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Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **17 (1944-1945)**

PDF erstellt am: **29.06.2024**

Persistenter Link: https://doi.org/10.5169/seals-16344

Fundamental Domains for Lattice Groups in Division Algebras II[1]

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C. AN OLD STORY RETOLD (WITH SOME MINOR ADAPTATIONS)

§ 5. Minkowski's Fundamental Inequality

Express the symmetric bilinear form

$$\mathrm{tr}\,(\bar{\xi}\eta) = x' T_0 y$$

in terms of that basis. We compute the determinant d of its coefficients $t^0_{ik} = \mathrm{tr}\,(\bar{\omega}_i \omega_k)$ as follows. If J represents the operation $\xi \to \bar{\xi}$ in terms of the basis ω_i, (11), then

$$\mathrm{tr}\,(\bar{\omega}_i \omega_k) = \Sigma_l j_{li}\, \mathrm{tr}\,(\omega_l \omega_k)$$

or

$$\|\, \mathrm{tr}\,(\bar{\omega}_i \omega_k) \| = J' \cdot \|\, \mathrm{tr}\,(\omega_i \omega_k) \| ,$$

hence

$$d = \pm |\, \mathrm{tr}\,(\omega_i \omega_k) | \, .$$

d is necessarily positive. $\mathrm{tr}\,(\omega_i \omega_k)$ are the coefficients of the bilinear form $\mathrm{tr}\,(\xi\eta)$, which is also symmetric because $\mathrm{tr}\,(XY)$ depends symmetrically on the two matrices X, Y. Thus we find that d is the absolute value of the determinant $|\,\mathrm{tr}\,(\omega_i\omega_k)|$. This absolute value is independent of the choise of the minimal basis $\omega_1, \ldots, \omega_g$ of $\{\mathfrak{F}\}$ and is therefore known as the *discriminant* of $\{\mathfrak{F}\}$. Computation of $\mathrm{tr}\,(\omega_i \omega_k)$ by means of the basis ω_l itself shows that these coefficients and therefore d are rational integers. The non-degeneracy of the symmetric bilinear form $\mathrm{tr}(\xi\eta)$ implied by $d \neq 0$ is an important fact which concerns the division algebra $\mathfrak{F}$ over k (though our proof passes through $\mathfrak{F}_K$ by means of the conjugation $\xi \to \bar{\xi}$).

Lemma 5.1. The determinant d of the quadratic form

$$\mathrm{tr}\,(\bar{\xi}\xi) = x' T_0 x = T_0[x]$$

equals the discriminant of $\{\mathfrak{F}\}$ and is a positive rational integer.

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[1]) *Anmerkung der Redaktion.* Der I. Teil der Arbeit ist in der Festschrift für Andreas Speiser, Orell-Füßli, Zürich, 1945, p. 218, erschienen.

Since the conjugate of $\bar{\xi}\gamma\eta$ is $\bar{\eta}\gamma\xi$, provided γ is symmetric, tr $(\bar{\xi}\gamma\eta) = x'Ty$ is a symmetric bilinear form. Its coefficients are $t_{ik} = \mathrm{tr}\,(\bar{\omega}_i\gamma\omega_k)$. We have

$$\gamma\omega_k = \Sigma_l g_{lk}\omega_l\,,$$

$$\mathrm{tr}\,(\bar{\omega}_i\gamma\omega_k) = \Sigma_l\,\mathrm{tr}\,(\bar{\omega}_i\omega_l)\cdot g_{lk}\,,$$

$$|\,\mathrm{tr}\,(\bar{\omega}_i\gamma\omega_k)| = |\,\mathrm{tr}\,(\bar{\omega}_i\omega_k)|\cdot|\,G\,|\,.$$

Thus:

Lemma 5.2. The determinant of the quadratic form tr $(\bar{\xi}\gamma\xi) = T_\gamma[x]$ equals $d\cdot\mathrm{Nm}\,\gamma$.

In terms of the fixed minimal basis $\omega_1,\ldots,\omega_g$ of the order $\{\mathfrak{F}\}$ we express each component ξ_μ of a vector $\mathfrak{x} = (\xi_1,\ldots,\xi_n)$ by the column of its coordinates $x_{\mu i}$, $\xi_\mu = \Sigma_i x_{\mu i}\omega_i$, and now use the $N = ng$ quantities $x_{\mu i}(\mu = 1,\ldots,n;\ i = 1,\ldots,g)$ as coordinates of $\mathfrak{x}$; they follow one another in the order $\mu i = 11,\ldots,1g;\ 21,\ldots,2g;\ldots.$ The Jacobi transformation (17) then appears as a linear transformation

$$z_\mu = x_\mu + \Sigma_{\nu>\mu}D_{\mu\nu}x_\nu \tag{21}$$

which connects the coordinates $z_{\mu i}$ with $x_{\mu i}$ and has the triangular matrix

$$D = \begin{Vmatrix} E\,, & D_{12}\,, & \ldots\,, & D_{1n} \\ 0\,, & E\,, & \ldots\,, & D_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0\,, & 0\,, & \ldots\,, & E \end{Vmatrix}. \tag{22}$$

Hence (16) and Lemma 5.2 prove that the determinant of the quadratic form $t[\mathfrak{x}] = t_T[\mathfrak{x}]$ of the variables $x_{\mu i}$ equals

$$d^n\cdot\mathrm{Nm}\,\varkappa_1\ldots\mathrm{Nm}\,\varkappa_n\,.$$

A lattice $\mathfrak{A}$ is given, and $\{\mathfrak{F}\}$ with the minimal basis $\omega_1,\ldots,\omega_g$ is the order of its multipliers. Since for any positive t there is only a finite number of lattice vectors $\mathfrak{x}$ for which $t[\mathfrak{x}] \leqslant t$ we can construct the successive minima of $t[\mathfrak{x}]$ as follows: Among all lattice vectors $\mathfrak{x} \neq \mathfrak{o}$ the minimum t_1 of $t[\mathfrak{x}]$ is attained for $\mathfrak{x} = \mathfrak{d}_1$; among all lattice vectors not in $[\mathfrak{d}_1]$ the minimum t_2 of $t[\mathfrak{x}]$ is attained for $\mathfrak{x} = \mathfrak{d}_2$; etc. For any m, $1 \leqslant m \leqslant n$, let $\mathfrak{x}$ range over all lattice vectors not in $[\mathfrak{d}_1,\ldots,\mathfrak{d}_{m-1}]$;

$\boldsymbol{t}[\mathfrak{x}]$ then assumes its minimum t_m for a certain $\mathfrak{x} = \mathfrak{d}_m$. The $\mathfrak{d}_1, \ldots, \mathfrak{d}_n$ thus obtained by induction constitute a semi-basis for $\mathfrak{A}$ such that

$$\boldsymbol{t}[\mathfrak{x}] \geqslant \boldsymbol{t}[\mathfrak{d}_m]$$

whenever $\mathfrak{x}$ is in $\mathfrak{A}$ but not in $[\mathfrak{d}_1, \ldots, \mathfrak{d}_{m-1}]$; $m = 1, \ldots, n$. Let us express this by saying that $\mathfrak{d}_1, \ldots, \mathfrak{d}_n$ is a *reduced semi-basis of* $\mathfrak{A}$ *with respect to* Γ; we have proved the existence of such a basis. The consecutive minima $t_m = \boldsymbol{t}[\mathfrak{d}_m]$ increase with the index m, $0 < t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$.

By carrying out the transformation (6) $\mathfrak{A}$ is turned into a lattice $\mathfrak{L}$ containing the unit vectors $\mathfrak{e}_\mu$, the form Γ into a form of the variables η. Denoting the new form by Γ again, and the variables by ξ instead of η, we are facing the following situation: $\mathfrak{L}$ is a given admissible lattice and, for any $m = 1, \ldots, n$,

$$\boldsymbol{t}_\Gamma[\mathfrak{x}] \geqslant \boldsymbol{t}_\Gamma[\mathfrak{e}_m] \tag{23}$$

whenever $\mathfrak{x} = (\xi_1, \ldots, \xi_n)$ is in $\mathfrak{L}$ and $(\xi_m, \ldots, \xi_n) \neq (o, \ldots, o)$. Let us say under these circumstances that Γ is an *$\mathfrak{L}$-reduced form.* Set again $\boldsymbol{t}[\mathfrak{e}_m] = t_m$ and observe that the N-dimensional "sphere" defined by

$$f^2(\mathfrak{x}) = t_1^{-1} \cdot \mathrm{tr}\,(\bar{\zeta}_1 \varkappa_1 \zeta_1) + \cdots + t_n^{-1} \cdot \mathrm{tr}\,(\bar{\zeta}_n \varkappa_n \zeta_n) < 1 \tag{24}$$

contains no lattice vector $\mathfrak{x} \neq \mathfrak{o}$, provided Γ is reduced[2]). Indeed, let ξ_m be the last non-vanishing component of the non-vanishing lattice vector $\mathfrak{x}$. We have

$$f^2(\mathfrak{x}) = \textstyle\sum_{\nu=1}^m t_\nu^{-1} \cdot \mathrm{tr}\,(\bar{\zeta}_\nu\, \varkappa_\nu\, \zeta_\nu) \geqslant t_m^{-1} \cdot \sum_{\nu=1}^m \mathrm{tr}\,(\bar{\zeta}_\nu\, \varkappa_\nu\, \zeta_\nu) = t_m^{-1} \cdot \boldsymbol{t}[\mathfrak{x}],$$

but by (23) $\boldsymbol{t}[\mathfrak{x}] \geqslant t_m$. The "length" $f(\mathfrak{x})$ is a gauge function. If v_n is the volume of the n-dimensional unit sphere, $v_n = \{\Gamma(\frac{1}{2})\}^n / \Gamma\left(1 + \frac{n}{2}\right)$, then the volume of the sphere (24) equals

$$v_{ng} \cdot \left\{ \frac{t_1^g \ldots t_n^g}{d^n\, \mathrm{Nm}\,\varkappa_1 \ldots \mathrm{Nm}\,\varkappa_n} \right\}^{\frac{1}{2}},$$

and hence the inequality (9) yields

$$4^{-ng} v_{ng}^2 d^{-n} t_1^g \ldots t_n^g [\mathfrak{L} : \mathfrak{J}]^2 \leqslant \mathrm{Nm}\,\varkappa_1 \ldots \mathrm{Nm}\,\varkappa_n .$$

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[2]) In this way *Minkowski* himself proceeded for quadratic forms; see Geometrie der Zahlen, 1896, pp. 196—199. About his general inequality $S_1 \ldots S_n V \leq 2^n$ ibid., pp. 211—219; compare *H. Davenport,* Quarterly Jour. of Math. **10** (1939), 119—121, *H. Weyl,* Proc. London Math. Soc., ser. 2, vol. **47** (1942), 270—279.

