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# Linear Accessibility of Boundary Points of a Jordan Region

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A problem of long standing in the theory of conformal mapping concerns the finding of conditions on a set $E$ on a Jordan curve $C$ that are necessary and sufficient in order that the conformal mapping of the region bounded by $C$ onto the unit disc should cast the set $E$ into a set of Lebesgue measure zero on the unit circle $C_1$. Seidel and Walsh [2], and Tsuji [3], have shown that the condition of not being accessible from the interior by curves of finite length is sufficient. On the other hand, Lohwater and Seidel [1] have shown that the condition of being of linear measure zero is not necessary; they have constructed a Jordan region $G$ whose boundary contains a collinear set $E$ of positive measure with the property that the image of $E$ on $C_1$ is of measure zero. In this particular example, the points in $E$ happen not to be lineary accessible (see definition below), and it becomes natural to ask whether in the general case the set of linearly inaccessible points on $C$ is always mapped into a set of measure zero on $C_1$. The present note describes an example that answers this question in the negative.

**Definition.** *A point $P$ on the boundary of a region $G$ is linearly accessible provided there exists a point $Q$ such that the rectilinear segment $PQ$ is interior to $G$, save for the point $P$.*

**Theorem.** *There exists a function, regular in the open unit disc, and schlicht and continuous in the closed unit disc, which maps the unit disc into a region bounded by a simple closed Jordan curve $C$ in such a way that the set of linearly accessible points on $C$ is the image of a set of measure zero on the unit circle.*

Let $k$ be a real constant $(0<k<1)$; let $\{n_j\}$ be a sequence of integers satisfying the conditions

$$n_1 = 1 ,$$
$$n_{j+1} \geqslant 8 n_j^3\, 2^j\, (1-k)^{-4j} \qquad (j = 1, 2, \ldots) \qquad (1)$$

and let

$$f_1(z) = z \ ,$$
$$f_j(z) = f_{j-1}(z) + k\, z^{n_j} f'_{j-1}(z)/n_j \qquad (j = 2, 3, \ldots) \ . \qquad (2)$$

It will be shown that the sequence $\{f_j(z)\}$ converges to a function $f(z)$ which is schlicht in the open unit disc and continuous on the closed unit disc, and that it maps a set of measure $2\pi$ into a set of linearly inaccessible boundary points. It seems fairly obvious that the function also maps the unit circle into a Jordan curve; but to avoid tedious arithmetic in the proof, it will be necessary to assume that the index $n_2$ is large enough so that certain inequalities are satisfied.

To sketch the intuitive ideas that suggested the construction and the proof, we consider first the functions

$$\Phi_1(z) = z \ ,$$
$$\Phi_j(z) = z + k \sum_{r=2}^{j} \frac{z^{n_r}}{n_r} \ .$$

The maps $\Gamma_j$ of the unit circle $C_1$ by these functions have the following properties: the curve $\Gamma_2$ can be obtained by putting into the unit circle $n_2 - 1$ waves of appropriate shape and of amplitude $2k/n_2$; the curve $\Gamma_j$ can be obtained by imposing $n_j - 1$ waves of appropriate shape and of amplitude $2k/n_j$ upon the curve $\Gamma_{j-1}$. But a difficulty arises with regard to schlichtness: a major portion of those arcs of $\Gamma_r$ which lie outside (inside) of the curve $\Gamma_{r-1}$ represents stretched (compressed) arcs of $\Gamma_{r-1}$; and no matter how small the constant $k$ is chosen, every curve $\Gamma_r$ of sufficiently high index contains arcs constituting such highly compressed images of the corresponding arcs on $C_1$ that some of the "waves" of $\Gamma_{r+1}$ become loops. In case of the functions $f_j(z)$, this difficulty is overcome by means of the factor $f'_{j-1}(z)$ that occurs in the right member of (2). Geometrically, this factor has the following effect: If $C_j$ is the map of $C_1$ by $f_j(z)$, and if the arc $A$ on $C_{r-1}$ corresponds to a stretched (compressed) portion of $C_1$, the waves imposed on $A$ in the formation of $C_r$ are relatively large (small); all the waves are of much the same shape, and looping of the approximation curves $C_j$ and their limit curve $C$ is avoided.

At each point on certain arcs of $C_{r+1}$, the tangent line to $C_{r+1}$ makes an angle of approximately $\sin^{-1} k$ with the tangent line at the corresponding point on $C_r$. Certain portions of these arcs on $C_{r+1}$ are images of arcs on $C_{r+2}$ that stand in a similar relation to $C_{r+1}$; etc. Upon drawing a sketch of the curve $C_3$ (with $k$ near unity), it becomes apparent how a certain portion of every arc of the limit curve $C$ is twisted so that its points are linearly inaccessible from the interior, except by means of line segments shorter than the diameter of the arc. The rest is obvious, and it remains only to sketch the details of the analytical version of the proof.

