On the use of a complex (Quaternion) velocity potential in three dimensions.

On the use of a complex (Quaternion) velocity potential in three dimensions

By ALAN ROSE, Manchester

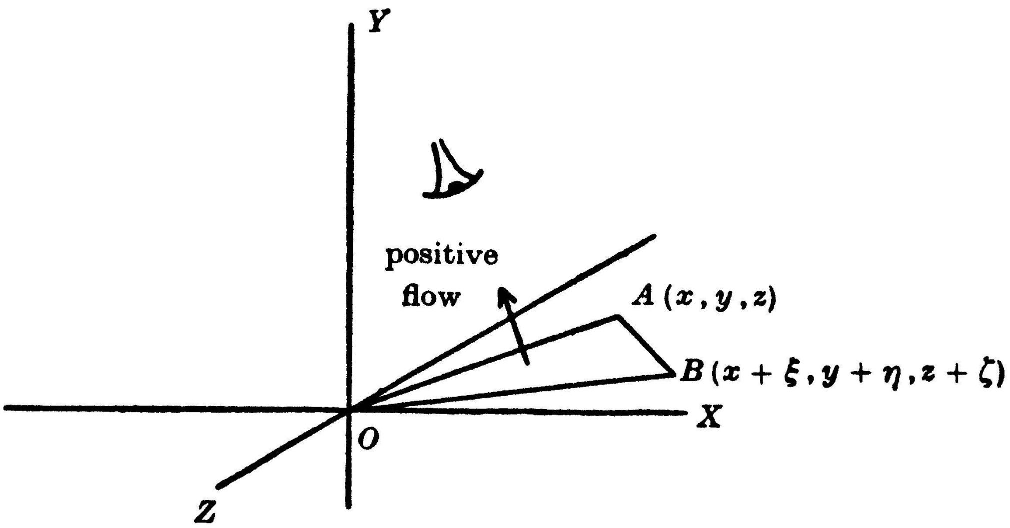
The following investigations concern the use of a stream-function which can be defined for any three-dimensional motion. It appears that in the case of motion which is axially symmetrical, if we take a set of rectangular axes OX, OY, OZ and an imaginary fourth axis OW perpendicular to the other three, then in terms of the velocity-potential Φ and the stream-function Ψ we can define a function

$$\Phi + i\,\Psi_1 + j\,\Psi_2 + k\,\Psi_3$$

which is a right-regular quaternion function of $w + ix + jy + kz$.

The theory is used to determine the effect of placing a point-source on the axis of symmetry of an arbitrary solid of revolution.

Definition: We define the stream-function $\Psi(x, y, z, \xi, \eta, \zeta)$ of the pair of points $A(x, y, z)$, $B(x + \xi, y + \eta, z + \zeta)$ to be the rate of flow of fluid across the triangle formed by these two points and the origin. We make the sign convention that if when an observer views the triangle OAB with AB appearing horizontal, O below AB, B to the right of A, and the plane of the triangle appearing vertical the flow is towards the observer, then Ψ is positive.



Velocity at the point (x, y, z) in terms of Ψ

We make the convention that all partial derivatives are considered to be evaluated at $(x, y, z, \xi, \eta, \zeta) = (x, y, z, 0, 0, 0)$.

Lemma: Since we can regard the point (x, y, z) as the point $(x + \xi - \xi, y, z)$ we have

$$\Psi(x, y, z, \xi, 0, 0) = -\Psi(x + \xi, y, z, -\xi, 0, 0) .$$

Hence, by Taylor's Theorem, since for all x, y, z $\Psi(x, y, z, 0, 0, 0) = 0$

$$\xi \frac{\partial \Psi}{\partial \xi} + \tfrac{1}{2} \xi^2 \frac{\partial^2 \Psi}{\partial \xi^2} - \xi \frac{\partial \Psi}{\partial \xi} + \tfrac{1}{2} \xi^2 \frac{\partial^2 \Psi}{\partial \xi^2} - \xi^2 \frac{\partial^2 \Psi}{\partial x\, \partial \xi} = O(\xi^3) .$$

Hence, proceeding to the limit $\xi \to 0$ we have

$$\frac{\partial^2 \Psi}{\partial \xi^2} = \frac{\partial^2 \Psi}{\partial x\, \partial \xi}$$

Similarly

$$\frac{\partial^2 \Psi}{\partial \eta^2} = \frac{\partial^2 \Psi}{\partial y\, \partial \eta}$$

and

$$\frac{\partial^2 \Psi}{\partial \zeta^2} = \frac{\partial^2 \Psi}{\partial z\, \partial \zeta} .$$

Let the components of velocity in directions parallel to OX, OY, OZ be respectively v_x, v_y, v_z.