If one knows a number $\tau_n \leqslant 1$ such that congruent non-overlapping spheres in a lattice arrangement cannot occupy more than the part τ_n of the total n-dimensional space, then v_N may here be replaced by the larger constant $\pi_N = v_N/\tau_N$. Blichfeldt has shown that, for instance, $\tau_n = (n+2)\cdot 2^{-\frac{1}{2}(n+2)}$ is a legitimate choice[3]). Setting

$$(g/4)^{ng}\pi_{ng}^2 d^{-n} = A_n ,$$

$[\mathfrak{L}:\mathfrak{J}] = j = j_n$, $c_n = A_n j_n^2$, we have arrived at the following fundamental inequality:

$$c_n \cdot \Pi_{\nu=1}^n (g^{-1} \operatorname{tr} \gamma_{\nu\nu})^g \leqslant \Pi_{\nu=1}^n \operatorname{Nm} \varkappa_\nu . \tag{25_n}$$

The docked form arising from the reduced $\Gamma[\mathfrak{x}]$ by setting

$$\xi_{m+1} = \cdots = \xi_n = o$$

is an $\mathfrak{L}_m$-reduced form of the vector $(\xi_1, \ldots, \xi_m)$. Hence a similar inequality holds for every $m = 1, \ldots, n$:

$$c_m \cdot \Pi_{\nu=1}^m (g^{-1} \operatorname{tr} \gamma_{\nu\nu})^g \leqslant \Pi_{\nu=1}^m \operatorname{Nm} \varkappa_\nu \tag{25_m}$$

where $c_m = A_m j_m^2$. From Lemma 3.5 and (19) we learn that

$$(g^{-1}\cdot \operatorname{tr} \gamma_{\nu\nu})^g \geqslant (g^{-1}\cdot \operatorname{tr} \varkappa_\nu)^g \geqslant \operatorname{Nm} \varkappa_\nu .$$

Applying this in (25_m) for $\nu = 1, \ldots, m$ we obtain the important inequality

$$c_m = A_m j_m^2 \leqslant 1 \qquad (m = 1, \ldots, n) ; \tag{26}$$

if, however, we apply it for $\nu = 1, \ldots, m-1$ only, we find

$$\operatorname{Nm} \varkappa_m \geqslant c_m (g^{-1} \operatorname{tr} \gamma_{mm})^g , \tag{27}$$

a fortiori, in view of (19)

$$\operatorname{Nm} \varkappa_m \geqslant c_m (g^{-1} \operatorname{tr} \varkappa_m)^g . \tag{27'}$$

From (26) upper bounds depending on $\{\mathfrak{F}\}$ only result for the indices $j_1, \ldots, j_n$. As there is but a finite number of lattices $\mathfrak{L}$ over $\mathfrak{J}$ with given indices, we have thus proved the

[3]) Math. Ann. **101**, 1929, 605—608.

FIRST THEOREM OF FINITENESS: *There exist $\mathfrak{L}$-reduced forms for not more than a finite number of admissible lattices $\mathfrak{L}$.*

(27) gives occasion to apply Lemmas 3.6 and 3.7 by identifying $\varkappa_m$ and $g^{-1} \cdot \mathrm{tr}\, \gamma_{mm} = t_m / g$ with the quantities γ and t of the lemmas. Let therefore r_m and r_m^* be the least and the largest eigenvalue of the positive symmetric matrix K_m and set $e(c_m) = e_m$, $b(c_m) = b_m$, $B(c_m) = B_m$. Then

$$r_m \geqslant b_m t_m / g\,, \quad r_m^* \leqslant B_m t_m / g \quad (\text{for } g \geqslant 1)\,; \tag{28}$$

$$r_m^* / r_m \leqslant e_m \quad (\text{for } g \geqslant 2)\,. \tag{29}$$

In the following we denote by M constants depending on $\mathfrak{L}$ only, not always the same. b_m, B_m, e_m are of this nature.

By Lemma 3.4

$$\mathrm{tr}\,(\bar{\zeta}_m \varkappa_m \zeta_m) \geqslant r_m \cdot \mathrm{tr}\,(\bar{\zeta}_m \zeta_m) \geqslant g^{-1} \cdot b_m t_m \cdot \mathrm{tr}\,(\bar{\zeta}_m \zeta_m) \tag{30}$$

and

$$\mathrm{tr}\,(\bar{\zeta}_m \varkappa_m \zeta_m) \leqslant g^{-1} \cdot B_m t_m \cdot \mathrm{tr}\,(\bar{\zeta}_m \zeta_m)\,. \tag{31}$$

§ 6. The Pyramid of Reduced Forms

Our next goal is two-fold: we shall derive upper bounds M for $\mathrm{abs}\,\delta_{\mu\nu} (\mu < \nu)$ and show that the "cell" Z_0 of the $\mathfrak{L}$-reduced forms Γ is defined within H^+ by a *finite* number of linear inequalities and hence is a convex pyramid.

Let m be one of the numbers $1, \ldots, n$, Γ an $\mathfrak{L}$-reduced form, $\mathbf{t} = \mathbf{t}_\Gamma$, and $\mathfrak{x} = (\xi_1, \ldots, \xi_n)$ a lattice vector for which

$$(\xi_m, \ldots, \xi_n) \neq (o, \ldots, o) \tag{32}$$

and

$$\mathbf{t}[\mathfrak{x}] = t_m \tag{33}$$

(equality, not inequality). We maintain that under these circumstances upper bounds M may be ascertained for all $\mathrm{abs}\,\zeta_\nu$. Indeed, our equation (33) reads

$$\Sigma_{\nu=1}^n \mathrm{tr}\,(\bar{\zeta}_\nu \varkappa_\nu \zeta_\nu) = t_m\,,$$

hence

$$\mathrm{tr}\,(\bar{\zeta}_\nu \varkappa_\nu \zeta_\nu) \leqslant t_m \leqslant t_\nu \qquad \text{for} \qquad \nu \geqslant m.$$

By (30)

$$\operatorname{tr}(\bar{\zeta}_\nu \varkappa_\nu \zeta_\nu) \geqslant g^{-1} b_\nu t_\nu \cdot \operatorname{tr}(\bar{\zeta}_\nu \zeta_\nu) .$$

Consequently

$$b_\nu \cdot \operatorname{tr}(\bar{\zeta}_\nu \zeta_\nu) \leqslant g \qquad (\text{for} \quad \nu = m, \ldots, n) . \tag{34}$$

Similar bounds for $\nu < m$ depend on the fact that Γ is reduced. Any element of $\mathfrak{J}_K$ is congruent mod $\{\mathfrak{J}\}$ to a "reduced" element

$$\xi = x_1 \omega_1 + \cdots + x_g \omega_g ,$$

i. e. one for which $-\frac{1}{2} < x_i \leqslant \frac{1}{2}$. Let ϱ^2 be an upper bound of $\operatorname{tr}(\bar{\xi}\xi) = T_0[x]$ in the unit cube $-\frac{1}{2} \leqslant x_i \leqslant \frac{1}{2}$. If $\mathfrak{J}_K$ is endowed with a metric by the positive form $T_0[x]$, then the g-dimensional linear space $\mathfrak{J}_K$ is completely covered by circles of radius ϱ around all elements of $\{\mathfrak{J}\}$. Let $\nu < m$. The lattice vector $\mathfrak{x}$ is changed into another lattice vector $\mathfrak{x}^0 = \mathfrak{x} - \mathfrak{a}$ by subtracting a vector of the form $\mathfrak{a} = (\alpha_1, \ldots, \alpha_\nu, o, \ldots o)$, the components $\alpha_1, \ldots, \alpha_\nu$ of which lie in $\{\mathfrak{J}\}$. The components $\xi_{\nu+1}, \ldots, \xi_n$ are not affected, $\xi^0_\mu = \xi_\mu$ for $\mu > \nu$; hence

$$(\xi^0_m, \ldots, \xi^0_n) = (\xi_m, \ldots, \xi_n) \neq (o, \ldots, o) .$$

Because Γ is reduced, we then must have

$$\boldsymbol{t}[\mathfrak{x}^0] \geqslant t_m = \boldsymbol{t}[\mathfrak{x}] ,$$

and this gives

$$\textstyle\sum_{\mu=1}^n \operatorname{tr}(\bar{\zeta}_\mu \varkappa_\mu \zeta_\mu) \leqslant \sum_{\mu=1}^n \operatorname{tr}(\bar{\zeta}^0_\mu \varkappa_\mu \zeta^0_\mu) .$$

