On the homotopy groups of unions of spaces

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Dedicated to Prof. H. Hopf on his 60th birthday

1. Introduction

A useful technique in calculating the homotopy groups of a space is to replace the given space by one of the same homotopy type whose structure lends itself more readily to computation. In particular, Chang has shown in [2], that an A_n^2-polyhedron, $n > 2$, in the sense of J. H. C. Whitehead [11], is of the same homotopy type as the union of a (finite) number of "elementary" cell-complexes with a single common point. Thus attention is drawn to the question: given two connected spaces X and Y, what do we know of the homotopy groups of $X \vee Y$, the union of X and Y with a single common point? Of course the homology groups of $X \vee Y$ are given by the simple relation

$$H_n (X \vee Y) = H_n (X) + H_n (Y), n > 0 \ , \tag{1.1}$$

but the corresponding relation for homotopy groups is more complicated, in general. For 'good' X, Y, $\Pi_1(X \vee Y)$ is the free product of $\Pi_1(X), \Pi_1(Y)$. If $n > 1$, then certainly $\Pi_n(X)$, $\Pi_n(Y)$ inject into $\Pi_n (X \vee Y)$ as direct factors, but a third term appears on the right hand side of (1.1). This term may be called the *cross-term* and is the isomorphic image, under the boundary operator, of the group $\Pi_{n+1} (X \times Y, X \vee Y)$, $X \vee Y$ being embedded in $X \times Y$ in a natural way. Thus

$$\Pi_n (X \vee Y) \approx \Pi_n (X) + \Pi_n (Y) + \Pi_{n+1} (X \times Y, X \vee Y), n > 1 \ . \tag{1.2}$$

The object of this paper is to study the cross-term under certain restrictions on X, Y (and n). The first non-trivial case considered, that in which $X = S^p$, $Y = S^q$, $n = p + q - 1$, $p \geqslant 2$, $q \geqslant 2$, was discussed by J. H. C. Whitehead in [10], the paper in which he introduced the Whitehead product. In fact, the cross-term is in this case cyclic infinite, generated by the Whitehead product $[\iota_p, \iota_q]$, where ι_p generates $\Pi_p (S^p)$, ι_q generates $\Pi_q (S^q)$. This result has been generalized by G. W. Whitehead in the form

$$\Pi_n (S^p \vee S^q) \approx \Pi_n (S^p) + \Pi_n (S^q) + \Pi_n (S^{p+q-1}) \ , \tag{1.3}$$

if $1 < n \leqslant p+q+\min(p,q) - 3$ (see [9][1]). Further generalizations in this direction are due to the author, J. C. Moore and others (see [6], [7]).

J. H. C. Whitehead showed in [11] that if X is an arcwise-connected space whose first $(p-1)$ homotopy groups vanish then

$$\Pi_n(X \vee S^q) = \Pi_n(X) + \Pi_n(S^q),\ 1 < n < p+q-1\ , \tag{1.4}$$

and Chang in [3] generalized J. H. C. Whitehead's original result by proving that, under the additional assumption that $p \leqslant q$,

$$\Pi_{p+q-1}(X \vee S^q) = \Pi_{p+q-1}(X) + \Pi_{p+q-1}(S^q) + [\Pi_p(X),\ \Pi_q(S^q)] \tag{1.5}$$

the group $[\Pi_p(X), \Pi_q(S^q)]$ being generated by Whitehead products $[\alpha, \iota_q]$, $\alpha \in \Pi_p(X)$. Our first result (in section 2) generalizes (1.4) and (1.5) by replacing S^q by an arbitrary arcwise-connected space, Y, whose first $(q-1)$ homotopy groups vanish. Naturally the restriction $p \leqslant q$ disappears in this generalization. We also characterize the cross-term as being isomorphic to the tensor product $H_p(X) \otimes H_q(Y)$.