**Lemma 1.** *The degree of the polynomial* $f_j(z)$ *is less than* $2n_j$.

This lemma follows from the relations (1) and (2) by induction. It implies that

$$f_j(z) = \sum_{m=1}^{2n_j-1} a_m z^m ,$$

where the coefficients $a_m$ are real, non-negative, and independent of the index $j$.

**Lemma 2.** *When* $j = 2, 3, \ldots,$ *the coefficients* $a_m$ *satisfy the conditions*

$$\begin{aligned} a_m &< \tfrac{1}{4} n_{j-1}^{-2} (1-k)^{4(j-1)} && (n_j \leqslant m < n_j + 2n_{j-1} - 3) \quad , \\ a_m &= 0 && (n_j + 2n_{j-1} - 3 \leqslant m < n_{j+1}) \ . \end{aligned}$$

The second condition follows from Lemma 1, the first condition from the inequality (1) (by induction, and with the aid of term-by-term differentiation of the polynomial $f_{j-1}$).

**Lemma 3.** $\sum_{m=1}^{\infty} a_m < \infty$ .

This result follows from the inequality

$$\sum_{m=n_j}^{n_j+2n_{j-1}} a_m < \tfrac{1}{2} n_{j-1}^{-1} (1-k)^{4(j-1)} ,$$

which in turn follows from Lemma 2. The sum of the infinite series $\Sigma\, a_m z^m$ $(|z| \leqslant 1)$ shall henceforth be denoted by $f(z)$.

**Lemma 4.** *If* $|z| < 1$, $f'(z) \neq 0$ .

With the notation $D_j^{j+p} = df_{j+p}(z)/df_j(z)$, the identity

$$D_j^{j+1} = 1 + kz^{n_{j+1}-1} + k \frac{z^{n_j+1}}{n_{j+1}} \frac{f_j''(z)}{f_j'(z)} \tag{3}$$

gives the inequality

$$|f_j'(z)| \geqslant (1-k)^{2(j-1)} ; \tag{4}$$

for $j = 1$, (4) is trivial; if (4) holds for a specific index $j$, term-by-term differentiation of the polynomial $f_j'(z)$ gives the estimate

$$\left| k \frac{z^{n_{j+1}}}{n_{j+1}} \frac{f_j''(z)}{f_j'(z)} \right| \leqslant 2^{-j} k (1-k)^{2j+2} , \tag{5}$$

from which it follows that $|D_j^{j+1}| > (1-k)^2$, and (4) is established for all indices $j$. Since $\lim f_j'(z) = f'(z)$, uniformly in every closed region in the open unit disc, the lemma is proved.

**Lemma 5.** *If* $|z| \leqslant 1$, $|f(z) - f_j(z)| < 2^{j-1}/n_{j+1}$ .

The estimate (5) gives the inequality $|D_j^{j+1}| < 2$, and it follows that

$$\left| f(z) - f_j(z) \right| \leqslant k \sum_{r=j}^{\infty} \left| f_r'(z) \right| / n_{r+1} \leqslant k \sum_{r=j}^{\infty} 2^{r-1} / n_{r+1}$$

$$< k\, 2^{j-1} [1 + (1-k)^4 + (1-k)^8 + \cdots ] / n_{j+1} ,$$

and the result follows immediately.

**Lemma 6.** *If $u$ and $v$ are two complex numbers* $(|u| < |v|)$, *and if the appropriate branches of the respective arguments are chosen, then*

$$\left| \arg (u+v) - \arg v \right| < \frac{\pi}{2} \left| u/v \right| .$$

This follows from the law of sines.