We consider the inward rates of flow across the faces of the tetrahedron $ABCO$ where the coordinates of A, B, C are (x, y, z), $(x, y + \eta, z)$, $(x, y, z + \zeta)$ respectively. Since the algebraic sum of these rates of flow is zero, we have

$$\Psi(x, y + \eta, z, 0, -\eta, 0) + \Psi(x, y, z, 0, 0, \zeta) + \Psi(x, y, z + \zeta, 0, \eta, -\zeta)$$

$-$ rate of flow across ABC in positive direction of $OX = 0$.

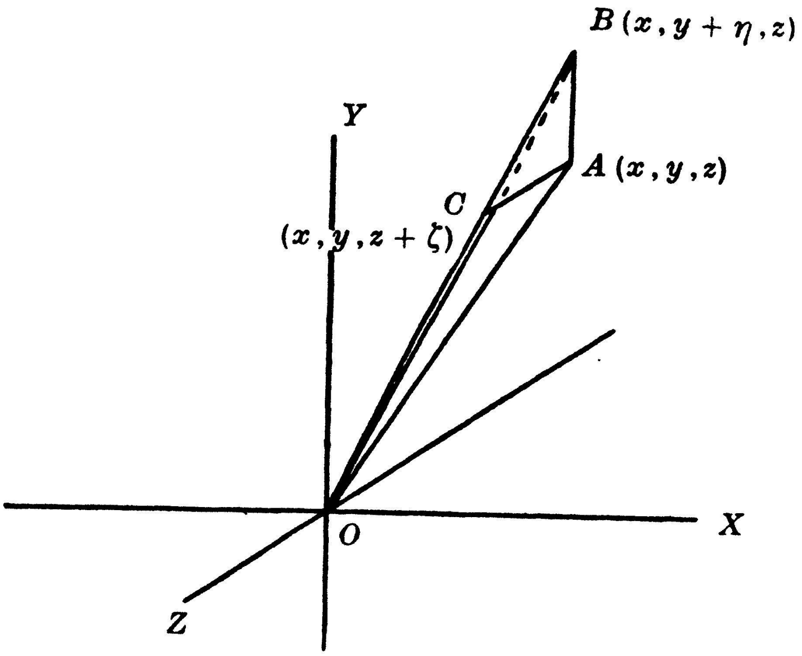
Hence we have by Taylor's Theorem

$$-\eta \frac{\partial \Psi}{\partial \eta} + \tfrac{1}{2} \eta^2 \frac{\partial^2 \Psi}{\partial \eta^2} - \eta^2 \frac{\partial^2 \Psi}{\partial y\, \partial \eta} + \zeta \frac{\partial \Psi}{\partial \zeta} + \tfrac{1}{2} \zeta^2 \frac{\partial^2 \Psi}{\partial \zeta^2} + \eta \frac{\partial \Psi}{\partial \eta} + \tfrac{1}{2} \eta^2 \frac{\partial^2 \Psi}{\partial \eta^2}$$

$$-\frac{\zeta \partial \cdot \Psi}{\partial \zeta} + \tfrac{1}{2} \zeta^2 \frac{\partial^2 \Psi}{\partial \zeta^2} - \eta\zeta \frac{\partial^2 \Psi}{\partial \eta\, \partial \zeta} - \zeta^2 \frac{\partial^2 \Psi}{\partial z\, \partial \zeta} + \eta\zeta \frac{\partial^2 \Psi}{\partial z\, \partial \eta} + \text{terms of degree 3}$$

or more in η, ζ = rate of flow across ABC in positive direction of OX.

Hence, by the lemma



Flow across $ABC = \eta\zeta\left(\frac{\partial^2 \Psi}{\partial z\,\partial\eta} - \frac{\partial^2 \Psi}{\partial\eta\,\partial\zeta}\right)$ + terms of degree 3 or more in η, ζ. Hence, dividing by $\frac{1}{2}\eta\zeta$ and proceeding to the limit $\eta, \zeta \to 0$, we have

$$v_x = 2\left(\frac{\partial^2 \Psi}{\partial z\,\partial\eta} - \frac{\partial^2 \Psi}{\partial\eta\,\partial\zeta}\right) .$$

Similarly, by considering outward rates of flow, we have

$$v_x = -2\left(\frac{\partial^2 \Psi}{\partial y\,\partial\zeta} - \frac{\partial^2 \Psi}{\partial\eta\,\partial\zeta}\right) .$$

Hence, by addition

$$v_x = \frac{\partial^2 \Psi}{\partial z\,\partial\eta} - \frac{\partial^2 \Psi}{\partial y\,\partial\zeta} .$$

If we take the velocity potential convention that

$$\boldsymbol{v} = -\operatorname{grad} \Phi$$

we have

$$\frac{\partial\Phi}{\partial x} + \frac{\partial^2 \Psi}{\partial z\partial\eta} - \frac{\partial^2 \Psi}{\partial y\,\partial\zeta} = 0 .$$