The next five sections of the paper are devoted to a study of $\Pi_{p+q}(X \vee Y)$, under the further restriction that X, Y are CW-complexes in the sense of [10], and $p, q \geqslant 3$. The restriction to CW-complexes (or at any rate to spaces of the homotopy type of CW-complexes) is implicit in the method, but the restriction on p, q (i. e., the omission of the cases $p = 2$, $q \geqslant 2$) may be removed at the cost of additional complication in the results. The method is based on the exact sequence of J. H. C. Whitehead [14] and we calculate $\Gamma_{p+q+1}(X \times Y, X \vee Y)$ in section 4. Though the results are expressed in invariant form, the proofs are frequently based on special choices of X and Y from their homotopy types; in particular, we often find it convenient to assume that the $(p-1)$-dimensional skeleton of X and the $(q-1)$-dimensional skeleton of Y are single points. It would be satisfactory if methods could be devised which did not depend on such choices. On the other hand, the methods used in this paper do indicate the great advantage for computation of cell-complexes.

It is of interest to note that, in general, the cross-term does not consist only of sums of Whitehead products. If we take $X = S^3 \cup e^4$, where e^4 is attached by a map of degree 2, and Y a replica of X, then

$$\Pi_6(X \vee Y) = \Pi_6(X) + \Pi_6(Y) + Z_4;$$

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[1]) The case $n = p + q + \min(p, q) - 3$ is omitted in the statement of (1.3) in [9], but it is shown in [1] (or [6]) that it may be included. The subgroup $\Pi_n(S^{p+q-1})$ is embedded in $\Pi_n(S^p \vee S^q)$ by composition with $[\iota_p, \iota_q]$.

the group Z_4 (cyclic group of order 4) is not generated by a Whitehead product but twice a generator is a Whitehead product. The generator is characterized in section 7.

The last section is devoted to the consideration of the union of more than two spaces; under certain restrictions the cross-term is the direct sum of the cross-terms arising from pairs of spaces in the union. The results, stated for a finite union, are valid also for an infinite union, provided that the union-space is given the weak topology and the direct sum is interpreted as the weak direct sum.

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2. The Chang-Whitehead theorem

Let X, Y be arcwise-connected and simply-connected topological spaces such that

$$\Pi_r(X) = 0, \quad r = 1, \ldots, p-1,$$
$$\Pi_s(Y) = 0, \quad s = 1, \ldots, q-1.$$

Let x_0, y_0 be base-points in X, Y and let $X \vee Y$ be the subspace $X \times y_0 \cup x_0 \times Y$ of $X \times Y$. We describe $X \vee Y$ as the *union* of X and Y with a single common point arising from the identification of x_0 and y_0. Let

$$\varrho_1 \colon \Pi_r(X) \to H_r(X)$$
$$\varrho_2 \colon \Pi_r(Y) \to H_r(Y)$$
$$\varrho \colon \Pi_r(X \times Y, X \vee Y) \to H_r(X \times Y, X \vee Y)$$

be the natural homomorphisms of the homotopy groups into the singular homology groups, and let $A \otimes B$, $\operatorname{Tor}(A, B)$ stand for the tensor product and torsion product[2]) of the abelian groups A, B. Finally let $[\alpha, \beta] \in \Pi_{m+n-1}(Z)$ stand, as usual, for the Whitehead product of elements $\alpha \in \Pi_m(Z)$, $\beta \in \Pi_n(Z)$, for any space Z.

We generalize theorem 3(b) of [13] and theorem 2 of [3] by proving

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[2]) Given two additive abelian groups A, B, the group $A \otimes B$ is generated by the pairs (a, b), $a \in A$, $b \in B$, with relations

$$(a_1 + a_2, b) = (a_1, b) + (a_2, b)$$
$$(a, b_1 + b_2) = (a, b_1) + (a, b_2).$$

Let A be represented as the difference group $F - R$, where F is free abelian. Then $\operatorname{Tor}(A, B)$ is the kernel of the natural homomorphism $R \otimes B \to F \otimes B$.

