resistance vector, $\{D_k\}$, the nodal displacement vector, and $\{P\}$, the external nodal loading ratio vector, respectively. Eq.(7) shows that the goal failure load factor, γ_{cr}, is the lowest factor derived from possible failure modes given by both elementary failure modes and their linear combinations. This implies that for a number of elementary failure modes their combinations become exponentially increasing, which is subjected to a combinatorial searching technique.

In order to implement effective generation-and-test of failure modes it is necessary to develop certain heuristic rules which can prune futile alternatives. In the following the similarity index is used as a heuristic rule. It is difficult to implement effective search of predominant modes either by the exhaustive enumeration method with the generate-and-test or the conventional approximate searching approaches with reduced search space by the depth-first or the breadth-first. This depends upon the fact that the more test of predominance is necessary for the larger number of generation of modes. Thus, if an approximate estimation of effective combination is implemented before actual combination procedure the amount of calculation decreases drastically even for a large sized structural systems. A combination of the smaller internal virtual work to the external one becomes predominant. Hence, when plastic hinge rotations decrease by the combination of appropriate modes the corresponding internal work decreases. This can be more accomplished for two candidate modes whose common hinges become larger in number, in other words, whose hinge distribution becomes more similar. Thus, pruning of the futile searching space is attained if it is possible to evaluate an extent of similarity of hinge distribution with less burden. Such evaluation is established by enumeration of both common and non-common plastic hinges between the i- and j-th modes, and satisfied to some extent by the following similarity index, S_{ij}:

$$S_{ij} = \sum_s min(h_{is},\ h_{js}) / \sum_s max(h_{is},\ h_{js}); \quad i,j = 1,\ldots,k \tag{8}$$

where h_{is} is a binary parameter given by:

$$h_{is} = \begin{cases} 0, & \text{if } \tau_{pis} = 0; \\ 1, & \text{otherwise.} \end{cases} \tag{9}$$

Eq.(8) provides the ratio of the numbers of common and non-common hinges between any two modes. Thus, $h_{is} = 1$ corresponds to a plastic hinge at the s section in the i-th failure mode. Furthermore, $0 \leq S_{ij} \leq 1$ is valid. Applying Eq.(8) to all of k elementary failure modes the plastic similarity index matrix, $k \times k$, can be obtained which is reflexive and symmetric but not transitive like of the fuzzy similarity. Consequently the present heuristics becomes: First, before implementation of the conventional generate-and-test for the exhaustive enumeration the similarity index, S_{ij}, by Eq.(8) should be firstly evaluated and pruned if not tolerable. Empirically, $0.3 \sim 0.4 \leq S_{ij}$ is preferable. This is applicable to further higher order combination modes whose similarity indices are easily evaluated by Eq.(8) recursively with substitution of both $i = n$ and $j = n-1$. Second, whenever there exists no common plastic hinge for combination of more than three modes by Eqs.(5) and (6), further searching can bound even with $S_{ij} \neq 0$.

4. NUMERICAL SIMULATION

A 12 story, 3 bay rectangular frame with 120 members subjected to vertical and horizontal proportional loading(Fig.1) is analyzed by the present method that is described in Prolog language on PC9801 personal computer. Prolog predicate has non-determinism by its backtracking ability which easily generate combination modes and automatically implement branching operation subject to generation rule. As a side-effect due to non-determinism a number of futile alternatives(combinations) appear, and should be pruned. It is advantageous to avoid floating calculation as far as possible. Furthermore, Prolog predicates of recursive rule with non-determinism can play role of both the generate and the test, which is significantly effective. Fig. 1 shows a typical combinatorial searching from 132 elementary modes around the optimal combination with $\gamma_{opt} = 3.176$. However, generation of elementary modes is irrelevant to loading condition which can change order of γ corresponding to the elementary modes. This implies that the elementary mode that provides the lowest load factor between elementary modes is expected to participate combinations which include γ_{opt} or its vicinities. The present heuristic bounding by S_{ij} limitation can effectively prune futile alternatives(modes) although it does not guarantee optimality. Thus, this heuristics provides an upper bound, and is effective for lower order combinations such as two-mode combination. S_{ij} becomes smaller with higher order combination with an elementary mode. The heuristic bounding by Eq.(5) prune combination that has not at least a common hinge even with $S_{ij} \neq 0$ (Note that the combination of elementary modes, [1+49+55], becomes fail in Fig.1). This becomes more prominent when the