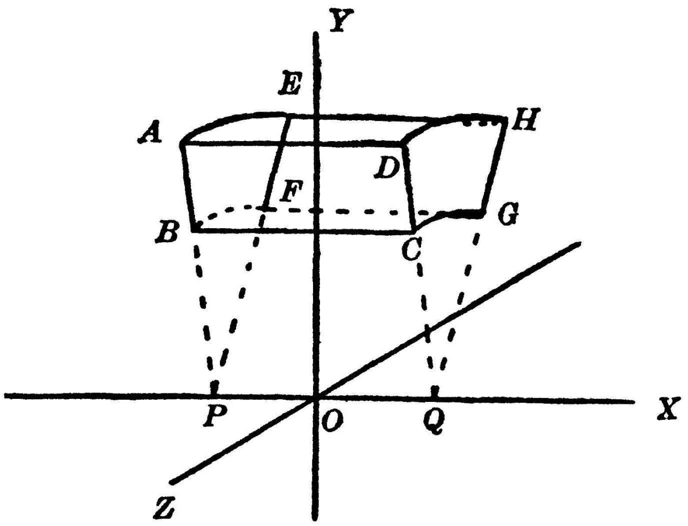
Similarly

$$\frac{\partial\Phi}{\partial y} + \frac{\partial^2 \Psi}{\partial x\,\partial\zeta} - \frac{\partial^2 \Psi}{\partial z\,\partial\xi} = 0$$

and

$$\frac{\partial\Phi}{\partial z} + \frac{\partial^2 \Psi}{\partial y\,\partial\xi} - \frac{\partial^2 \Psi}{\partial x\,\partial\eta} = 0 .$$

Since all functions occurring so far are independent of w, all partial derivatives with respect to w are zero. Hence, if we define $\Psi_1 = \frac{\partial \Psi}{\partial \xi}$, $\Psi_2 = \frac{\partial \Psi}{\partial \eta}$, $\Psi_3 = \frac{\partial \Psi}{\partial \zeta}$, the last three of the four conditions for $\Phi + i\Psi_1 + j\Psi_2 + k\Psi_3$ to be a right-regular function of $w + ix + jy + kz$ are satisfied.



Axially symmetrical motion, the fourth regularity condition

We define the vector Ψ to be the vector whose components at any point (x, y, z) in directions parallel to OX, OY, OZ respectively, are the values of $\frac{\partial \Psi}{\partial \xi}$, $\frac{\partial \Psi}{\partial \eta}$, $\frac{\partial \Psi}{\partial \zeta}$ at that point.

Since the motion is axially symmetrical, the stream-function of any pair of points in an axial plane is zero. Hence the vector Ψ at any point will be perpendicular to the axial plane through that point. Also, in view of the axial symmetry, Ψ will have the same value at any two points whose cylindrical coordinates differ only in the value of the angular coordinate. We now consider the integral $\iint \boldsymbol{\Psi} \cdot \boldsymbol{n} dS$ where $\boldsymbol{n}$ is the unit normal vector over the surface $ABCDEFGH$, where $ABCD$, $EFGH$ are portions of axial planes and $BCGF$, $ADHE$ are portions of the surfaces of cylinders whose axes are coincident with the axis of symmetry of the motion. In view of what we have proved about the direction of Ψ, the contributions of all parts of the surface other than $ABCD$ and $EFGH$ will be zero, and in view of our result about cylindrical coordinates the contributions of these two faces to the integral will be equal and opposite. Hence the integral $\iint \boldsymbol{\Psi} \cdot \boldsymbol{n} dS$ over the surface is zero. Hence the integral of div. Ψ over the volume of the interior of the surface will be zero by Green's Theorem. Since this is true for all bodies $ABCDEFGH$ of this type, wherever situated and however small, the value of div. Ψ will be zero everywhere. But since $\dfrac{\partial \Phi}{\partial w}$ is zero everywhere, we have

$$\frac{\partial \Phi}{\partial w} - \frac{\partial \Psi_1}{dx} - \frac{\partial \Psi_2}{\partial y} - \frac{\partial \Psi_3}{\partial z} = 0 ,$$

which is the remaining condition required for $\Phi + i\,\Psi_1 + j\,\Psi_2 + k\,\Psi_3$ to be a right-regular function of $w + ix + jy + kz$.

The Stream-function of a Uniform Stream

We consider a uniform stream moving with velocity V in the positive direction of the x-axis. The area of the triangle formed by the origin and the points (x, y, z), $(x + \xi, y + \eta, z + \zeta)$ expressed vectorially is

$$-\tfrac{1}{2}(x, y, z) \wedge (x + \xi, y + \eta, z + \zeta)$$

i. e.