Theorem 2.1. *Let* $\iota_1 : \Pi_r(X) \to \Pi_r(X \vee Y)$, $\iota_2 : \Pi_r(Y) \to \Pi_r(X \vee Y)$ *be injections. Then* ι_1, ι_2 *are univalent*[3]) *and*

$$\Pi_r(X \vee Y) = \iota_1 \Pi_r(X) + \iota_2 \Pi_r(Y), \quad r < p + q - 1, \ldots \tag{2.2}$$

$$\Pi_{p+q-1}(X \vee Y) = \iota_1 \Pi_{p+q-1}(X) + \iota_2 \Pi_{p+q-1}(Y) + \tau(H_p(X) \otimes H_q(Y)) \tag{2.3}$$

where τ *is univalent and is given by*

$$\tau(\xi \otimes \eta) = [\varrho_1^{-1}(\xi), \varrho_2^{-1}(\eta)], \quad \xi \in H_p(X), \; \eta \in H_q(Y) .$$

It is well-known that ι_1, ι_2 are univalent and that, for any $r > 1$,

$$\Pi_r(X \vee Y) = \iota_1 \Pi_r(X) + \iota_2 \Pi_r(Y) + d\Pi_{r+1}(X \times Y, X \vee Y), \ldots \tag{2.4}$$

where d is the homotopy boundary homomorphism

$$d\colon \Pi_{r+1}(X \times Y, X \vee Y) \to \Pi_r(X \vee Y)$$

and is univalent.

Since X and Y are simply-connected, $X \vee Y$ is simply-connected and (2.2) is trivial if $r = 1$. We assume $r > 1$, and note that $p \geqslant 2$, $q \geqslant 2$, $p + q \geqslant 4$. It is also well-known that the singular homology groups (with integer coefficients) of a topological product $X \times Y$ are given in terms of the singular homology groups of X and Y by the formula[4])

$$H_n(X \times Y) \approx \sum_{i+j=n} H_i(X) \otimes H_j(Y) + \sum_{i+j=n-1} \mathrm{Tor}(H_i(X), H_j(Y)) \tag{2.5}$$

Consider the exact homology sequence

$$\ldots \to H_n(X \vee Y) \xrightarrow{i} H_n(X \times Y) \xrightarrow{j} H_n(X \times Y, X \vee Y) \xrightarrow{\partial} H_{n-1}(X \vee Y) \to \ldots$$

Now $H_n(X \vee Y) = H_n(X) \otimes H_0(Y) + H_0(X) \otimes H_n(Y)$, $n \geqslant 1$, and i is univalent.

By the exactness of the sequence, j maps $H_n(X \times Y)$ on to $H_n(X \times Y, X \vee Y)$ and $H_n(X) \otimes H_0(Y) + H_0(X) \otimes H_n(Y)$ is the kernel of j. Thus

$$H_n(X \times Y, X \vee Y) \approx \sum_{\substack{i+j=n \\ i \neq 0, j \neq 0}} H_i(X) \otimes H_j(Y) + \sum_{i+j=n-1} \mathrm{Tor}(H_i(X), H_j(Y)). \tag{2.6}$$

Since $\Pi_r(X) = 0$, $r = 1, \ldots, p-1$, and $\Pi_s(Y) = 0$, $s = 1, \ldots, q-1$, it follows that

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[3]) We use the word "univalent" for a $(1 - 1)$ mapping.

[4]) A nice proof appears in [8].

