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# A partition formula connected with Abelian groups

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Let $p$ be a given prime. The object of this note is to prove the following rather curious result.

*The sum of the reciprocals of the orders of all the Abelian groups of order a power of $p$ is equal to the sum of the reciprocals of the orders of their groups of automorphisms.*

It is well known that the Abelian groups of order $p^n$ stand in (1 — 1) correspondence with the $\omega(n)$ unrestricted partitions of $n$, the partition corresponding to a given Abelian group being called its *type*.

Thus the sum of the reciprocals of the orders of all the Abelian groups of order a power of $p$ is equal to

$$\sum_{n=0}^{\infty} \frac{\omega(n)}{p^n} . \qquad (1)$$

Writing

$$f_n(x) \equiv (1-x)(1-x^2)(1-x^3)\ldots(1-x^n) , \qquad f_0(x) = 1 , \qquad (2)$$

and

$$\varrho = \frac{1}{p} , \qquad (3)$$

the value of the sum (1) is easily seen to be

$$\frac{1}{f_\infty(\varrho)} . \qquad (4)$$

But, by an identity due to Euler, this is the same as

$$\sum_{n=0}^{\infty} \frac{\varrho^n}{f_n(\varrho)} . \qquad (5)$$

And the theorem mentioned above will accordingly follow once we have shown that *the sum of the reciprocals of the orders of the groups of automorphisms of the $\omega(n)$ Abelian groups of order $p^n$ is equal to $\varrho^n | f_n(\varrho)$.*

For partitions we use the notation of Macmahon. Thus, the Abelian group $G$ of order $p^n$ and type

$$(1^{\lambda_1} 2^{\lambda_2} 3^{\lambda_3} \cdots) \qquad (6)$$

is the direct product of cyclic groups, $\lambda_1$ of which are of order $p$, $\lambda_2$ of order $p^2$, $\lambda_3$ of order $p^3$, and so on. Clearly,

$$n = \lambda_1 + 2\lambda_2 + 3\lambda_3 + \cdots . \qquad (7)$$

The partition of $n$ which is *associated* with the partition (6) has the parts $\mu_1, \mu_2, \ldots$ given by

$$\mu_i = \lambda_i + \lambda_{i+1} + \lambda_{i+2} + \cdots \qquad (i = 1, 2, \ldots) \ . \tag{8}$$

Thus, since $\lambda_i \geqslant 0$ for each $i$, we have

$$\mu_1 \geqslant \mu_2 \geqslant \cdots \qquad \geqslant 0 \ . \tag{9}$$

And plainly, from (7) and (8),

$$\mu_1 + \mu_2 + \cdots \qquad = n \ . \tag{10}$$

Conversely, given any partition of $n$ in the form (9), (10), the associated partition (6) is obtained at once by the rule that

$$\lambda_i = \mu_i - \mu_{i+1} \qquad (i = 1, 2, \ldots) \ . \tag{11}$$

The associated partition has a simple meaning for the group $G$. Let $G_k$ denote the characteristic subgroup of $G$ which consists of all elements of $G$ of order $p^k$ or less. Then

$$1 = G_0 < G_1 < G_2 < \cdots < G_m = G \ ,$$

where $m$ is the largest of the type invariants[1]) of $G$, and *the order of* $G_k \mid G_{k-1}$ *is precisely* $p^{\mu_k}$.

Now the order of the group of automorphisms of $G$ can be expressed very simply in terms of the "associated invariants" $\mu_k$. It is[2])

$$\frac{f_{\mu_1-\mu_2}(\varrho)\, f_{\mu_2-\mu_3}(\varrho)\ \cdots}{\varrho^{\mu_1^2+\mu_2^2+\cdots}} \ . \tag{12}$$

And the result we require to prove is the case $x = \varrho$ of the identity

$$\frac{x^n}{f_n(x)} = \sum_{(\mu)} \frac{x^{\mu_1^2+\mu_2^2+\cdots}}{f_{\mu_1-\mu_2}(x)\, f_{\mu_2-\mu_3}(x)\cdots} \ , \tag{13}$$

the sum being taken over all $\omega(n)$ partitions (9), (10) of the number $n$.

The various terms of (13) may be regarded as the generating functions of partitions or compositions of certain definite kinds. For example, the coefficient of $x^N$ on the left of (13) is equal to the number of partitions of $N$ for which the greatest part is exactly $n$. As a first step in the proof of the identity, we shall connect every such partition of $N$ with a particular

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[1]) I. e. $\lambda_m > 0$, $\lambda_{m+1} = \lambda_{m+2} = \cdots = 0$.

[2]) Cf. *A. Speiser*, Theorie der Gruppen von endlicher Ordnung, 3er. Aufl., § 43, Satz 114.

one of the $\omega(n)$ partitions (9), (10) of $n$, and thereby with a particular one of the $\omega(n)$ summands on the right of (13).