$$-\tfrac{1}{2}(x, y, z) \wedge (\xi, \eta, \zeta) \ .$$

Hence the stream-function is the scalar product of this vector with the vector $(V, 0, 0)$, so that we have

$$\Psi = \tfrac{1}{2} V(-\zeta y + \eta z)$$

The complex potential of the motion will therefore be

$$V(-x + \tfrac{1}{2} jz - \tfrac{1}{2} ky) \ .$$

The condition that f(x, y, z, ξ, η, ζ) be a stream-function

Let the pseudo-velocity corresponding to $f(x, y, z, \xi, \eta, \zeta)$ have components v_x, v_y, v_z, and let the actual stream-function of this motion be $\Psi(x, y, z, \xi, \eta, \zeta)$. Then

$$\frac{\partial^2(\Psi - f)}{\partial z\, \partial \eta} - \frac{\partial^2(\Psi - f)}{\partial y\, \partial \zeta} = v_x - v_x = 0$$

$$\frac{\partial^2(\Psi - f)}{\partial x\, \partial \zeta} - \frac{\partial^2(\Psi - f)}{\partial z\, \partial \xi} = 0 , \qquad \frac{\partial^2(\Psi - f)}{\partial y\, \partial \xi} - \frac{\partial^2(\Psi - f)}{\partial x\, \partial \eta} = 0 .$$

From now on we will choose our axes so that OX is the axis of symmetry. Since the stream vector (Ψ) of any point is perpendicular to the axial plane through that point, we have

$$\frac{\partial \Psi}{\partial \xi} = 0 \ , \quad \frac{\partial \Psi}{\partial \eta} \Big/ \frac{\partial \Psi}{\partial \zeta} = - \frac{z}{y} \ .$$

Hence, in view of the axial symmetry[1])

$$\frac{\partial (\Psi - f)}{\partial \eta} = z \chi (x, y^2 + z^2) \ , \qquad \frac{\partial (\Psi - f)}{\partial \zeta} = - y \chi (x, y^2 + z^2)$$

so that, writing g for $\Psi - f$, and R for $x^2 + y^2$ we have

$$\chi + 2z^2 \frac{\partial \chi}{\partial R} + \chi + 2y^2 \frac{\partial \chi}{\partial R} = 0 \ , \quad y \frac{\partial \chi}{\partial x} = 0 \ \text{ so that } \ \chi + R \frac{d\chi}{dR} = 0$$

whence

$$\chi (x, y^2 + z^2) = k / (y^2 + z^2)$$

where k is a constant. Hence

$$\Psi = f + g = f + \frac{kz}{y^2 + z^2} \eta - \frac{ky}{y^2 + z^2} \zeta \ .$$

It is usually possible to determine k by considering the value of Ψ at infinity.

The stream-function of a unit point source at the point $(a, 0, 0)$

In view of the axial symmetry, we have from the above that Ψ must be of the form

$$(\eta z - \zeta y) \chi (x, y^2 + z^2) \ .$$

Since the velocity vector at (x, y, z) is

$$\left(\frac{x - a}{((x-a)^2 + y^2 + z^2)^{3/2}} \ , \ \frac{y}{((x-a)^2 + y^2 + z^2)^{3/2}} \ , \ \frac{z}{((x-a)^2 + y^2 + z^2)^{3/2}} \right)$$

we have

$$y \frac{\partial \chi}{\partial x} = - \frac{y}{((x-a)^2 + y^2 + z^2)^{3/2}} \ , \quad \chi = \frac{-x + a}{(y^2 + z^2)((x-a)^2 + y^2 + z^2)^{1/2}} + \\ + f(y^2 + z^2) \ ,$$

[1]) We are assuming here that f is also of this form, otherwise it is obviously not a stream-function.

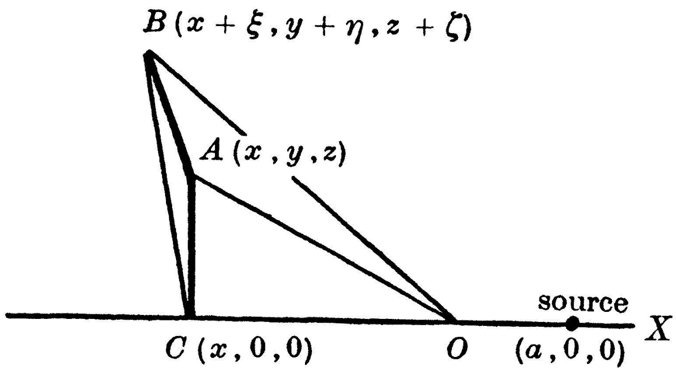
where f is an arbitrary function, so that a value of Ψ which on differentiation yields correct values for the y- and z-components of velocity is

$$\frac{(\eta z - \zeta y)(-x + a)}{(y^2 + z^2)\left((x - a)^2 + y^2 + z^2\right)^{1/2}} .$$

By differentiation it is readily seen that in this way this value of Ψ also yields the correct value for the x-component of velocity. Hence for some value of k the correct value of the stream-function is

$$\frac{(-\eta z + \zeta y)(x - a)}{(y^2 + z^2)\left((x - a)^2 + y^2 + z^2\right)^{1/2}} + \frac{k}{y^2 + z^2}(\eta z - \zeta y) .$$

The value of k depends upon whether a is greater or less than 0. We consider the tetrahedron formed by the origin, the points (x, y, z) and $(x + \xi, y + \eta, z + \zeta)$, and the point $(x, 0, 0)$.