**Lemma 7.** *If* $|z| \leqslant 1$, *appropriate choice of branches gives the relations*

$$\arg f_1'(z) = 0 ,$$

$$\arg f_j'(z) = w_j(z) + \sum_{r=2}^{j} \arg (1 + k z^{n_r - 1}) \quad (j = 2, 3, \ldots) , \tag{6}$$

*where*

$$|w_j(z)| < \frac{\pi k}{4} (1-k)^5 .$$

For $j = 2, 3, \ldots,$ equation (3), the estimate (5), and Lemma 6 imply that

$$\arg f_j'(z) = \sum_{r=1}^{j-1} \arg \left[ 1 + k z^{n_{r+1}-1} + \frac{k z^{n_{r+1}}}{n_{r+1}} \frac{f_r''(z)}{f_r'(z)} \right]$$

$$= w_j(z) + \sum_{r=1}^{j-1} \arg (1 + k z^{n_{r+1}-1})$$

where $w_2(z) = 0$, and (for $j > 2$)

$$|w_j(z)| < \frac{\pi}{2} k \sum_{r=2}^{j-1} 2^{-r}(1-k)^{2r+1} < \frac{\pi k}{4}(1-k)^5 .$$

**Lemma 8.** If $u = \tan^{-1} \dfrac{k \sin\theta}{1 + k\cos\theta}$, then $\left|\dfrac{du}{d\theta}\right| \leqslant \dfrac{k}{1-k}$.

The proof is an exercise in the differential calculus.

**Lemma 9.** *If* $2\pi n_2^{-2/3} \leqslant \psi - \varphi \leqslant \pi$, *then* $|f(e^{i\psi}) - f(e^{i\varphi})| > 2n_2^{-2/3}$.

Let $e^{i\psi} = u$, $e^{i\Phi} = v$. Then Lemma 5 gives the inequality

$$|f(u) - f(v)| > |f_1(u) - f_1(v)| - 2/n_2$$

$$= 2\sin\frac{\psi - \varphi}{2} - 2/n_2$$

$$> 2(\sin \pi n_2^{-2/3} - n_2^{-1}) .$$

**Lemma 10.** *If* $2\pi n_{j+1}^{-2/3} \leqslant \psi - \varphi \leqslant 2\pi n_j^{-2/3}$, *then*

$$|f(e^{i\psi}) - f(e^{i\varphi})| > n_{j+1}^{-1/2} \qquad (j = 2, 3, \ldots),$$

*provided $n_2$ exceeds a certain number which is independent of the index $j$.*

Again, with the notation as above,

$$|f(u) - f(v)| > |f_j(u) - f_j(v)| - 2^j/n_{j+1} .$$

With the notation of Lemma 7,

$$f_j(u) - f_j(v) = \int_u^v f_j'(z)\, dz =$$

$$= \int_\varphi^\psi \exp\left\{i\left[\frac{\pi}{2} + \theta + w_j(e^{i\theta}) + \sum_{r=2}^{j} \arg(1 + k e^{i\theta(n_r - 1)})\right]\right\} |f_j'(e^{i\theta})|\, d\theta .$$

In order to obtain a lower bound for the modulus of the integral, we need an upper bound on the length $\lambda_j$ of the interval over which the quantity in brackets varies as $\theta$ varies from $\varphi$ to $\psi$. The respective contributions to $\lambda_j$ from $\theta$, $w_j$, and the last term under the summation are less than

$$2\pi n_j^{-2/3}, \ \tfrac{1}{2}\pi k(1-k)^5, \ \text{and} \ 2\sin^{-1} k ;$$

and, by Lemma 8, the contribution from the remaining terms is less than

$$\frac{2\pi}{n_j^{2/3}}\frac{k}{1-k}\left[n_2 - 1 + n_3 - 1 + \cdots + n_{j-1} - 1\right] = O(n_j^{-1/3}) .$$

It follows that

$$\lambda_j < 2\sin^{-1} k + \tfrac{1}{2}\left[\pi k(1-k)^5\right] + O(n_j^{-1/3}) < 2\eta ,$$

where $\eta$ is independent of $j$, and less than $\pi/2$ if $n_2$ is sufficiently large. Therefore, for sufficiently large values of $n_2$ ,

$$\begin{aligned} |f_j(u) - f_j(v)| &> (\psi - \varphi)\cos\left(\frac{\pi}{2} - \eta\right) \min_{|z|=1} |f_j'(z)| \\ &\geqslant 2\pi n_{j+1}^{-2/3} \cos\left(\frac{\pi}{2} - \eta\right)(1-k)^{2(j-1)} \\ &> 2n_{j+1}^{-1/2} , \end{aligned}$$

and the lemma follows immediately.

**Lemma 11.** *If $n_2$ is sufficiently large, the function $f(z)$ maps the unit circle into a Jordan curve $C$.*

By Lemma 3, the mapping of $C_1$ by $f(z)$ is continuous and single-valued. By Lemmas 9 and 10, the same is true for the inverse mapping, and the lemma is established.