$H_r(X) = 0,\ r = 1, \ldots, p-1$, and $H_s(Y) = 0,\ s = 1, \ldots q-1$, and

$$\varrho_1 : \Pi_p(X) \approx H_p(X)\,,$$
$$\varrho_2 : \Pi_q(Y) \approx H_q(Y)\,. \tag{2.7}$$

It now follows from (2.6) that

$$H_n(X \times Y,\ X \vee Y) = 0,\quad n = 1, \ldots\ p+q-1\ , \tag{2.8}$$

and
$$H_{p+q}(X \times Y,\ X \vee Y) \approx H_p(X) \otimes H_q(Y)\ . \tag{2.9}$$

Since $\Pi_1\ (X \vee Y) = 0$, and $\Pi_1(X \times Y) = 0$ it follows from (2.8) by the Hurewicz isomorphism theorem (in the relative case) that

$$\Pi_n(X \times Y, X \vee Y) = 0, n = 2, \ldots, p+q-1\ , \tag{2.10}$$

and
$$p : \Pi_{p+q}(X \times Y,\ X \vee Y) \approx H_{p+q}(X \times Y, X \vee Y)\ . \tag{2.11}$$

(2.2) now follows immediately from (2.4) and (2.10). To complete the proof of (2.3), it is only necessary to study the precise nature of the isomorphism (2.9). Let

$$f_1 : I^i \to X,\quad f_2 : I^j \to Y,\quad i > 0,\ j > 0\ ,$$

be singular cubes of X, Y respectively. Let $f : I^{i+j} \to X \times Y$ be given by

$$f(a, b) = (f_1(a),\ f_2(b)),\quad a \in I^i,\ b \in I^j\ . \tag{2.12}$$

Then f is a singular $(i+j)$-cube of $X \times Y$. The mapping

$$(f_1, f_2) \to f$$

induces a chain mapping $C_i(X) \otimes C_j(Y) \to C_{i+j}(X \times Y)$, and this in turn, induces the isomorphic embedding of $H_i(X) \otimes H_j(Y)$ in $H_{i+j}(X \times Y,\ X \vee Y)$.

Now let $f_1 : I^p, \dot{I}^p \to X, x_0$, $f_2 : I^q, \dot{I}^q \to Y, y_0$ be maps representing $\alpha \in \Pi_p(X)$, $\beta \in \Pi_q(Y)$ respectively, and let the element of $\Pi_{p+q}(X \times Y,$ $X \vee Y)$ represented by f (defined as in (2.12)) be written $\alpha \cdot \beta$. We note that

$$d(\alpha \cdot \beta) = [\alpha, \beta]\ . \tag{2.13}$$

Now $\varrho_1(\alpha)$ is the homology class of the cycle (f_1, I^p) and $\varrho_2(\beta)$ is the homology class of the cycle (f_2, I^q). Then the element $\varrho_1(\alpha) \otimes \varrho_2(\beta)$ $\in H_p(X) \otimes H_q(Y)$ is to be identified in (2.9) with the homology class of the relative cycle (f, I^{p+q}). In other words,

$$\varrho(\alpha \cdot \beta) = \varrho_1\alpha \otimes \varrho_2\beta\ . \tag{2.14}$$

when the left and right-hand sides of (2.9) are identified. Now $H_p(X) \otimes H_q(Y)$ is embedded in $\Pi_{p+q-1}(X \vee Y)$ by the univalent mapping $d\varrho^{-1}$. From (2.7), (2.11), (2.13) and (2.14) it follows that, if $\xi \in H_p(X)$, $\eta \in H_q(Y)$, then

$$\begin{aligned} d\varrho^{-1}(\xi \otimes \eta) &= d(\varrho_1^{-1}\xi \cdot \varrho_2^{-1}\eta) \\ &= [\varrho_1^{-1}\xi, \varrho_2^{-1}\eta] , \end{aligned}$$

and the theorem is proved.

We may immediately generalize theorem 2.1 to the following

Theorem 2.15. *Let Z be the union of the spaces X_i with a single common point, $i = 1, \ldots, k$, and let*

$$\Pi_r(X_i) = 0 , \; r = 0, 1, \ldots, p_i - 1 ,$$

where $p_1 \leqslant p_2 \leqslant \ldots \leqslant p_k$. Then

$$\Pi_r(Z) = \iota_1\Pi_r(X_1) + \ldots + \iota_k\Pi_r(X_k) , \quad r < p_1 + p_2 - 1 , \qquad (2.16)$$

and

$$\begin{aligned} \Pi_{p_1+p_2-1}(Z) &= \iota_1\Pi_{p_1+p_2-1}(X_1) + \ldots + \iota_k\Pi_{p_1+p_2-1}(X_k) \\ &+ \tau \sum_{i<j} (H_{p_1}(X_i) \otimes H_{p_2}(X_j)) \end{aligned} \qquad (2.17)$$

where $\iota_1, \ldots, \iota_k, \tau$ are defined as above and Σ represents a direct sum.