We consider the case $a > 0$, and consider $ABCO$ for large negative x. Since the source is outside the tetrahedron, the algebraic sum of the rates of flow across the faces of $ABCO$ is zero. Since ACO, BCO are portions of axial planes, the rates of flow across these faces are zero. Hence $\Psi(x, y, z, \xi, \eta, \zeta)$ is equal to the rate of flow across the face ABC. But if we let $x \to -\infty$ while y and z remain finite, the area of ABC will remain finite and the velocity at points of ABC will tend to zero. Hence Ψ will tend to zero. Now as $x \to -\infty$ the above value of Ψ tends to

$$\frac{(\eta z - \zeta y)(1 + k)}{y^2 + z^2}$$

Hence $k = -1$. Thus

$$\Psi = \frac{(-\eta z + \zeta y)}{y^2 + z^2}\left(\frac{x - a}{\left((x - a)^2 + y^2 + z^2\right)^{1/2}} + 1\right) .$$

Similarly when $a > 0$

$$\Psi = \frac{(-\eta z + \zeta y)}{y^2 + z^2}\left(\frac{x - a}{((x-a)^2 + y^2 + z^2)^{1/2}} - 1\right) .$$

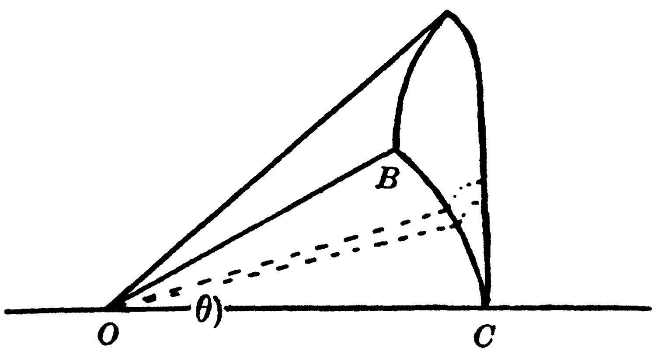
When $a = 0$ we have a singularity at the origin and so this method cannot be used. This difficulty can be overcome, however, by an ϵ-method similar to that of the next two sections.

The stream-function corresponding to the velocity potential $r^{-n-1} P_n(\theta)$

Since the motion is axially symmetrical Ψ will be of the form

$$(\eta z - \zeta y)\, \chi(x, y^2 + z^2) .$$

We consider the figure formed by an isosceles triangle whose equal sides are equally inclined to the x-axis, the circular arcs, centre O, joining the extremities of these sides to the x-axis and the portion of spherical surface determined by them, and the plane portions determined by these two sides, the arcs and the x-axis.



We first consider a velocity potential of the form $(r - a)^{-n-1} P_n(\theta)$ where a is small and positive and take $(a, 0, 0)$ as centre of the sphere while leaving O as a vertex.

Since

$$\int_0^\pi P_n(\theta) \sin\theta \, d\theta .$$

is zero, the algebraic sum of the rates of flow across the faces of $OABC$ is zero, and since OAC, OBC are portions of axial planes the rates of flow across these are zero. When A and B are near together we can identify the surface ABO with the triangle ABO, so that $\Psi(x, y, z, \xi, \eta, \zeta)$

is equal to the rate of flow across the spherical surface ABC. Now since OA and OB are equal in length and equally inclined to OX, the spherical angle ACB is equal to

$$\frac{\eta z - \zeta y}{y^2 + z^2}\ .$$

Hence the rate of flow across ABC is[2])

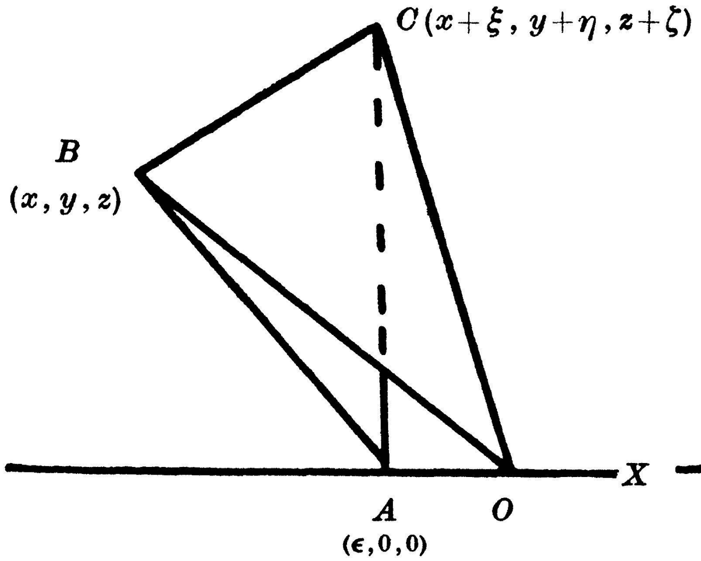
$$\int_0^{\tan^{-1}\left(\frac{\sqrt{(y^2+z^2)}}{x-a}\right)} \frac{\eta z - \zeta y}{y^2 + z^2} \sqrt{((x-a)^2 + y^2 + z^2)}$$
$$\sin\theta \sqrt{((x-a)^2 + y^2 + z^2)}\, d\theta \frac{(-2\cos\theta)}{((x-a)^2 + y^2 + z^2)^{3/2}}$$