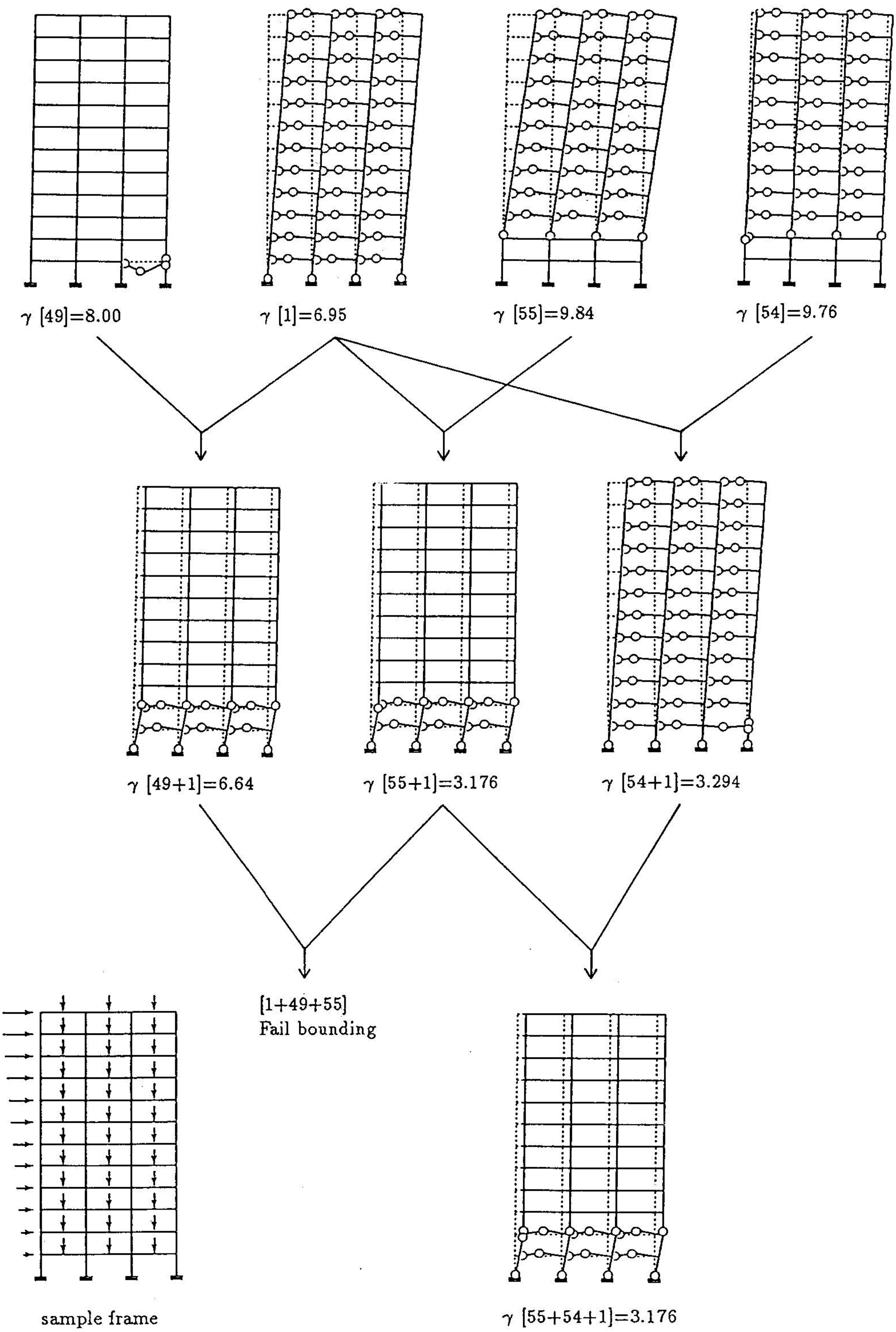

Fig.1 Example and failure modes

order of combination becomes higher. Numerical results suggest superiority of the depth-first combination from elementary modes in order of ascent of γ.