Let $Z_t = X_1 \vee X_2 \vee \ldots \vee X_t$, $1 \leqslant t \leqslant k$. By an obvious induction, using (2.2), we have

$$\Pi_r(Z_t) = 0 , \; r = 0 , 1, \ldots, p_1 - 1 . \qquad (2.18)$$

Now (2.16) is trivial if $k = 1$. Let us assume that

$$\Pi_r(Z_{k-1}) = \iota_1\Pi_r(X_1) + \ldots + \iota_{k-1}\Pi_r(X_{k-1}) , \; r < p_1 + p_2 - 1 , \; k \geqslant 2.$$

Then, since $\Pi_r(X_k) = 0$, $r = 0, 1, \ldots, p_2 - 1$, it follows from (2.18) and (2.2) that

$\Pi_r(Z) = \iota_1\Pi_r(X_1) + \ldots + \iota_{k-1}\Pi_r(X_{k-1}) + \iota_k\Pi_r(X_k)$, and (2.16) is proved. (2.17) is trivial if $k = 1$. Let us assume that

$$\begin{aligned} \Pi_{p_1+p_2-1}(Z_{k-1}) &= \iota_1\Pi_{p_1+p_2-1}(X_1) + \ldots + \iota_{k-1}\Pi_{p_1+p_2-1}(X_{k-1}) \\ &+ \tau \sum_{i<j\leqslant k-1} (H_{p_1}(X_i) \otimes H_{p_2}(X_j)) . \end{aligned}$$

Then, by (2.18) and (2.3),

$$\begin{aligned} \Pi_{p_1+p_2-1}(Z) &= \iota\Pi_{p_1+p_2-1}(Z_{k-1}) + \iota_k\Pi_{p_1+p_2-1}(X_k) \\ &+ \tau(H_{p_1}(Z_{k-1}) \otimes H_{p_2}(X_k)) , \end{aligned}$$

where $\iota : \Pi_r(Z_{k-1}) \to \Pi_r(Z)$ is the (univalent) injection.

How $H_{p_1}(Z_{k-1}) = \sum_{j=1}^{k-1} H_{p_1}(X_j)$ so that (2.17) follows immediately. Of course, in (2.16) and (2.17), $\iota_1, \ldots, \iota_k, \tau$ are univalent.

Finally, we give a generalization in a different direction. Let Z be a connected CW-complex which is the union of the two CW-complexes X, Y. Let $\Pi_r(X) = 0$, $r = 0, 1, \ldots, p-1$, and $\Pi_s(Y) = 0$, $s = 0, 1, \ldots, q-1$. Then we have

Theorem 2.19. *Let $X \frown Y$ be contractible over itself. Then*

$$\Pi_r(Z) = j_1 \Pi_r(X) + j_2 \Pi_r(Y)\,, \quad r < p+q-1\,, \tag{2.20}$$

$$\Pi_{p+q-1}(Z) = j_1 \Pi_{p+q-1}(X) + j_2 \Pi_{p+q-1}(Y) + \tau^*(H_p(X) \otimes H_q(Y)) \tag{2.21}$$

where $j_1 : \Pi_r(X) \to \Pi_r(Z)$, $j_2 : \Pi_r(Y) \to \Pi_r(Z)$ are univalent injections and τ^* is univalent and is given by

$$\tau^*(\xi \otimes \eta) = \{\varrho_1^{-1}\xi, \varrho_2^{-1}\eta\}\,, \quad \xi \in H_p(X),\ \eta \in H_q(Y)\,,$$

$\{\ \}$ being the Whitehead product taken in Z.