But the terms in the left and right sums coincide for $\mu > \nu$. Therefore

$$\textstyle\sum_{\mu=1}^\nu \operatorname{tr}(\bar{\zeta}_\mu \varkappa_\mu \zeta_\mu) \leqslant \sum_{\mu=1}^\nu \operatorname{tr}(\bar{\zeta}^0_\mu \varkappa_\mu \zeta^0_\mu) . \tag{35}$$

One may choose $\alpha_\nu, \ldots, \alpha_1$ in $\{\mathfrak{J}\}$ one after the other so that $\zeta^0_\nu, \ldots, \zeta^0_1$ are reduced mod $\{\mathfrak{J}\}$. Then

$$\operatorname{tr}(\bar{\zeta}^0_\mu \varkappa_\mu \zeta^0_\mu) \leqslant g^{-1} B_\mu t_\mu \cdot \operatorname{tr}(\bar{\zeta}^0_\mu \zeta^0_\mu) \leqslant g^{-1} B_\mu t_\mu \varrho^2 \quad (\mu \leqslant \nu) ,$$

and consequently the right member of (35) does not exceed

$$g^{-1} \varrho^2 (B_1 t_1 + \cdots + B_\nu t_\nu) \leqslant g^{-1} \varrho^2 (B_1 + \cdots + B_\nu) t_\nu .$$

In the left member we retain only the last term

$$\operatorname{tr}(\bar{\zeta}_\nu \varkappa_\nu \zeta_\nu) \geqslant g^{-1} b_\nu t_\nu \cdot \operatorname{tr}(\bar{\zeta}_\nu \zeta_\nu) .$$

Hence

$$b_\nu \cdot \operatorname{tr}(\bar{\zeta}_\nu \zeta_\nu) \leqslant \varrho^2 (B_1 + \cdots + B_\nu) \quad \text{for } \nu = 1, \ldots, m-1 . \tag{34*}$$

Given the reduced Γ, the equation (33) holds in particular for $\mathfrak{x} = \mathfrak{e}_m$; therefore by (34*)

$$\operatorname{abs}^2 \delta_{\nu\mu} \leqslant \varrho^2 (B_1 + \cdots + B_\nu) / b_\nu \quad (\nu < \mu) . \tag{36}$$

In view of Lemma 3.2 and the recursion formulas

$$\begin{aligned} &\xi_n = \zeta_n , \\ &\xi_{n-1} + \delta_{n-1,n} \xi_n = \zeta_{n-1} , \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

our upper bounds M for $\operatorname{abs} \zeta_\nu$ and $\operatorname{abs} \delta_{\nu\mu}$ as given by (34), (34*), (36) entail similar bounds for $\operatorname{abs} \xi_\nu$,

$$\operatorname{abs}^2 \xi_\nu \leqslant M .$$

Representing ξ_ν by its column $x_\nu = (x_{\nu 1}, \ldots, x_{\nu g})$ in terms of the basis $\omega_1, \ldots, \omega_g$ we thus find

$$T_0[x_\nu] \leqslant M ,$$

and since $T_0[x]$ is a positive quadratic form completely determined by $\{\mathfrak{F}\}$ we obtain upper bounds M for all the coordinates $x_{\nu i}$ of $\mathfrak{x}$,

$$|x_{\nu i}| \leqslant M . \tag{37}$$

$\mathfrak{J}$ is a subgroup of index j of the additive Abelian group $\mathfrak{L}$; hence the multiple j of any vector of $\mathfrak{L}$ lies in $\mathfrak{J}$, or in other words the $x_{\nu i}$ are integers divided by j. Thus (37) leaves only a finite number of possibilities for $\mathfrak{x}$, independent of the special reduced form Γ.

For a more careful formulation of this result let m again be one of the numbers $1, \ldots, n$. We call a vector $\mathfrak{x} = (\xi_1, \ldots, \xi_n)$ in $\mathfrak{L}$ an essential lattice vector of rank m if $(\xi_m, \ldots, \xi_n) \neq (o, \ldots, o)$ and if *there exists* an $\mathfrak{L}$- reduced form Γ such that

$$t_\Gamma[\mathfrak{x}] = t_\Gamma[\mathfrak{e}_m] .$$

Then we have proved that there is only a finite number of essential lattice vectors $\mathfrak{x}$ of rank m.

The cell Z_0 of the reduced forms Γ is defined within H^+ by an infinite number of linear inequalities $L(\Gamma) \geqslant 0$: for each $m \leqslant n$ and each lattice vector $\mathfrak{x}$ satisfying (32) we have such an L, namely

$$L(\Gamma) = t_\Gamma[\mathfrak{x}] - t_\Gamma[\mathfrak{e}_m] = \sum_{\mu,\nu} \operatorname{tr}(\bar{\xi}_\mu \gamma_{\mu\nu} \xi_\nu) - \operatorname{tr} \gamma_{mm} \geqslant 0 . \qquad (38)$$

We speak of the positive forms Γ as *points* in H^+. Let Γ_0 be a point in Z_0, Γ outside Z_0. Then there is at least one inequality L which is not satisfied by Γ, $L(\Gamma) < 0$. But as there is only a finite number of lattice vectors $\mathfrak{x}$ for which $t_\Gamma[\mathfrak{x}] \leqslant t_\Gamma[\mathfrak{e}_1]$ or $\leqslant t_\Gamma[\mathfrak{e}_2]$ or ..., there is not more than a finite number of inequalities $L = L_1, \ldots, L_h$ which are violated by Γ: the planes L_p ,,separate,, Γ_0 from Γ. Traveling from Γ_0 to Γ along the straight segment $\overrightarrow{\Gamma_0 \Gamma}$ the variable point $u\Gamma_0 + (1-u)\Gamma$ $(0 \leqslant u < 1)$ will cross the plane L_p at $u = u_p, u_p L_p(\Gamma_0) + (1-u_p) L_p(\Gamma) = 0$. Let u_1 be the least of the numbers $u_1, \ldots, u_h$, and $u_1\Gamma_0 + (1-u_1)\Gamma$ the corresponding point Γ_1. Then Γ_1 is obviously reduced, but satisfies the equation $L_1(\Gamma_1) = 0$. Hence L_1 is essential in the sense that there exists a reduced form Γ_1 for which the equation $L_1(\Gamma_1) = 0$ holds. However, $L_1(\Gamma) < 0$. Thus we have shown: If Γ violates any of the inequalities L it violates in particular an essential L. Or formulated in the positive way: a point Γ satisfying the essential among the inequalities L, satisfies them all. But we know that there is only a finite number of essential inequalities L (which correspond to what we called above the essential lattice vectors $\mathfrak{x}$ of ranks $m = 1, \ldots, n$). This finishes the proof of the

SECOND THEOREM OF FINITENESS: *The cell Z_0 of $\mathfrak{L}$- reduced forms is defined within the space H^+ of all positive forms by means of a finite number of linear inequalities; in this sense it is a convex pyramid.*

(The proof makes use of the assumption that Z_0 is not empty. Of course an empty Z_0 may also be defined by a finite number of linear inequalities, but one must not choose them from among the inequalities L.)

§ 7. The Pattern of Cells

The following geometric terminology suggests itself. Any semi-basis $\mathfrak{d}_1, \ldots, \mathfrak{d}_n$ of $\mathfrak{A}$ determines a *cell* $Z = Z(\mathfrak{d}_1, \ldots, \mathfrak{d}_n)$; the point Γ *lies in the cell* if $\mathfrak{d}_1, \ldots, \mathfrak{d}_n$ is reduced with respect to Γ, i. e. if

$$t_\Gamma[\mathfrak{x}] \geqslant t_\Gamma[\mathfrak{d}_m]$$

for all vectors $\mathfrak{x}$ that are in $\mathfrak{A}$ but not in $[\mathfrak{d}_1, \ldots, \mathfrak{d}_{m-1}]$ $(m = 1, \ldots, n)$. Because there exists a reduced semi-basis of $\mathfrak{A}$ with respect to any given positive form Γ, each point Γ lies in at least one cell Z: *the cells cover H^+ without gaps*. To each cell $Z(\mathfrak{d}_1, \ldots, \mathfrak{d}_n)$ there corresponds an admissible lattice $\mathfrak{L}$, namely the representation of $\mathfrak{A}$ in terms of $\mathfrak{d}_1, \ldots, \mathfrak{d}_n$; the same lattice to two cells if and only if they arise from each other by a lattice substitution s,

$$Z = Z(\mathfrak{d}_1, \ldots, \mathfrak{d}_n), \qquad Z^s = Z(\mathfrak{d}_1^s, \ldots, \mathfrak{d}_n^s) .$$

s carries Γ into the form Γ^s defined by

$$\Gamma^s[\mathfrak{x}^s] = \Gamma[\mathfrak{x}] \quad \text{or} \quad \Gamma^s[\mathfrak{x}] = \Gamma[\mathfrak{x}^{s^{-1}}] ;$$

Γ^s lies in Z^s when Γ lies in Z. We distinguish the different admissible lattices $\mathfrak{L}$ by different colors and paint the cells accordingly. Provided we omit the empty cells (the lattices $\mathfrak{L}$ for which there are no $\mathfrak{L}$- reduced forms) only a finite number of colors is needed. A lattice transformation s carries this pattern of cells including its coloring into itself. We know the non-empty cells to be convex pyramids.