**Lemma 12.** *The set of all points on $C_1$ whose image by $f(z)$ is linearly accessible from the interior of $C$ is of Lebesgue measure zero.*

It will be convenient to say that a point $P$ on $C$ is *$R$-accessible* provided there exists a rectilinear segment $PQ$, of length $R$ and interior to $C$ except for the point $P$. Lemma 12 will be proved when it is shown that for every positive number $R$ the set of $R$-accessible points on $C$ has measure zero. This in turn will be established when it is shown that if $A$ is an arc of length $L$ on $C_1$, there exists on $A$ a collection of finitely many subarcs, of total length $\varepsilon L$ (where $\varepsilon$ is independent of $L$), such that the image of each point on any of these subarcs is not $R$-accessible.

Let $R$ be a positive number; let $\beta = \sin^{-1} k$, $\alpha = (1-h)\beta$, where $h$ is a small positive constant, and let $s$ be an integer greater than $3\pi/\alpha$. For every positive integer $n$, the set of points $z$ on $C_1$ at which the inequality

$$\alpha \leqslant \arg(1 + kz^n) \leqslant \beta$$

holds is composed of $n$ symmetrically distributed arcs whose total lenght $\lambda$ is independent of $n$. It follows that if $A$ is an arc of length $L$, and $j$ is a positive integer $(\sqrt{n_{j-1}} > 1/L)$, there exists a set $A_j^*$ on $A$, composed of arcs of total length $\lambda L(1 + O(n_j^{-1}))$, such that

$$\alpha \leqslant \arg (1 + k z^{n_j - 1}) \leqslant \beta$$

on $A_j^*$. Let the two outermost arcs of $A_j^*$ be deleted; and from each of the remaining arcs, let the two outer thirds be deleted. The remainder, of total length $\lambda L \left[1 + O\left(n_j^{-1}\right)\right] / 3$, shall be denoted by $A_j$.

Again, each arc of $A_j$ contains arcs on which

$$\alpha \leqslant \arg (1 + k z^{n_{j+1} - 1}) \leqslant \beta .$$

These arcs occur in groups of approximately $n_{j+1}/n_j$ arcs. From each of these groups, let the two outermost arcs be deleted; from each of the remaining arcs, let the two outer thirds be deleted; and let the remainder be denoted by $A_{j+1}$.

If this process is carried out $s$ times, there results a set $B_j = A_{j+s-1}$, composed of arcs of total length $(\lambda/3)^s L\,(1 + O\,(1/n_j))$. Let $u$ be any point in $B_j$; then $u$ is the end point of arcs $\gamma_1, \gamma_2, \ldots, \gamma_j$ of respective lengths

$$\frac{2\pi}{3n_j}, \frac{2\pi}{3n_{j+1}}, \ldots, \frac{2\pi}{3n_{j+s-1}},$$

on which the respective inequalities

$$\alpha \leqslant \arg (1 + k z^{n_{j+r} - 1}) \leqslant \beta \qquad (r = 0, 1, \ldots, s-1)$$

hold. Moreover, it may be assumed that $u$ is either the left end point of each of these arcs, or the right end point of each of these arcs.

Suppose, to be definite, that $u = e^{i\psi}$, and that the point $v = e^{i\varphi}$ lies in the arc $\gamma_1$ when $0 \leqslant \psi - \varphi \leqslant 2\pi/3n_j$. Then, when

$$\frac{2\pi}{n_{j+t+1}^{2/3}} \leqslant \psi - \varphi \leqslant \frac{2\pi}{n_{j+t}^{2/3}},$$

$$\begin{aligned} \arg\left(f(u) - f(v)\right) &= \arg\left[f_{j+t}(u) - f_{j+t}(v)\right] + o(1) \\ &= \frac{\pi}{2} + \psi + w(u, v) + \sum_{r=2}^{j-1} \arg\left(1 + k e^{i\theta(n_r - 1)}\right) \\ &\quad + \sum_{r=j}^{j+t} \arg\left(1 + k e^{i\theta(n_r - 1)}\right) + o(1) . \end{aligned}$$

The first four terms of the last member are nearly constant in $\gamma_1$, except for the quantity $w(u, v)$, which will be discussed presently. The last term lies between

$$(j + t)(1 - h)\beta \qquad \text{and} \qquad (j + t + 1)\beta \ .$$

The term $w(u, v)$ varies fairly widely; but its modulus never exceeds $\frac{\pi}{4} k(1 - k)^5$ .

It follows that as $v$ runs through $\gamma_1$ (which includes $\gamma_2, \gamma_3, \ldots, \gamma_{s-1}$), the argument of $f(u) - f(v)$ runs through a range of approximately $3\pi$. Because the map of $C_1$ is continuous, the proof of the lemma is complete, and the Theorem is established.

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