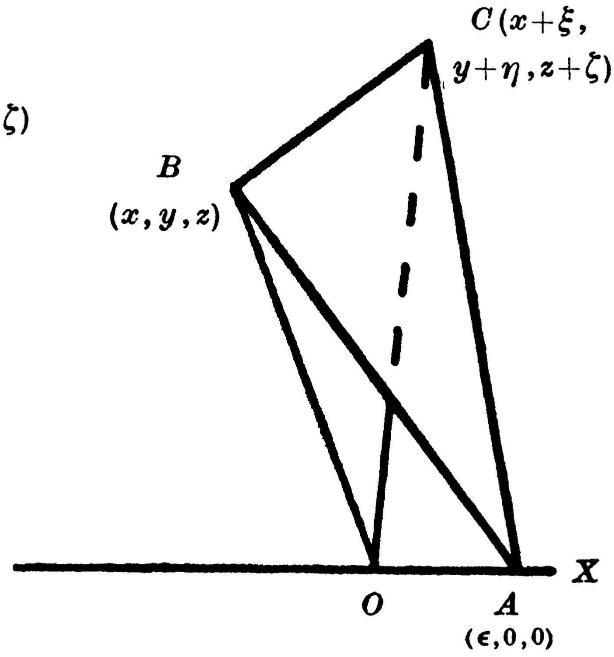
$$= \frac{\eta z - \zeta y}{y^2 + z^2} \frac{1}{((x-a)^2 + y^2 + z^2)^{1/2}} \left(\frac{x^2}{x^2 + y^2 + z^2} - 1\right) .$$

Thus

$$\Psi = \frac{\eta z - \zeta y}{((x-a)^2 + y^2 + z^2)^{3/2}}\ .$$

Similarly this holds if a is a little less than 0. However this method does not apply when $a = 0$, since in this case we have a singularity at the origin. If, however we take the point $(\epsilon, 0, 0)$, where $\epsilon \neq a, a \neq 0$ instead of the origin, as a vertex of the triangle associated with the definition of Ψ, the value of Ψ for $\Phi = (r - a)^{-2} P_1(\theta)$ will be the same as before.





[2]) for $n = 1$; for other values the method is similar.

For since $\epsilon \neq a$, the algebraic sum of the rate of outward flow of fluid across the faces of the tetrahedron $OABC$ is zero.[3])

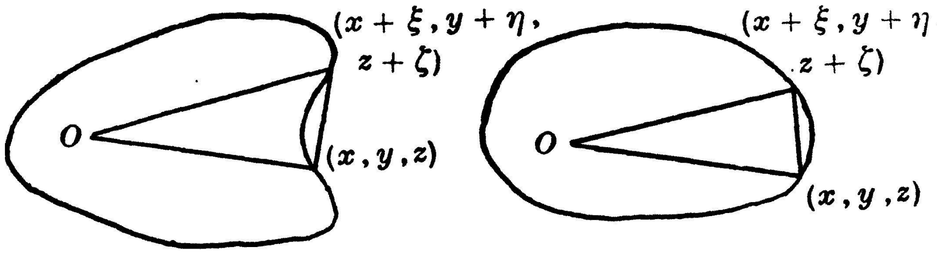
But as the motion is axially symmetrical the rate of $\begin{matrix}\text{outward (L. H. Diagram)}\\ \text{inward (R. H. Diagram)}\end{matrix}$ flow across OBC is equal to the rate of $\begin{matrix}\text{inward (L. H. Diagram)}\\ \text{outward (R. H. Diagram)}\end{matrix}$ flow across ABC, so that the value of Ψ is unaltered if we replace O by A. Hence when $a \neq 0$ we have that if $\epsilon \neq 0$ and $\epsilon \neq a$, then

$$\Psi = \frac{\eta z - \zeta y}{((x-a)^2 + y^2 + z^2)^{3/2}}$$

and by continuity, if $a = 0$ then

$$\Psi = \frac{\eta z - \zeta y}{(x^2 + y^2 + z^2)^{3/2}} .$$

The boundary conditions satisfied on the surface of an obstacle containing the origin



Since the stream-function of a pair of points on the surface of the obstacle is small compared with ξ, η and ζ when ξ, η and ζ are small (and is in fact zero when the body is convex), the tangential components of the stream-vector are zero. (This boundary condition does not apply to bodies for which lines can be drawn through the origin meeting the body more than twice.) Hence the stream-vector is either the null-vector or else it is along the normal to the surface at the point under consideration. But we have already shown that the stream-vector is either the null-vector or else it is perpendicular to the axial plane through the point under consideration. Hence the stream-vector is the null-vector, i. e. the boundary conditions are

$$\frac{\partial \Psi}{\partial \xi} = 0 , \quad \frac{\partial \Psi}{\partial \eta} = 0 , \quad \frac{\partial \Psi}{\partial \zeta} = 0 .$$

[3]) There may be a singularity at an interior point of OA, but this will have no effect since $\int_0^\pi P_n(\theta) \sin\theta \, d\theta = 0$.