There will, however, be overlappings: a point Γ may belong to a number of distinct cells. Indeed, the two cells $Z(\mathfrak{d}_1, \ldots, \mathfrak{d}_n)$ and $Z(\mathfrak{d}_1\alpha_1, \ldots, \mathfrak{d}_n\alpha_n)$ completely cover one another if the α_μ are unitary factors. Here an element α of $\mathfrak{F}$ (or of $\mathfrak{F}_K$) is said to be unitary if

$$\bar{\alpha}\,\alpha = \varepsilon .$$

The norm of a unitary element α equals ± 1 and its reciprocal $\alpha^{-1} = \bar{\alpha}$. Hence $\operatorname{tr}(\bar{\alpha}\tau\alpha) = \operatorname{tr}\tau$ for every element τ of $\mathfrak{F}_K$; in particular the value (14) is not changed by passing from the argument $\mathfrak{x}$ to $\mathfrak{x}\alpha$, whatever the quadratic form $\Gamma = ||\gamma_{\mu\nu}||$. The unitary elements form a group.

As above, we consider the cells as entities *sui generis*, not as point sets; but we identify certain cells according to the rule: *Any two semi-bases* $\mathfrak{d}_1, \ldots, \mathfrak{d}_n$ *and* $\mathfrak{c}_1, \ldots, \mathfrak{c}_n$ belong to the same family or *determine the same cell if* $\mathfrak{c}_\mu = \mathfrak{d}_\mu\alpha_\mu$ *where* α_μ *is unitary*. In this case $\mathfrak{c}_1, \ldots, \mathfrak{c}_n$ is a reduced basis with respect to the quadratic form Γ whenever $\mathfrak{d}_1, \ldots, \mathfrak{d}_n$ is; hence whether or not a point Γ lies in a cell Z does not depend on the basis, $\mathfrak{d}_1, \ldots, \mathfrak{d}_n$ or $\mathfrak{c}_1, \ldots, \mathfrak{c}_n$, by which Z is defined. It is also true that the image Z^s of Z by a given lattice substitution s is independent of the defining basis because $\mathfrak{c}_\mu = \mathfrak{d}_\mu\alpha_\mu$ implies $\mathfrak{c}_\mu^s = \mathfrak{d}_\mu^s\alpha_\mu$. Two admissible lattices $\mathfrak{L}$ and $\mathfrak{L}^*$ are said to belong to the same family if they arise from each other by a "special substitution"

$$\xi_\mu = \alpha_\mu \xi_\mu^* \qquad (\alpha_\mu \text{ unitary}) . \tag{39}$$

There is a one-to-one correspondence between the classes of equivalent cells and the families of lattices $\mathfrak{L}$, and we now paint all lattices $\mathfrak{L}$ of the same family and all cells of the corresponding class with the same color. Given an admissible $\mathfrak{L}$, the number of special substitutions (39) we have to reckon with is *a priori* limited. Indeed, since $\mathfrak{e}_\mu$ is contained in $\mathfrak{L}^*$, the vectors

$$(\alpha_1, o, \ldots, o), \quad (o, \alpha_2, \ldots, o), \ldots, (o, o, \ldots, \alpha_n)$$

must be in $\mathfrak{L}$, and the inequalities

$$\operatorname{tr}(\bar{\alpha}_\mu \alpha_\mu) \leqslant g$$

will leave but a finite number of possibilities open for them. This shows two things: (1) a family of admissible lattices contains only a finite number of members; (2) *the group of special substitutions carrying $\mathfrak{L}$ into itself is finite.* If one writes out the substitution

$$\xi_1 = \alpha_1 \xi_1^*, \ldots, \xi_m = \alpha_m \xi_m^*$$

in terms of the coordinates $x_{\mu i} (\mu = 1, \ldots, m;\ i = 1, \ldots, g)$ of $\xi_1, \ldots, \xi_m$ its determinant equals $\operatorname{Nm} \alpha_1 \ldots \operatorname{Nm} \alpha_m = \pm 1$. Hence the fundamental parallelotopes of $\mathfrak{L}_m$ and $\mathfrak{L}_m^*$ have the same (mg)-dimensional volume provided $\mathfrak{L}$ and $\mathfrak{L}^*$ belong to the same family. Thus two lattices of the same family have the same row of indices $j_1, \ldots, j_n$.

$\mathfrak{d}_1, \ldots, \mathfrak{d}_n$ is said to be a *properly reduced* semi-basis of $\mathfrak{A}$ with respect to Γ, and Γ is said to *belong to the core* of $Z(\mathfrak{d}_1, \ldots, \mathfrak{d}_n)$ if

$$t_\Gamma[\mathfrak{x}] > t_\Gamma[\mathfrak{d}_m]$$

for all vectors $\mathfrak{x}$ in $\mathfrak{A}$ which are not in $[\mathfrak{d}_1, \ldots, \mathfrak{d}_{m-1}]$ and not of the special form $\mathfrak{d}_m \alpha$ (α unitary) $(m = 1, \ldots, n)$. One proves at once: Is $\mathfrak{d}_1, \ldots, \mathfrak{d}_n$ a properly reduced and $\mathfrak{d}_1^*, \ldots, \mathfrak{d}_n^*$ a reduced semi-basis with respect to Γ, then $\mathfrak{d}_\mu^* = \mathfrak{d}_\mu \alpha_\mu$, the α_μ being unitary. Or: *A point belonging to the core of a cell lies in no other cell.*

Does this mean that there are no overlappings? Considering a cell

$Z = Z(\mathfrak{d}_1, \ldots, \mathfrak{d}_n)$ as the set of points Γ lying in Z we must show that any *inner* point of Z belongs to its core.

Again, we make the substitution

$$\mathfrak{x} = \mathfrak{d}_1 \eta_1 + \cdots + \mathfrak{d}_n \eta_n \tag{40}$$

and afterwards write ξ for η, with the effect that $\mathfrak{e}_\mu$, $\mathfrak{L}$, Z_0 take the place of $\mathfrak{d}_\mu$, $\mathfrak{A}$, $Z(\mathfrak{d}_1, \ldots, \mathfrak{d}_n)$. For each $m = 1, \ldots, n$ and each vector $\mathfrak{x}$ in $\mathfrak{L}$ outside $[\mathfrak{e}_1, \ldots, \mathfrak{e}_{m-1}]$ which is not of the form $\mathfrak{e}_m \alpha$ (α unitary) we set up the linear form

$$L(\Gamma) = \Sigma_{\mu,\nu} \operatorname{tr} (\bar{\xi}_\mu \gamma_{\mu\nu} \xi_\nu) - \operatorname{tr} (\gamma_{mm}) \tag{41}$$

If it is sure that none of these L vanishes identically, then an inner point Γ of Z_0 necessarily satisfies the strict inequalities $L(\Gamma) > 0$ and hence belongs to the core of Z_0. Thus we must prove that, given a vector $\mathfrak{x}$ of Z, (41) vanishes identically in Γ only if $\mathfrak{x} = \mathfrak{e}_m \alpha$, α unitary. It suffices to do this for $m = 1$. Let γ be any symmetric element of $\mathfrak{F}_K$. Choosing for Γ in succession the n diagonal matrices $||\gamma_{\mu\nu}||$ with the elements

$$\{\gamma, o, \ldots, o\}, \quad \{o, \varepsilon, o, \ldots, o\}, \quad \ldots, \quad \{o, o, \ldots, \varepsilon)$$

along the diagonal we deduce from the identity $L(\Gamma) \equiv 0$:

$$\operatorname{tr} (\bar{\xi}_1 \gamma \xi_1) = \operatorname{tr} \gamma, \quad \operatorname{tr} (\bar{\xi}_2 \xi_2) = 0, \ldots, \quad \operatorname{tr} (\bar{\xi}_n \xi_n) = 0,$$

therefore $\xi_2 = \cdots = \xi_n = o$, $\mathfrak{x} = \mathfrak{e}_1 \alpha$, and $\xi_1 = \alpha$ satisfies the equation $\operatorname{tr} (\bar{\alpha} \gamma \alpha) = \operatorname{tr} \gamma$ for every symmetric γ. Specialize further by setting first $\gamma = \varepsilon$ and then $\gamma = \alpha \bar{\alpha}$:

$$\operatorname{tr} \varepsilon = g, \quad \operatorname{tr} (\bar{\alpha} \alpha) = g, \quad \operatorname{tr} (\bar{\alpha} \alpha \bar{\alpha} \alpha) = \operatorname{tr} (\alpha \bar{\alpha}) = \operatorname{tr} (\bar{\alpha} \alpha) = g.$$

Consequently the trace of the square of $\beta = \bar{\beta} = \bar{\alpha}\alpha - \varepsilon$ is zero; but $\operatorname{tr} (\bar{\beta} \beta) = 0$ implies $\beta = o$, hence α is unitary.

Thus there is no overlapping inasmuch as no inner point of one cell lies in any other cell of our pattern. In the next section we shall show that there is no clustering of cells inside H^+. For this reason we still have a covering of H^+ without gaps even when we retain only the cells with inner points; these are solid pyramids.