5. CONCLUDING REMARKS

The failure load analysis based on statically admissible field is a typical combinatorial optimality problem, which can be approached by the generate-and-test with heuristics in AI technology. Hence, the following concluding remarks are obtained:

a) The present searching is implemented on a multi-branch tree so that the corresponding generate-and-test can be accomplished by parallel procedure. Consequently, such declarative language as PARALOG is expected more drastic acceleration of searching for practical system.
b) Practically LP requires a larger memory size. While the present method can generate predominant modes (smaller γ_i) with a smaller memory size that are applicable to reliability analysis by the mode approach.
c) Although the present evaluation function by Eq.(7) does not guarantee monotony after recursive combination by Eq.(5), its pruning can realize significant decrease of searching space.
d) It is expected that topological measurement of frames can accelerate further pruning of futile alternatives.
e) $S_{ij} = 0$ corresponds to the exhaustive enumeration that can provide the optimal solution or the lowest load factor. Practically, to save computing time a tolerable value is taken with a result of upper bound.

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APPENDIX

Eqs.(1) and (2) can be derived by topology and geometry of a frame as follows: Any member in a frame corresponds to an oriented edge and by introduction of an imaginary member at supports connecting to the fixed point, O, resulting in an oriented graph. The compatibility condition of rigid body displacements of each member at a failure state becomes the closing condition that the sum of rigid body rotations at a circuit in an oriented graph should be zero:

$$\begin{aligned} \sum_{circuit} du_{ij} &= \sum_{circuit} \mu_{ij}\psi_{ij}l_{ij} = 0 \\ \sum_{circuit} dv_{ij} &= \sum_{circuit} \lambda_{ij}\psi_{ij}l_{ij} = 0 \end{aligned} \qquad (a)$$

where du_{ij} and dv_{ij} mean displacements of a member, i-j, in x- and y-directions due to rigid rotation, ψ_{ij}. λ_{ij} and μ_{ij} are the x- and y-direction cosines and l_{ij}, the member length, respectively. For all independent circuits the following relations are obtained:

$$\begin{aligned} [R][\mu][L]\{\psi\} &= 0 \\ [R][\lambda][L]\{\psi\} &= 0 \end{aligned} \qquad (b)$$

where $\{\psi\}$ means the rigid body rotation vector and $[\mu]$, $[\lambda]$ and $[L]$, the diagonal matrices with elements, μ_{ij}, λ_{ij} and l_{ij}, respectively. A fundamental circuit matrix, $[R]$, has the following elements:

$$R(c,e) = \begin{cases} 1, & \text{when a fundamental circuit, c, includes an edge, e, positively,} \\ -1, & \text{when a fundamental circuit, c, includes an edge, e, negatively,} \\ 0, & \text{otherwise.} \end{cases} \qquad (c)$$

In Eq.(b) the rigid body rotations of imaginary members are assumed zero. The number of fundamental circuits becomes $(m-n)$, and the size of $[R]$, $(m-n)\times(m+n_r)$. The plastic hinge rotations at the ends of a member, i-j, correspond to $\tau_{pij} = \theta_i - \psi_{ij}$ and $\tau_{pji} = \theta_j - \psi_{ij}$. By applying the connection matrix, $[D_m]$, of the expanded graph with introduction of a new node at the middle of an edge this rotation vector becomes:

$$\{\tau_p\} = [D_m]^t\{\{\theta\}^t, \{\psi\}^t\} \qquad (d)$$

where $\{\theta\}$ means the nodal rotation vector. The connection matrix, $[D_m]$ has the following elements:

$$D_m(v,e) = \begin{cases} 1, & \text{when an edge, } e\text{, leaves a node, } v, \\ -1, & \text{when an edge, } e\text{, enters a node, } v, \\ 0, & \text{otherwise.} \end{cases} \qquad (e)$$