Regular functions associated with motion symmetrical about OX

It has been shown that if $f(z)$ is an analytic function of the quaternion variable z, then $\Delta(f(z))$ is a right- and left-regular function of z.

Corresponding to $x + iy$ in two variables, we have in four variables the first degree regular polynomial $\Delta(z^3)$, i. e. $-4(3w + ix + jy + kz)$.

For axially symmetrical motions we need functions in which the coefficients of j and k are in the ratio $z : -y$, and the coefficient of i is zero. As the highest coefficients in the two above cases are 1 and 3 respectively, and the other coefficients are all 1, this suggests that a corresponding function for three real variables may be

$$2x - jz + ky \ .$$

This can easily be shown to be, in fact, right-regular.

Corresponding to $(x + iy)^{-1}$ or $\dfrac{x - iy}{x^2 + y^2}$ in two variables we have in four variables $\Delta(z^{-1})$ or

$$\frac{-4(w - ix - jy - kz)}{(w^2 + x^2 + y^2 + z^2)^2} \ .$$

Since the numerators both have 1 for the first coefficient and -1 for all others, and the denominators are of degree 2 and 4 respectively, this suggests as a corresponding function in three variables

$$\frac{x + jz - ky}{(x^2 + y^2 + z^2)^{3/2}}$$

and this can easily be verified to be right-regular.

These stream-functions are in fact the complex potentials evaluated above for a uniform stream of velocity $(-2, 0, 0)$ and for the velocity potential $r^{-2} P_1(\theta)$.

Determination of the effect on a uniform stream of a spherical obstacle, by a method analogous to the complex-variable method for the cylinder [4])

We consider the sphere $x^2 + y^2 + z^2 = 1$ in a stream of velocity $(-2, 0, 0)$. In the cylindrical case we find that by transforming

$$x + iy + \frac{x - iy}{x^2 + y^2} \quad \text{into} \quad \lambda + i\mu$$

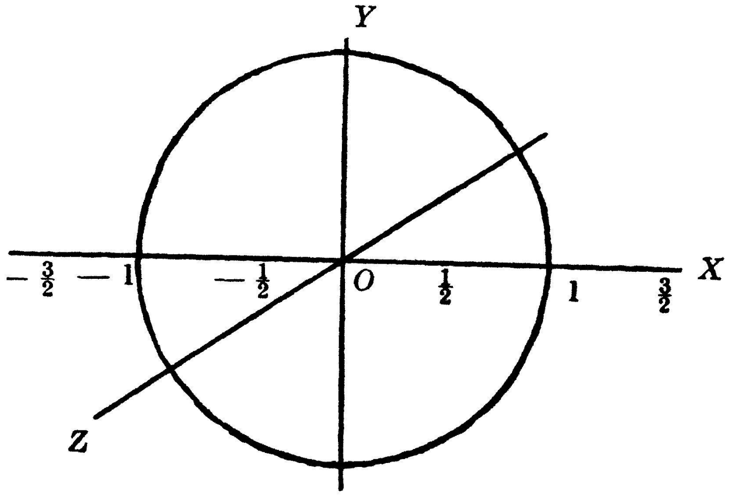
[4]) We are here considering the case $|\epsilon|(|a| - |\epsilon|) > 0$.

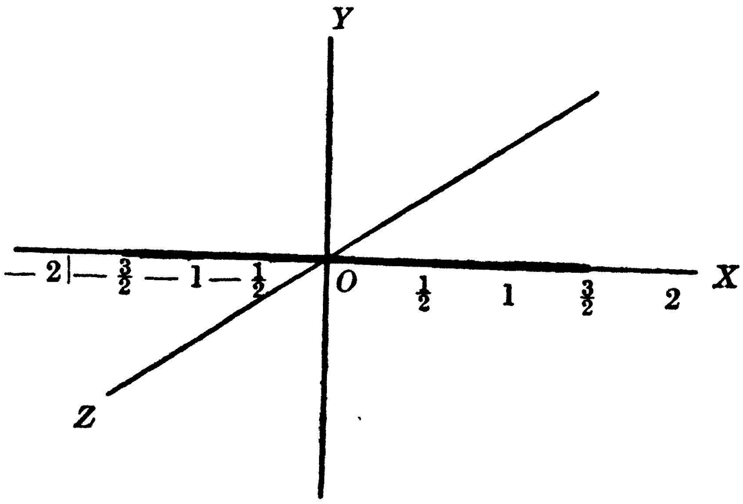
the cylinder is transformed into a plane surface of finite width. Taking the corresponding functions of three variables, we find that by transforming