An assembly of cells $Z = Z(\mathfrak{d}_1, \ldots, \mathfrak{d}_n)$ in which each color is represented by one member would constitute a fundamental domain for the lattice group if the group $\mathfrak{g}_Z$ of lattice transformations carrying Z into itself consisted of the identity only. As this will not be so, generally speaking, we first have to whittle down Z to a fundamental domain within Z of the finite group $\mathfrak{g}_Z$. In view of our rule of identification for cells a lattice transformation s carrying $Z(\mathfrak{d}_1, \dots, \mathfrak{d}_n)$ into itself must be of the form $\mathfrak{d}_\mu \to \mathfrak{d}_\mu^s = \mathfrak{d}_\mu \alpha_\mu$ where the factors α_μ are unitary. It transforms the vector $\mathfrak{d}_1 \eta_1 + \cdots + \mathfrak{d}_n \eta_n$ into

$$\mathfrak{d}_1^s \eta_1 + \cdots + \mathfrak{d}_n^s \eta_n = \mathfrak{d}_1 \eta_1^s + \cdots + \mathfrak{d}_n \eta_n^s$$

where $\eta_\mu^s = \alpha_\mu \eta_\mu$. Hence after the substitution (40) which replaces $\mathfrak{A}$ and $Z(\mathfrak{d}_1, \dots, \mathfrak{d}_n)$ by $\mathfrak{L}$ and Z_0 respectively, the group $\mathfrak{g}_Z$ is made up of those special transformations

$$\{\alpha_1, \dots, \alpha_n\}: \quad \xi_\mu^s = \alpha_\mu \xi_\mu \quad (\alpha_\mu \text{ unitary})$$

which carry $\mathfrak{L}$ into itself. They induce a group $\mathfrak{g}$ of linear transformations in the space H of quadratic forms $\Gamma = \|\gamma_{\mu\nu}\|$:

$$\Gamma \to \Gamma^s, \quad \gamma_{\mu\nu} \to \bar{\alpha}_\mu^{-1} \gamma_{\mu\nu} \alpha_\nu^{-1} = \alpha_\mu \gamma_{\mu\nu} \bar{\alpha}_\nu \,. \tag{42}$$

Denote by $\mathfrak{g}_2$ the invariant subgroup of $\mathfrak{g}_1 = \mathfrak{g}$ the elements of which leave γ_{12} unchanged, by $\mathfrak{g}_3$ the invariant subgroup of $\mathfrak{g}_2$ whose elements leave γ_{13} unchanged, ..., by $\mathfrak{g}_n$ the invariant subgroup of $\mathfrak{g}_{n-1}$ the elements of which leave γ_{1n} unchanged. $\mathfrak{g}_n$ consists of the identity only. Indeed, for any element $\{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n\}$ of $\mathfrak{g}_2$ the substitution $\xi \to \alpha_1 \xi \bar{\alpha}_2$ is the identity, therefore $\alpha_1 \bar{\alpha}_2 = \varepsilon$ or $\alpha_1 = \alpha_2$. Hence all elements of $\mathfrak{g}_n$ are of the form $\{\alpha, \dots, \alpha\}$ and $\xi \to \alpha \xi \bar{\alpha}$ is the identity. But then (42) is the identity. — In its influence upon γ_{12} the group $\mathfrak{g}_1$ is actually $\mathfrak{g}_1/\mathfrak{g}_2$. We endow the g-dimensional space of the variable "point" $\xi = \gamma_{12}$ with a metric by means of the positive form $\operatorname{tr}(\bar{\xi}\xi)$. The operations of $\mathfrak{g}_1/\mathfrak{g}_2$ are linear metric-preserving ("orthogonal") mappings of the ξ-space. In familiar fashion we construct a fundamental domain for this finite group as follows. We choose a point $\xi = \pi_0$ which is carried into $h+1$ distinct points $\pi_0, \pi_1, \dots, \pi_h$ by the $h+1$ operations of $\mathfrak{g}_1/\mathfrak{g}_2$ and set up the h linear inequalities expressing that the variable point ξ lies at least as near to π_0 as to $\pi_1, \dots, \pi_h$:

$$l^{(r)}(\xi) \equiv \operatorname{tr}\{(\bar{\pi}_0 - \bar{\pi}_r)\xi\} \geqslant 0 \quad (r = 1, \dots, h)\,.$$

By adding these h inequalities $l(\gamma_{12}) \geqslant 0$ to the ones $L(\Gamma) \geqslant 0$ defining Z_0, we obtain a convex part $Z_0^{(2)}$ of Z which is invariant under the group $\mathfrak{g}_2$ but whose $h+1$ images generated by the operations of $\mathfrak{g}_1/\mathfrak{g}_2$ cover Z_0 without gaps and overlappings. We then carry out the same construction for γ_{13} with respect to the group $\mathfrak{g}_2/\mathfrak{g}_3, \ldots,$ for γ_{1n} with respect to the group $\mathfrak{g}_{n-1}/\mathfrak{g}_n = \mathfrak{g}_{n-1}$. Thus by a number of linear inequalities

$$l_2^{(r_2)}(\gamma_{12}) \geqslant 0, \ldots, l_n^{(r_n)}(\gamma_{1n}) \geqslant 0 \quad (1 \leqslant r_2 \leqslant h_2, \ldots, 1 \leqslant r_n \leqslant h_n),$$

each concerning only one coefficient of Γ, a convex part $Z_0^\bullet$ of Z_0 is constructed, the images of which by the mappings s of $\mathfrak{g}$ cover Z_0 without gaps and overlappings. Denote the corresponding part of $Z(\mathfrak{d}_1, \ldots, \mathfrak{d}_n)$ by $Z^\bullet(\mathfrak{d}_1, \ldots, \mathfrak{d}_n)$: an assembly of such $Z^\bullet(\mathfrak{d}_1, \ldots, \mathfrak{d}_n)$ in which each color is represented by one member constitutes a fundamental domain for the lattice group. (In the case $n=1$, one has of course to proceed in the same manner in the g^+-dimensional space of a symmetric variable $\gamma = \gamma_{11}$ rather than in the g-dimensional spaces of $\gamma_{12}, \ldots, \gamma_{1n}$.)

§ 8. The Third Theorem of Finiteness

To make sure that the pattern of cells shows no inner clustering in H^+ it is not sufficient, as Minkowski seems to have believed, to prove that each cell borders on not more than a finite number of neighbors. Rather, one has to introduce a variable subregion H_t of H^+ depending on a real parameter $t > 0$ in such manner that it grows as t increases and sweeps over the whole region H^+ as t tends to infinity, and then to prove that there is only a finite number of lattice substitutions s carrying a given cell $Z = Z(\mathfrak{d}_1, \ldots, \mathfrak{d}_n)$ into cells Z^s which have points in common with H_t.

In analyzing Minkowski's proof I came to adopt the following definition of the expanding subregion. Given $p \geqslant 1$, $w > 0$ and a semibasis $\mathfrak{c}_1, \ldots, \mathfrak{c}_n$ of $\mathfrak{A}$, we say that the positive form Γ lies in

$$Z(\mathfrak{c}_1, \ldots, \mathfrak{c}_n | p, w)$$

if

$$t_\Gamma[\mathfrak{x}] \geqslant p^{-1} \cdot t_\Gamma[\mathfrak{c}_m]$$

for every $m = 1, \ldots, n$ and every vector $\mathfrak{x}$ that is in $\mathfrak{A}$ but not in $[\mathfrak{c}_1, \ldots, \mathfrak{c}_{m-1}]$, and if moreover

$$t_\Gamma[\mathfrak{c}_m - \mathfrak{y}] \geqslant t_\Gamma[\mathfrak{c}_m] - w \cdot t_\Gamma[\mathfrak{c}_\mu]$$

for $\mu < m$ whenever $\mathfrak{y}$ is in $\mathfrak{A}$ and in $[\mathfrak{c}_1, \ldots, \mathfrak{c}_\mu]$. While p and w increase to infinity, the set $Z(\mathfrak{c}_1, \ldots, \mathfrak{c}_n | p, w)$ grows, and any given point Γ of H^+ will finally come to lie in it. Instead of p and w one could of course introduce a lot more parameters, but could also reduce the two parameters to one, for instance by setting $p = \exp w$; it makes little difference either way.

THE THEOREM OF DISCONTINUITY: *There is only a finite number of lattice substitutions s carrying a given cell $Z = Z(\mathfrak{d}_1, \ldots, \mathfrak{d}_n)$ into cells Z^s that have points in common with*

$$H_{p,w} = Z(\mathfrak{c}_1, \ldots, \mathfrak{c}_n | p, w) : Z^s \cap H_{p,w} \neq 0 .$$

From the beginning we may assume $\mathfrak{d}_\mu = \mathfrak{e}_\mu$. Then $\mathfrak{A}$ coincides with $\mathfrak{L}$ and $Z(\mathfrak{d}_1, \ldots, \mathfrak{d}_n)$ with Z_0. The image Z_0^s has points in common with $H_{p,w}$ if $Z_0 \cap H_{p,w}^{s^{-1}} \neq 0$. Let s^{-1} carry $\mathfrak{c}_1, \ldots, \mathfrak{c}_n$ into $\mathfrak{e}_1^*, \ldots, \mathfrak{e}_n^*$, and Γ be a common point of Z_0 and $Z(\mathfrak{e}_1^*, \ldots, \mathfrak{e}_n^* | p, w)$. Then the $\mathfrak{e}_\mu^*$ are vectors in $\mathfrak{L}$. Write $t_\Gamma[\mathfrak{e}_m] = t_m$, $t_\Gamma[\mathfrak{e}_m^*] = t_m^*$. The form Γ is $\mathfrak{L}$-reduced whereas

$$t_\Gamma[\mathfrak{x}] \geqslant p^{-1} \cdot t_\Gamma[\mathfrak{e}_m^*] \tag{43}$$

whenever $\mathfrak{x}$ is in $\mathfrak{L}$ and outside $[\mathfrak{e}_1^*, \ldots, \mathfrak{e}_{m-1}^*]$, and

$$t_\Gamma[\mathfrak{e}_m^* - \mathfrak{y}] \geqslant t_\Gamma[\mathfrak{e}_m^*] - w \cdot t_\Gamma[\mathfrak{e}_\mu^*]$$

whenever $\mathfrak{y}$ is in $\mathfrak{L}$ and in $[\mathfrak{e}_1^*, \ldots, \mathfrak{e}_\mu^*]$ $(\mu < m \leqslant n)$. From these facts bounds M which do not depend on Γ are to be derived for the vectors $\mathfrak{e}_m^*$. We omit the subscript Γ in t_Γ. Two lemmas point the way:

Lemma 8.1.

$$t_m^* \leqslant p t_m . \tag{44}$$

Proof. At least one of the m vectors $\mathfrak{e}_1, \ldots, \mathfrak{e}_m$, say $\mathfrak{e}_\mu$, lies outside $[\mathfrak{e}_1^*, \ldots, \mathfrak{e}_{m-1}^*]$. Then by (43)

$$t[\mathfrak{e}_\mu] \geqslant p^{-1} \cdot t[\mathfrak{e}_m^*] \quad \text{or} \quad t_\mu \geqslant p^{-1} t_m^* ,$$

and since $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_m$ *a fortiori* $t_m \geqslant p^{-1} t_m^*$.