This may be done most conveniently by means of the *graph*[3]) of the partition of $N$ in question. Let the parts of this partition, arranged in descending order of magnitude be $N_1, N_2, \ldots$, so that we have

$$\begin{aligned} n = N_1 \geqslant N_2 \geqslant \cdots , \\ N_1 + N_2 + \cdots = N . \end{aligned} \tag{14}$$

Then its graph may be defined to consist of a set of $N$ coplanar lattice-points, viz. all those points whose Cartesian coordinates $(x, y)$ are positive integers satisfying

$$x \leqslant N_y . \tag{15}$$

(When $y$ exceeds the number of parts of (14), we take $N_y = 0$.)

We are now able to define, successively, the numbers $\mu_1, \mu_2, \ldots.$, which correspond to the partition (14).

We take $\mu_1$ to be the greatest integer such that the point $(\mu_1, \mu_1)$ belongs to the graph (15). Next, supposing that $\mu_1, \mu_2, \ldots, \mu_{i-1}$ have already been defined, and that their sum is less than $n$, we define $\mu_i$ to be the greatest integer such that $(\mu_1 + \mu_2 + \cdots + \mu_i, \mu_i)$ is a point of the graph.

It follows at once, from (14) and (15), that the numbers $\mu_i$ so defined satisfy (9) and (10). Plainly, also, the square of $\mu_i^2$ lattice-points having for opposite corners the points $(\mu_1 + \cdots + \mu_{i-1} + 1, 1)$ and $(\mu_1 + \cdots + \mu_i, \mu_i)$ belongs entirely to the graph. Thus, if we write

$$M = N - \mu_1^2 - \mu_2^2 - \cdots , \tag{16}$$

there remain, outside the squares just mentioned, precisely $M$ further points of the graph.

We divide these $M$ remaining points into sets, according to the values of their $x$-coordinates. Let the number of them which lie in the strip $0 < x \leqslant \mu_1$ be $M_1$. And, for any $i > 1$, let the number which lie in the strip $\mu_{i-1} < x \leqslant \mu_i$ be $M_i$. If the number of $\mu$'s is $r$, we obtain in this way a definite composition[4]) of $M$,

$$M = M_1 + M_2 + \cdots + M_r , \tag{17}$$

into $r$ non-negative integers, this composition, like the partition (9), (10), being uniquely determined by the original partition (14) of $N$.

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[3]) *P. A. Macmahon*, Combinatory Analysis, II, 3. Our graph reads upwards, not downwards as in Macmahon.

[4]) A composition is a partition in which the order of the summands is important.

As a final consequence of (14), (15), we remark that (for each $i = 1, 2, \ldots, r$) the $M_i$ points of the $i$-th strip constitute, a translation apart, the graph of a certain partition $P_i$ of $M_i$, these partitions $P_i$ being, just as much as the numbers $M_i$ themselves, uniquely determined by (14). Further, for each $i$, the greatest part of $P_i$ is not greater than $\mu_i$. And, for each $i > 1$, the number of parts of $P_i$ is not greater that $\mu_{i-1} - \mu_i$.

But, conversely, suppose that we choose any set of positive integers $\mu_i$ satisfying (9) and (10), the sum of whose squares does not exceed $N$, and define $M$ by (16); then choose any composition (17) of $M$, one part $M_i$ corresponding to each $\mu_i$, taking care that

$$M_i \leqslant \mu_i (\mu_{i-1} - \mu_i) \qquad (i > 1) \;;$$

and finally, for each $M_i$, choose arbitrarily a partition $P_i$ having its greatest part not greater than $\mu_i$ and having (for $i > 1$) not more than $\mu_{i-1} - \mu_i$ parts.

Then it is obvious that we can reverse our former construction at every step, and arrive at a definite partition (14) of $N$, which has $n$ as its greatest part, and for which the corresponding $\mu$'s, $M$'s and $P$'s are precisely the ones we have chosen.

If, then, we denote by $\psi_{a,b}(x)$ the generating function for the partitions of $N$ into at most $a$ parts none of which exceed $b$, we have proved the identity

$$\frac{x^n}{f_n(x)} = \sum_{(\mu)} \psi_{\infty,\mu_1}(x)\, \psi_{\mu_1-\mu_2,\mu_2}(x)\, \psi_{\mu_2-\mu_3,\mu_3}(x) \ldots x^{\mu_1^2+\mu_2^2+\cdots}, \tag{18}$$

the sum being taken over all partitions (9), (10) of $n$. But it is known[1]) that, for finite $a$ and $b$,

$$\psi_{a,b}(x) = \frac{f_{a+b}(x)}{f_a(x)\, f_b(x)},$$

while

$$\psi_{\infty,b}(x) = \frac{1}{f_b(x)}.$$

Substituting these values in (18), we obtain the required identity (13). This concludes the proof.

(Eingegangen den 17. September 1938.)

[5]) *P. A. Macmahon*, loc. cit., 5.