Since the orthogonality, $[R_m][D_m]^t = [0]$, is valid for the circuit and connection matrices, by premultiplying the fundamental circuit matrix to Eq.(d) the following expression is derived:

$$[R_m]\{\tau_p\} = \{0\} \qquad (f)$$

Eq.(f) implies thus the compatibility of the vector, $\{\tau_p\}$, the closing condition that the sum of rotation at plastic hinges in a fundamental circuit becomes zero. When a set of edges of the expanded graph are separated into those of trees and cotrees, the path matrix, $[H_{mt}]$, can be derived. Since a fundamental circuit consists of the tree and cotrees, the relation, $[H_{mt}][D_m]^t = [E]$, where $[E]$ means a unit matrix, is defined. Thus, by premultiplying $[H_{mt}]$ to Eq.(d), the following equation is obtained:

$$[H_{mt}]\{\tau_{pt}\} = \begin{bmatrix} [_jH_{mt}] \\ [_cH_{mt}] \end{bmatrix} \{\tau_{pt}\} = \begin{Bmatrix} \{\theta\} \\ \{\psi\} \end{Bmatrix} \qquad (g)$$

where $[_jH_{mt}]$ and $[_cH_{mt}]$ mean the path matrices from the fixed point to a node and to a middle point of an edge on the tree of the expanded graph, respectively. $\{\tau_p\} = \{\{\tau_{pt}\}, \{\tau_{p\bar{t}}\}\}$, $\{\tau_{pt}\}$ and

$\{\tau_{p\bar{t}}\}$ mean plastic hinge rotation vectors corresponding to the tree and the cotree, respectively. Substitution of Eq.(g) into Eq.(b) describes that Eq.(b) has $(3n - m)$ independent solutions. Gaussian elimination provides the following relation:

$$\{{}_D\tau_{pt}\} = [C_I]\{{}_I\tau_{pt}\} \qquad (h)$$

where $\{\tau_{pt}\} = \{\{{}_D\tau_{pt}\}^t, \{{}_I\tau_{pt}\}^t\}^t$. Furthermore, when partitioned such $[R_m] = [[R_{mt}], [R_{m\bar{t}}]] = [[R_{mt}], [E]]$ and $[R_{mt}] = [[{}_D R_{mt}], [{}_I R_{mt}]]$, Eqs.(f) and (h) give:

$$\{\tau_p\} = \begin{Bmatrix} \{\tau_{p\bar{t}}\} \\ \{{}_D\tau_{pt}\} \\ \{{}_I\tau_{pt}\} \end{Bmatrix} = \begin{bmatrix} -[{}_D R_{mt}][C_I] - [{}_I R_{mt}] \\ [C_I] \\ [E] \end{bmatrix} \{{}_I\tau_{pt}\} = [C]\{{}_I\tau_{pt}\} \qquad (1)$$

The size of $[C]$ is $2m \times (3n - m)$, whose column vector, $\{C_i\}$, means the corresponding hinge rotations to ${}_I\tau_{pti} = 1$. Eq.(1) implies that the number of hinges with non-zero rotation cannot exceed $3(m-n)+1$, which is equal to the degree of redundancy plus one. The column elements of $[C]$ are independent of each other, and deformation elements thus expressed contribute directly to an elementary failure modes. The displacements at nodal points can be expressed by means of the path matrix, $[H]$:

$$\begin{aligned} \{D_x\} &= [H][\mu][L][{}_c H_{mt}]\{\tau_{pt}\} \\ \{D_y\} &= -[H][\lambda][L][{}_c H_{mt}]\{\tau_{pt}\} \end{aligned} \qquad (2)$$

where $\{D_x\}$ and $\{D_y\}$ mean nodal displacement vectors in x- and y-directions, respectively.