Lemma 8.2. If the two spaces $S^m = [\mathfrak{e}_1, \ldots, \mathfrak{e}_m]$ and $S_*^m = [\mathfrak{e}_1^*, \ldots, \mathfrak{e}_m^*]$ do not coincide, then

$$t_{m+1} \leqslant p t_m . \tag{45}$$

Indeed, if $\mathfrak{e}_\mu^*(\mu < m)$ lies outside $[\mathfrak{e}_1, \dots, \mathfrak{e}_m]$, then by the definition of reduction $t_\mu^* = t[\mathfrak{e}_\mu^*] \geqslant t_{m+1}$. Therefore by (44), $pt_\mu \geqslant t_{m+1}$ and *a fortiori* $pt_m \geqslant t_{m+1}$.

The second lemma suggests introduction of those numbers

$$m = l_0, l_1, \dots, l_v \quad (0 = l_0 < l_1 < \dots < l_v = n)$$

for which $S^m = S_*^m$ and to divide the row $m = 1, 2, \dots, n$ into the sections

$$l_0 < m \leqslant l_1, \quad l_1 < m \leqslant l_2, \dots, \quad l_{v-1} < m \leqslant l_v . \tag{46}$$

The inequality (45) holds if m and $m + 1$ belong to the same section; hence

$$t_\mu \leqslant p^{\mu-\nu} t_\nu \tag{47}$$

for $\mu > \nu$ provided ν and μ are in the same section.

From now on the proof follows closely the line of Minkowski's proof of the first theorem of finiteness. Again we use Jacobi's transformation for Γ and the notations that go with it. We consider the possibilities for $\mathfrak{e}_1^*, \dots, \mathfrak{e}_n^*$ which correspond to a definite partition (46) into sections. For a given m let λ be any of the numbers $l_1, \dots, l_{v-1}$ which is less than m. Because of (44) and $[\mathfrak{e}_1^*, \dots, \mathfrak{e}_\lambda^*] = [\mathfrak{e}_1, \dots, \mathfrak{e}_\lambda]$ the vector $\mathfrak{x} = \mathfrak{e}_m^*$ of $\mathfrak{L}$ satisfies the following inequalities

$$t[\mathfrak{x}] \leqslant pt_m ,$$

$$t[\mathfrak{x} - \mathfrak{a}] \geqslant t[\mathfrak{x}] - wpt_\lambda \tag{48}$$

whenever $\mathfrak{a}$ is in $\mathfrak{L}$ and $[\mathfrak{e}_1, \dots, \mathfrak{e}_\lambda]$. We maintain that they are reconcilable only with a finite number of possibilities for $\mathfrak{x}$. As in § 6 the first inequality yields

$$b_\nu \cdot \operatorname{tr}(\bar{\zeta}_\nu \zeta_\nu) \leqslant pg \qquad \text{for } \nu \geqslant m . \tag{49}$$

Owing to (47), the same argument still works for $\nu < m$ provided ν is in the same section as m, with the result

$$b_\nu \cdot \operatorname{tr}(\bar{\zeta}_\nu \zeta_\nu) \leqslant p^{m-\nu+1} \cdot g . \tag{49*}$$

If, however, ν is in a lower section than m, $l_{u-1} < \nu \leqslant l_u = \lambda < m$, we apply the inequality (48),

$$t[\mathfrak{x}] - t[\mathfrak{x} - \mathfrak{a}] \leqslant wpt_\lambda \tag{50}$$

to a vector $\mathfrak{a} = \mathfrak{e}_1\alpha_1 + \cdots + \mathfrak{e}_\nu\alpha_\nu$ the components $\alpha_1, \ldots, \alpha_\nu$ of which belong to the order $\{\mathfrak{J}\}$. Setting $\mathfrak{x}^0 = \mathfrak{x} - \mathfrak{a}$ we may ascertain $\alpha_\nu, \ldots, \alpha_1$ so that $\zeta_\nu^0, \ldots, \zeta_1^0$ become reduced mod $\{\mathfrak{J}\}$, and then (50) gives

$$t_\nu b_\nu \cdot \mathrm{tr}\,(\bar{\zeta}_\nu \zeta_\nu) \leqslant \Sigma_{\mu=1}^{\nu} t_\mu B_\mu \cdot \mathrm{tr}\,(\bar{\zeta}_\mu^0 \zeta_\mu^0) + wgpt_\lambda\,,$$

$$b_\nu \cdot \mathrm{tr}\,(\bar{\zeta}_\nu \zeta_\nu) \leqslant \varrho^2(B_1 + \cdots + B_\nu) + wgp^{\lambda-\nu+1}. \qquad (49^{**})$$

These bounds M for all $\mathrm{abs}^2\,\zeta_\nu$ lead by means of (36) and Lemma 3.2 to similar bounds for all $\mathrm{abs}^2\,\xi_\nu$. Since $\mathfrak{x} = \mathfrak{e}_m^*$ lies in S^l, provided the section to which m belongs ends with l, one might add to (49) the remark that ζ_ν and ξ_ν vanish for $\nu > l$.

D. SIEGEL'S RESULTS

§ 9. The Jacobi Transform of t_Γ

Siegel follows another procedure[4]). He carries through the Jacobi transformation of the quadratic form $t_\Gamma[\mathfrak{x}]$ of the N variables $x_{\mu i}$. By means of the substitution (21) we transformed it into

$$T_1[z_1] + \cdots + T_n[z_n]$$

where $T_m[z] = \mathrm{tr}\,(\bar{\zeta}\varkappa_m\zeta)$. Setting

$$\left\| \begin{matrix} T_1 & & & \\ & T_2 & & \\ & & \ddots & \\ & & & T_n \end{matrix} \right\| = T$$

besides (22) we have

$$D'\,T\,D = t_\Gamma$$

Upper bounds M that depend on $\mathfrak{L}$ only were found for $\mathrm{abs}^2\,\delta_{\mu\nu}$; do they imply upper bounds of the same nature for the coefficients of the representing matrices $D_{\mu\nu}$?

To ask the question is to answer it. Let ξ be an element of $\mathfrak{J}_K$ such that $\mathrm{abs}^2\,\xi = T_0[x] \leqslant M$. Write down the multiplication table of the basis ω_i:

$$\omega_i\,\omega_k = \Sigma_l e_{lk}^{(i)}\,\omega_l\,.$$

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[4]) Ann. of Math. **44**, 1943, p. 687.

Then the matrices representing $\omega_1, \dots, \omega_g$ in terms of this basis are $\|e_{kl}^{(1)}\|, \dots, \|e_{kl}^{(g)}\|$, hence $\xi = x_1\omega_1 + \cdots + x_g\omega_g$ is represented by

$$\|x_{ik}\| \quad \text{where} \quad x_{ik} = e_{ik}^{(1)}x_1 + \cdots + e_{ik}^{(g)}x_g \, .$$

$T_0[x] \leqslant M$ yields upper bounds M for the absolute values of the coordinates $x_1, \dots, x_g$ and thus for the absolute values of x_{ik}. However, this is not the most direct proof. Denote for a moment by $\|x_{ik}^0\|$, $\|x_{ik}\|$ the two matrices representing ξ in terms of the normal basis ω_i^0 and the basis ω_i respectively, and by $U = \|u_{ik}\|$, $U^{-1} = \|\tilde{u}_{ik}\|$ the transformation that leads from one to the other. Then

$$x_{ik} = \Sigma_{j,\,h}\, \tilde{u}_{ij} x_{jh}^0 u_{hk} \, ,$$

$$x_{ik}^2 \leqslant \Sigma_j\, \tilde{u}_{ij}^2 \cdot \Sigma_{j,h}(x_{jh}^0)^2 \cdot \Sigma_h\, u_{hk}^2 \, .$$

Here

$$\Sigma_{j,\,h}(x_{jh}^0)^2 = \operatorname{tr}\,(\bar{\xi}\,\xi) \leqslant M \, ,$$

and therefore

$$x_{ik}^2 \leqslant \tilde{u}_i^0\, M u_k^0$$

where u_i^0, $\tilde{u}_i^0$ are the elements in the diagonal of $U'U$ and $(U'U)^{-1}$.