Let $x_0 \in X \frown Y$ and left $f_t : X \frown Y \to X \frown Y$ be a homotopy such that $f_0 = 1$, $f_1(X \frown Y) = x_0$, $f_t(x_0) = x_0$, where 1 stands for an identity map. By the homotopy extension theorem, f_t has extensions $g_t : X \to X$, $h_t : Y \to Y$ such that $g_0 = 1$, $h_0 = 1$.

Define $\Phi : Z \to X \vee Y$ by $\Phi(x) = (g_1(x), y_0)$, $x \in X$, $\Phi(y) = (x_0, h_1(y))$, $y \in Y$, where $y_0 = x_0$. Φ is single-valued since $f_1(X \frown Y) = x_0$. Define $\Psi : X \vee Y \to Z$ by $\Psi(x, y_0) = x$, $\Psi(x_0, y) = y$. Then $\Phi\Psi \mid X \times y_0 : X \times y_0 \to X \times y_0$ is homotopic to the identity rel (x_0, y_0) and $\Phi\Psi \mid x_0 \times Y : x_0 \times Y \to x_0 \times Y$ is homotopic to the identity rel (x_0, y_0). Thus $\Phi\Psi \sim 1 : X \vee Y \to X \vee Y$. Similarly $\Psi\Phi \mid X : X \to X$ is homotopic to the identity and $\Psi\Phi \mid Y : Y \to Y$ is homotopic to the identity. Since the homotopies agree on $X \frown Y$, it follows that $\Psi\Phi \sim 1 : Z \to Z$. Thus Φ and Ψ are homotopy equivalences.

Now $\Pi_r(X \vee Y) = \iota_1 \Pi_r(X) + \iota_2 \Pi_r(Y)$, $r < p + q - 1$, and

$$\Pi_{p+q-1}(X \vee Y) = \iota_1 \Pi_{p+q-1}(X) + \iota_2 \Pi_{p+q-1}(Y) + \tau(H_p(X) \otimes H_q(Y))\,.$$

Let us write $j_1 : \Pi_r(X) \to \Pi_r(Z)$, $j_2 : \Pi_r(Y) \to \Pi_r(Z)$, for the injection homomorphisms. Since Ψ is a homotopy equivalence it induces an isomorphism $\Psi_* : \Pi_r(X \vee Y) \approx \Pi_r(Z)$, and it is clear that Ψ_* maps $\iota_1\Pi_r(X)$ isomorphically onto $j_1\Pi_r(X)$ and maps $\iota_2\Pi_r(Y)$ isomorphically onto $j_2\Pi_r(Y)$. In fact, $\Psi_* \iota_1 = j_1$, $\Psi_* \iota_2 = j_2$, so that j_1 and j_2 are univalent. We have shown that

$$\Pi_r(Z) = j_1 \Pi_r(X) + j_2 \Pi_r(Y)\,, \quad r < p + q - 1\,.$$

Let us write $\{\alpha, \beta\}$ for the Whitehead product, in Z, of $\alpha \in \Pi_p(X)$, $\beta \in \Pi_q(Y)$. Then it is clear that $\Psi_* [\alpha, \beta] = \{\alpha, \beta\}$.
It follows that

$$\Pi_{p+q-1}(Z) = j_1 \Pi_{p+q-1}(X) + j_2 \Pi_{p+q-1}(Y) + \tau^* (H_p(X) \otimes H_q(Y)) ,$$

where τ^* is univalent and is given by

$$\tau^* (\xi \otimes \eta) = \{\varrho_1^{-1} \xi, \varrho_2^{-1} \eta\}, \xi \in H_p(X), \eta \in H_q(Y) .$$

This completes the proof of the theorem.