$$2x - jz + ky + \frac{x + jz - ky}{(x^2 + y^2 + z^2)^{3/2}} \quad \text{into} \quad 2\lambda - j\nu + k\mu$$

the sphere $x^2 + y^2 + z^2 = 1$ is transformed into that part of the x-axis for which $-\frac{3}{2} \leqslant x \leqslant \frac{3}{2}$. For, when $x^2 + y^2 + z^2 = 1$,

$$2x - jz + ky + \frac{x + jz - ky}{(x^2 + y^2 + z^2)^{3/2}} = 3x = 2\lambda - j\nu + k\mu \ .$$





The boundary conditions are preserved for the same reasons as in the two dimensional case:

Since the needle-shaped body does not disturb the motion for the new system of coordinates, we have that the complex potential is

$$2\lambda - j\nu + k\mu \quad \text{or} \quad \frac{2x - jz + ky}{1} + \frac{x + jz - ky}{(x^2 + y^2 + z^2)^{3/2}}$$

so that the velocity potential is

$$2x + \frac{x}{(x^2 + y^2 + z^2)^{3/2}} \quad \text{or} \quad 2\left(r\cos\theta + \tfrac{1}{2} r^{-2}\cos\theta\right) .$$

Determination of the effect of placing a point source on the axis of symmetry of an arbitrary solid of revolution which can be made to satisfy the boundary conditions

We choose the axes so that the axis of symmetry is the x-axis and that the origin is inside the body. We consider only bodies for which this can be done in such a way that any line through the origin meets the body exactly twice.

We consider the domain bounded by the hyper-cylinders A and B, where the axes of the cylinders are the w-axis and their sections by the prime $w = 0$ are the body and the sphere $x^2 + y^2 + z^2 = r^2$ respectively, and r is chosen so that A and B have no point in common.

Let the equation of the surface of the body be $y^2 + z^2 = f(x)$, and the source be of strength m at the point $(a, 0, 0)$.

In view of the boundary conditions, the imaginary parts of the complex potential at points on the boundary of the obstacle are zero.

We define the complex potential at (x, y, z, w) to be the same as at $(x, y, z, 0)$, thus making the complex potential right-regular at all points outside the body except $(a, 0, 0, w)$. On the surface of A the real part of the complex potential can be regarded as a function of x only, since it is independent of w and the motion is axially symmetrical.

Thus we can express Φ as a Fourier series in x. We assume that all sin terms after the first n and all cos terms after the first n can be neglected. If the result obtained does not confirm this, we must start again, taking a larger value for n.

We consider the domain determined by A and B in the case when the w-coordinates of the plane faces of the ends of the cylinders all four tend to infinity, one tending to $+\infty$ and one to $-\infty$ for each cylinder. We remove from the domain a small hyper-cylinder C whose ends form part of the same planes as the ends of the other cylinders, and whose axis has the equations: $x = a$, $y = 0$, $z = 0$. Using Fueter's second Integral Theorem we calculate the complex potential at all points of the domain and express its value on the surface of A as a Fourier series in x to n sine terms and n cosine terms. Equating the n sine coefficients and the n cosine coefficients to the original values, we have $2n$ homogeneous equations in the $2n$ original unknown coefficients. These can therefore be solved to within a constant factor. This factor can then be determined by taking into consideration the boundary conditions. The integrals required for the application of the second Integral Theorem can be evaluated as follows. The complex potential tends to zero as the distance from the body of the point under consideration tends to infinity, so that as the plane ends of the outermost cylinder are of finite area, their contribution to the integral tends to zero. The contribution of the surface A can be evaluated in terms of the original unknown coefficients. Since the disturbing terms of the complex potential will be right-regular outside A, we have by Integral Theorems 1 and 2 that the contribution of that part of the value on B relating to the undisturbed motion and the contribution of C will together be the complex-potential of the undisturbed source. To find the effect of the rest of B's contribution, we consider the domain determined by B and a coaxial cylinder of large radius. As before, we let the lengths of the axes of the cylinders tend to infinity. Let us denote the outer surface by D. By the first Integral Theorem, the contribution of B is the same as that of D would have been. But since the disturbing terms must be of degree less than -1 in x, y and z, the contribution of D tends to zero as its radius tends to infinity. Hence the contribution of B is zero.

I should like to express my gratitude to Dr. K. Mahler and Prof. Dr. R. Fueter for two important suggestions relating to the earlier and latter parts of this paper respectively.

(Received the 11th July 1949.)

REFERENCE

Rud. Fueter, The function-theory of the differential equations $\Delta u = 0$ and $\Delta\Delta u = 0$ with four real variables. Comm. Math. Helv. 7 (1934/35), p. 307.