To complete our task we have to perform the Jacobi transformation on each of the forms $T = T_m$. We compare them with the form T_0 which is independent of Γ. Thus we are dealing with two positive quadratic forms $T[x]$ and $T_0[x]$ of g variables x and carry out their Jacobi transformation,

$$T = D'QD = Q[D] \quad \text{and} \quad T_0 = Q_0[D_0].$$

Q, Q_0 are diagonal matrices with the positive terms $q_1, \dots, q_g$; $q_1^0, \dots, q_g^0$ along the diagonal while D and D_0 are triangular,

$$D = \left\| \begin{matrix} 1, & d_{12}, & \dots, & d_{1n} \\ & 1, & \dots, & d_{2n} \\ & \dots & \dots & \dots \\ & & & 1 \end{matrix} \right\|, \quad D_0 = \left\| \begin{matrix} 1, & d_{12}^0, & \dots, & d_{1n}^0 \\ & 1, & \dots, & d_{2n}^0 \\ & \dots & \dots & \dots \\ & & & 1 \end{matrix} \right\| .$$

Introduce the triangular $C = DD_0^{-1} = \|c_{ik}\|$.

Lemma 9.1. Suppose we have two positive constants r, r^* such that

$$rT_0[x] \leqslant T[x] \leqslant r^*T_0[x] \, .$$

Then

$$rq_i^0 \leqslant q_i \leqslant r^* q_i^0$$

and

$$c_{ik}^2 \leqslant \left(\frac{r^*}{r} - 1\right) \cdot \frac{q_k^0}{q_i^0} \qquad (i < k) .$$

[This is a quantitative reinforcement of the qualitative statement that a form T determines its Jacobi transformation uniquely: $r^* = r = 1$ implies $Q = Q_0$, $(C = E,)$ $D = D_0$.]

Proof. Instead of $T \leqslant r^* T_0$ or $Q[D] \leqslant r^* Q_0[D_0]$ one may write $Q[C] \leqslant r^* Q_0$. This inequality for the matrices $Q[C]$ and Q_0 implies the corresponding inequalities for the elements in their diagonal:

$$q_k + \sum_{i<k} q_i c_{ik}^2 \leqslant r^* q_k^0 . \tag{51}$$

Therefore $q_k \leqslant r^* q_k^0$. Interchanging T and T_0 one finds in the same way $q_k^0 \leqslant \frac{1}{r} q_k$. By taking this result into account (51) yields

$$\sum_{i<k} q_i^0 c_{ik}^2 \leqslant \left(\frac{r^*}{r} - 1\right) q_k^0 .$$

Apply the lemma to our forms

$$T_m = Q_m[D_m] .$$

Returning to the notations of § 5, (28), (29), we have indeed

$$r_m T_0[x] \leqslant T_m[x] \leqslant r_m^* T_0[x]$$

and therefore obtain for the coefficients $c_{ik}^{(m)}$ of the triangular matrix $C_m = D_m D_0^{-1}$ the upper bounds

$$(c_{ik}^{(m)})^2 \leqslant (e_m - 1) \cdot q_k^0 / q_i^0 \qquad (i < k) . \tag{52}$$

Similar such bounds, which depend on $\mathfrak{L}$ only, follow then for $D_m = C_m D_0$ and ultimately for the triangular matrix

$$\left\| \begin{matrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_n \end{matrix} \right\| \cdot \left\| \begin{matrix} E, & D_{12}, & \ldots, & D_{1n} \\ & E, & \ldots, & D_{2n} \\ & \ldots & \ldots & \ldots \\ & & & E \end{matrix} \right\| = \tilde{D}$$

which effects the transformation of t_Γ into the diagonal matrix

$$Q = \left\| \begin{matrix} Q_1 & & & \\ & Q_2 & & \\ & & \ddots & \\ & & & Q_n \end{matrix} \right\| , \qquad t_\Gamma = Q[\tilde{D}] \,.$$

Denote the diagonal elements

$$q_{11}, \ldots, q_{1g} \mid q_{21}, \ldots, q_{2g} \mid \ldots \mid q_{n1}, \ldots, q_{ng}$$

of Q in this order by $q_1, \ldots, q_N$. For any two consecutive ones which do not jump the partitions $|$, like $q_{\nu i}$ and $q_{\nu,\, i+1}$ $(i = 1, \ldots, g-1)$, we find by Lemma 9.1

$$\frac{q_{\nu,i}}{q_{\nu,i+1}} \leqslant \frac{r_\nu^*}{r_\nu} \cdot \frac{q_i^0}{q_{i+1}^0} \,,$$

hence by Lemma 3.6 or formula (29)

$$\frac{q_{\nu,i}}{q_{\nu,i+1}} \leqslant e_\nu \cdot \frac{q_i^0}{q_{i+1}^0} \,. \tag{53}$$

In order to cross a partition, for instance from $q_{\nu g}$ to $q_{\nu+1,\,1}$ we appeal to (28) and Lemma 9.1:

$$q_{\nu g} \leqslant r_\nu^* q_g^0 \leqslant g^{-1} B_\nu t_\nu \cdot q_g^0 \,,$$

$$q_{\nu+1,\,1} \geqslant r_{\nu+1} q_1^0 \geqslant g^{-1} b_{\nu+1} t_{\nu+1} \cdot q_1^0 \,.$$

But $t_{\nu+1} \geqslant t_\nu$; therefore

$$\frac{q_{\nu g}}{q_{\nu+1,1}} \leqslant \frac{B_\nu}{b_{\nu+1}} \cdot \frac{q_g^0}{q_1^0} \,. \tag{53*}$$

We thus obtain bounds M for all the quotients

$$q_K \,/\, q_{K+1} \qquad (K = 1, \ldots, N-1)$$

and all the coefficients d_{KL} of $\tilde{D}$. Any positive quadratic form

$$\tilde{G}[\mathfrak{x}] = \Sigma_{K,L=1}^{N} g_{KL}\, x_K\, x_L$$

of N variables $x_1, \ldots, x_N$ determines uniquely its Jacobi transformation

$$z_K = x_K + \sum_{L>K} d_{KL}\, x_L\,,$$

$$\tilde{G}[\mathfrak{x}] = \sum_K q_K\, z_K^2\,.$$

Given a positive number t, let us say that $\tilde{G}$ belongs to the set $\mathfrak{R}_t$ if

$$q_K / q_{K+1} \leqslant t \quad \text{and} \quad d_{KL}^2 \leqslant t \quad (K < L)\,.$$

We have constructed a number a which depends on $\mathfrak{L}$ only, such that for every $\mathfrak{L}$-reduced form Γ the corresponding t_Γ belongs to $\mathfrak{R}_a$.

The quadratic forms of N real variables $x_1, \ldots, x_N$ with real coefficients form a linear space $\mathfrak{R}$ in which the positive ones form an open convex cone $\mathfrak{R}^+$. In Siegel's conception the theorem of discontinuity deals with this space $\mathfrak{R}$ of dimensionality $\frac{1}{2}N(N+1)$ rather than with the space of quadratic forms Γ in $\mathfrak{F}_K$ of dimensionality g_n. It is clear that with t increasing to infinity, $\mathfrak{R}_t$ will exhaust $\mathfrak{R}^+$, and the set H_t of all positive Γ for which the corresponding t_Γ lies in $\mathfrak{R}_t$ will exhaust H^+. A lattice transformation

$$s : (\xi_1, \ldots, \xi_n) \to (\xi_1^s, \ldots, \xi_n^s), \quad \xi_\mu^s = \sum_\nu \alpha_{\mu\nu}\, \xi_\nu \tag{54}$$

when expressed in terms of the N coordinates $x_{\mu i}$ appears as the linear transformation A, (20); it has the property that the coefficients both of A and its inverse A^{-1} are rational numbers with the common denominator j. A general principle of Siegel's[5]) asserts that, given $a > 0$ and $t > a$, there exists but a finite number of transformations A of this character which carry $\mathfrak{R}_a$ into sets that have points in common with $\mathfrak{R}_t$. This principle, which is a very powerful tool in all investigations concerning quadratic forms, including the indefinite ones, permits him to transfer the problem of the discontinuity of the lattice group from H to $\mathfrak{R}$. Compared to Minkowski's approach which the previous section followed in its outline, this method has the disadvantage of yielding undesirably high estimates for the number of such images Z_0^s of Z_0 as may be expected to have points in common with H_t. But it recommends itself by the generality of the underlying principle.

The lattice group consists of certain linear substitutions (54) and is, therefore, contained in the continuous group W of all non-singular linear substitutions $A = \|\alpha_{\mu\nu}\|$, $\alpha_{\mu\nu} \in \mathfrak{F}_K$. Consider continuous representations of W which become discontinuous under restriction of the

[5]) Abh. Math. Sem. Hansischen Univ. **13**, 1940, Satz 3, p. 217.

variable element s of W to the lattice group, in the sense that in the representation space no set of points which are equivalent under this group has an accumulation point. As the Theorem of Discontinuity in either of its two forms proves, the representation $\Gamma \to \Gamma^s = \Gamma[A^{-1}]$ in the space H^+ of all positive quadratic forms Γ is of this discontinuous nature. Siegel devotes the major part of his paper[6]) to developing a general principle from which it follows that among all representations of such nature our $\Gamma \to \Gamma^s$ is the "most compact" and therefore of least dimension. I shall not deal here with this side of the problem of reduction.

§ 10. Volume of the fundamental domain

Let us consider *that portion of the pyramid Z_0 of the $\mathfrak{L}$-reduced positive $\Gamma = ||\gamma_{\mu\nu}||$ whose points Γ satisfy the condition*

$$\mathrm{Nm}\,\Gamma \leqslant 1\,. \tag{55}$$

Using the Jacobi transformation of the coefficients $\gamma_{\mu\nu}$ into $\varkappa_\mu$, $\delta_{\mu\nu}$ ($\mu < \nu$) Siegel proved[7]) that this portion of Z_0 *has a finite volume V in the g_n-dimensional linear space H.* I describe here an alternative procedure which operates directly with the $\gamma_{\mu\nu}$ and leads to simpler estimates.