It should be observed that we restrict ourselves to CW-complexes only in order to have available the homotopy extension theorem. Any restriction on the spaces X, Y, Z which ensures the existence of the homotopies g_t, h_t will render the conclusion valid.

3. An exact sequence

Let A_r, C_r, $r = 2, 3, \ldots$ be two systems of abelian groups[5]) related by homomorphisms d_r, j_r, such that

(i) $d_r : C_r \to A_{r-1}$, $r > 2$, $d_2 C_2 = 0$

(ii) $j_r : A_r \to C_r$

(iii) $d_r^{-1}(0) = j_r A_r$.

Writing δ_r for $j_{r-1} d_r$, we have $\delta_r : C_r \to C_{r-1}$ and $\delta_{r-1}\delta_r = 0$. Thus we have a homology theory based on the "chain groups" C_r and we write $H_r = H(C_r; \delta_r)$. We write $\Pi_r = A_r - d_{r+1} C_{r+1}$, $\Gamma_r = j_r^{-1}(0) \subset A_r$. Then, as shewn in [14] there is an exact sequence

$$\ldots \Gamma_r \xrightarrow{\lambda} \Pi_r \xrightarrow{\varrho} H_r \xrightarrow{\mu} \Gamma_{r-1} \to \ldots \to \Gamma_2 \to \Pi_2 \to H_2 \to 0 , \qquad (3.1)$$

where λ is the projection $A_r \to \Pi_r$, restricted to Γ_r ,

ϱ is induced by $j_r : A_r \to C_r$,

and μ is induced by $d_r : C_r \to A_{r-1}$.

Now let P be a simply-connected CW-complex and Q a simply-connected subcomplex of P. We identify C_r with $\Pi_r(P^r \cup Q, P^{r-1} \cup Q)$, $r \geqslant 3$, that is, we identify C_r with the r^{th} chain group of P mod Q. We identify[6])

[5]) We specialize the situation in [14] by taking $A_r = 0$, $r \leqslant 1$, $C_r = 0$, $r \leqslant 1$.
[6]) Note that $\Pi_2 (P^2 \cup Q, Q)$ is abelian because $\Pi_1 (Q) = 0$.

A_r, $r \geqslant 2$, with $\Pi_r(P^r \cup Q, Q)$, and j_r, $r \geqslant 2$, with the injection homomorphism

$$j_r : \Pi_r(P_r \cup Q, Q) \to \Pi_r(P^r \cup Q, P^{r-1} \cup Q) .$$

We identify d_r, $r \geqslant 3$, with the homotopy boundary homomorphism

$$d_r : \Pi_r(P^r \cup Q, \; P^{r-1} \cup Q) \to \Pi_{r-1}(P^{r-1} \cup Q, Q) .$$

Finally we put $C_2 = j_2 \Pi_2(P^2 \cup Q, Q)$, $d_2 C_2 = 0$. Condition (iii) is immediately verified by reference to the exactness of the homotopy sequence of the triple $(P^r \cup Q, \; P^{r-1} \cup Q, Q)$. Let ι_r be the injection $\Pi_r(P^{r-1} \cup Q, Q) \to \Pi_r(P^r \cup Q, Q)$. Then $\Gamma_r = J_r^{-1}(0) = \iota_r \Pi_r(P^{r-1} \cup Q, Q)$; since δ_r, $r \geqslant 3$, is the homology boundary operator of P mod Q, it follows that $H_r = H_r(P, Q)$, $r \geqslant 3$; and

$$\Pi_r = \Pi_r(P^r \cup Q, Q) - d_{r+1}\Pi_{r+1}(P^{r+1} \cup Q, P^r \cup Q) = \Pi_r(P, Q) .$$

Moreover, as may readily be verified, the homomorhism ϱ then becomes, for $r \geqslant 3$, the natural homomorphism $\varrho : \Pi_r(P, Q) \to H_r(P, Q)$.