Dealing first with the lateral $\gamma_{\mu\nu}$ ($\mu < \nu$) set for a moment $-\gamma_{\mu\mu}^{-1}\gamma_{\mu\nu} = \beta_{\mu\nu}$ ($\mu < \nu$) and take $\beta_{25} = \beta$ as an example. Write $\gamma_{\mu\mu} = \gamma_\mu$. Choose any element ξ of $\{\mathfrak{J}\}$ and apply (38) to the lattice vector $\mathfrak{x} = (\xi_1, \ldots, \xi_n)$ of which all components ξ_μ vanish, except $\xi_2 = \xi$, $\xi_5 = \varepsilon$. One finds

$$\mathrm{tr}\,(\bar{\xi}\gamma_{22}\xi + \bar{\xi}\gamma_{25} + \gamma_{52}\xi) \geqslant 0 \qquad (\gamma_{52} = \bar{\gamma}_{25})$$

or

$$\mathrm{tr}\,\{(\bar{\beta} - \bar{\xi})\,\gamma_2(\beta - \xi)\} \geqslant \mathrm{tr}\,(\bar{\beta}\gamma_2\beta)\,. \tag{56}$$

Determine ξ in $\{\mathfrak{J}\}$ so that $\beta - \xi = \beta_0$ is reduced mod $\{\mathfrak{J}\}$. We make use of the upper bound furnished by Lemma 3.5, but replace the largest eigenvalue r^* by the sum $\mathrm{tr}\,\gamma$ of all eigenvalues:

$$\mathrm{tr}\,(\bar{\beta}_0\gamma_2\beta_0) \leqslant \mathrm{tr}\,\gamma_2 \cdot \mathrm{tr}\,(\bar{\beta}_0\beta_0) \leqslant \varrho^2 \cdot \mathrm{tr}\,\gamma_2\,.$$

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[6]) Discontinuous Groups, Ann. of Math. **44**, 1943, 674—684. Cf. also *M. Eichler*, Comm. Math. Helv. **11**, 1938/39, 253—272.

[7]) l. c.[6]), p. 688.

Hence (56) implies

$$\operatorname{tr}(\bar{\beta}_{\mu\nu}\gamma_\mu\beta_{\mu\nu}) \leqslant \varrho^2 \cdot \operatorname{tr}\gamma_\mu \qquad (\mu = 2,\ \nu = 5),$$

or more generally, for $\mu < \nu$,

$$\operatorname{tr}(\bar{\gamma}_{\mu\nu}\gamma_\mu^{-1}\gamma_{\mu\nu}) \leqslant \varrho^2 \cdot \operatorname{tr}\gamma_\mu\,. \tag{57}$$

For a fixed γ_μ we extend the integration with respect to the g real coefficients of the variable $\gamma_{\mu\nu}$ over the entire ellipsoid (57). Since by Lemma 5.2 $d \cdot (\operatorname{Nm}\gamma_\mu)^{-1}$ is the determinant of the quadratic form

$$\operatorname{tr}(\bar{\xi}\gamma_\mu^{-1}\xi) = T_\mu^*[x]$$

we find for the volume V_μ of this ellipsoid

$$V_\mu^2 = v_g^2\, d^{-1}\operatorname{Nm}\gamma_\mu \cdot \varrho^{2g}(\operatorname{tr}\gamma_\mu)^g$$

or, because of $\operatorname{Nm}\gamma_\mu \leqslant (g^{-1}\cdot\operatorname{tr}\gamma_\mu)^g$,

$$V_\mu \leqslant k(\operatorname{tr}\gamma_\mu)^g = k t_\mu^g$$

where

$$k = v_g\varrho^g(g^g\cdot d)^{-1/2}\,.$$

In view of (25_n) and (55) it remains to integrate

$$k^{\frac{1}{2}n(n-1)}\,(t_1^{n-1}t_2^{n-2}\ldots t_n^0)^g$$

with respect to the ng^+ coordinates of the variable symmetric positive $\gamma_1, \ldots, \gamma_n$ over the region described by

$$\left.\begin{array}{ll} 0 < t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n\,, & \\ t_1\ldots t_n \leqslant M_0^n & (M_0 = g\cdot c_n^{-1/ng}) \end{array}\right\}. \tag{58}$$

Let v_+/g^+ be the volume in the space of the g^+ coefficients of an arbitrary symmetric element γ of the bounded portion described by $\gamma > 0$, $\operatorname{tr}\gamma \leqslant 1$. Then the volume of the infinitely thin shell

$$\gamma > 0, \quad t \leqslant \operatorname{tr}\gamma \leqslant t + dt$$

is $v_+t^{g^+-1}dt$, and we obtain as an upper bound for V the integral of

$$k^{\frac{1}{2}n(n-1)}v_+^n\, t_1^{(n-1)g+(g^+-1)}\ldots t_n^{g^+-1}$$

extended with respect to the real variables $t_1, \ldots, t_n$ over (58). It is only at this last quite elementary step that we are dealing with a non-bounded domain. (58) implies

$$\begin{aligned} t_1 \ldots t_{n-1}\, t_n \quad &\leqslant M_0^n\,, \\ t_1 \ldots t_{n-2}\, t_{n-1}^2 &\leqslant M_0^n\,, \\ t_1 \ldots t_{n-3}\, t_{n-2}^3 &\leqslant M_0^n\,, \qquad t_\mu > 0\ . \\ \ldots\ldots\ldots\ldots&\ldots\ldots \\ t_1^n \qquad\qquad &\leqslant M_0^n\,, \end{aligned}$$

Integration over the larger domain described by these inequalities may be carried out step by step, first with respect to t_n,

$$0 < t_n \leqslant M_0^n\, t_1^{-1} \ldots t_{n-1}^{-1}\,, \qquad .$$

then with respect to t_{n-1},

$$0 < t_{n-1} \leqslant M_0^{n/2}\, t_1^{-1/2} \ldots t_{n-2}^{-1/2}$$

..., with the result

$$V \leqslant \frac{2^{n-1}}{n!\; g^+ g^{n-1}}\, k^{\frac{1}{2}n(n-1)}\, v_+^n\, M_0^{gn}\ .$$

For the rational case, $g = 1$, this upper limit is

$$\frac{2^{n-1}}{n\,!}\, c_n^{\,-\frac{1}{2}(n+1)}\ .$$

By following existing models for $g = 1$ one can find explicit expressions for the volume of the fundamental domain.

§ 11. The Group of Units in an Order of a Simple Algebra

From the fundamental domain of the lattice group and its images one obtains at once a fundamental domain for any subgroup of the lattice group of finite index. *This remark settles the problem for the group of units in any order of a simple algebra over k.*

According to Wedderburn, the elements A of such an algebra $\mathfrak{F}_n$ consist of the matrices

$$\left\| \begin{matrix} \alpha_{11}\,, & \ldots, & \alpha_{1n} \\ \ldots & \ldots & \ldots \\ \alpha_{n1}\,, & \ldots, & \alpha_{nn} \end{matrix} \right\|$$

formed by means of arbitrary elements $\alpha_{\mu\nu}$ of a division algebra $\mathfrak{F}$. Let again g be the rank of $\mathfrak{F}$ and set $N = ng$. $Nn = n^2 g$ is the rank of $\mathfrak{F}_n$. As previously, we interpret A as a linear mapping of the vector space $S^n/\mathfrak{F}$. Let $\{\mathfrak{F}_n\}$ be an order in $\mathfrak{F}_n$ and $\Omega_1, \ldots, \Omega_{Nn}$ a minimal basis of $\{\mathfrak{F}_n\}$. The first columns of all elements A of $\{\mathfrak{F}\}$ form a vector lattice $\mathfrak{A}$; the element A of $\{\mathfrak{F}_n\}$ is a mapping that carries every vector of the lattice $\mathfrak{A}$ into a vector of $\mathfrak{A}$. The *units* in $\{\mathfrak{F}_n\}$ are those elements A of $\{\mathfrak{F}_n\}$ which are one-to-one mappings of $\mathfrak{A}$ into itself. We now consider *all* elements B of $\mathfrak{F}_n$ which, interpreted as mappings, carry lattice vectors into lattice vectors; they form an order $\{\mathfrak{F}_n\}^* \supset \{\mathfrak{F}_n\}$, the units of which are our old lattice transformations s. We maintain that the group of units of $\{\mathfrak{F}_n\}$ is a subgroup *of finite index* within the group of units of $\{\mathfrak{F}_n\}^*$.

Indeed, express $\Omega_1, \ldots, \Omega_{Nn}$ in terms of a minimal basis $\Omega_1^*, \ldots, \Omega_{Nn}^*$ of $\{\mathfrak{F}_n\}^*$. The coefficients are rational integers; denote the absolute value of its determinant by h. Then the multiple hB of any element B of $\{\mathfrak{F}_n\}^*$ is an element A of $\{\mathfrak{F}_n\}$. Let B_1, B_2 be two units in $\{\mathfrak{F}_n\}^*$, the difference of which is of the form hB, B in $\{\mathfrak{F}_n\}^*$. Then

$$B_2 B_1^{-1} - E = h \cdot B B_1^{-1} = hB_3$$

or

$$B_2 B_1^{-1} = A \quad \text{where} \quad A = E + hB_3 .$$

Here B_3 lies in $\{\mathfrak{F}_n\}^*$, hence hB_3, A in $\{\mathfrak{F}_n\}$. Consequently the index whose finiteness we claim cannot exceed h^{Nn}.

(Received the 10 April 1945.