Let us write $\Gamma_r(P, Q)$ for Γ_r; then we have the exact sequence

$$\begin{aligned} \ldots \Gamma_r(P, Q) \to \Pi_r(P, Q) \to H_r(P, Q) \to \Gamma_{r-1}(P, Q) \to \ldots \to \Gamma_2(P, Q) \\ \to \Pi_2(P, Q) \to H_2 \to 0 \end{aligned} \tag{3.2}$$

where we leave H_2 unidentified[7]).

Lemma 3.3. *Let $\Pi_r(P, Q) = 0$, $r = 2, \ldots, k-1$. Then $\Gamma_r(P, Q) = 0$, $r = 3, \ldots, k$.*

It follows immediately from (3.2) and the Hurewicz isomorphism theorem that $\Gamma_r(P, Q) = 0$, $r = 3, \ldots, k-1$. However, we will prove that $\Gamma_k(P, Q) = 0$ (this, of course, is all that is needed to prove the lemma).

Let $P_0 = P^k \cup Q$. Then, since $k \geqslant 3$, P_0 is simply-connected. Also $\Pi_r(P_0, Q) = 0$, $r = 2, \ldots, k-1$. Now the isomorphism $\varrho : \Pi_k(P_0, Q) \approx H_k(P_0, Q)$, given by the Hurewicz isomorphism theorem, is a mapping of $\Pi_k(P^k \cup Q, Q)$ onto the group of relative k-cycles of P mod Q. This means that $j_k : \Pi_k(P^k \cup Q, Q) \to \Pi_k(P^k \cup Q, P^{k-1} \cup Q)$ is univalent, whence $\Gamma_k(P, Q) = \iota_k \Pi_k(P^{k-1} \cup Q, Q) = 0$.

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[7]) In our formulation H_2 cannot be identified with $H_2\,(P, Q)$ because C_2 was not the true 2-dimensional chain group of P mod Q. However, the somewhat obscure situation at the bottom end of the sequence will not concern us as we will be able to apply lemma 3.3.

From (3.2) we now get

Theorem 3.4 *Let P be a CW-complex and Q a subcomplex. Let $\Pi_1(P) = \Pi_1(Q) = 0$ and let $\Pi_r(P, Q) = 0$, $r = 2, \ldots, k-1$. Then we have an exact sequence*

$$\ldots \Gamma_n(P,Q) \xrightarrow{\lambda} \Pi_n(P,Q) \xrightarrow{\varrho} H_n(P,Q) \xrightarrow{\mu} \Gamma_{n-1}(P,Q) \rightarrow \ldots$$
$$\rightarrow \Pi_{k+1}(P,Q) \xrightarrow{\varrho} H_{k+1}(P,Q) \rightarrow 0 .$$

In particular, ϱ maps $\Pi_{k+1}(P,Q)$ onto $H_{k+1}(P,Q)$.

We note that we may replace the condition $\Pi_r(P,Q) = 0$, $r = 2, \ldots, k-1$, by the condition $H_r(P,Q) = 0$, $r = 2, \ldots, k-1$.

Now let X and Y be simply-connected CW-complexes. We wish to apply theorem 3.4 with $P = X \times Y$, $Q = X \vee Y$. However, it is not necessarily the case that $X \times Y$ is a CW-complex if X, Y are CW-complexes. On the other hand, we are able to use the methods of this section because $X \times Y$ inherits from X and Y the property that a compact subset is contained in a finite subcomplex.

Lemma 3.5. *Let X, Y be cell-complexes with the property that a compact subset is contained in a finite subcomplex. Then $X \times Y$ also has this property.*

For let $F \subset X \times Y$ be compact and let F_1, F_2 be the projections of F on X, Y. Then F_1, F_2 are compact, so that $F_1 \subset K, F_2 \subset L$, where K, L are finite subcomplexes of X, Y respectively. Then $F \subset F_1 \times F_2 \subset K \times L$, and $K \times L$ is a finite subcomplex of $X \times Y$.

It may now readily be verified that the arguments of this section remain valid when we replace P, Q by $X \times Y, X \vee Y$. The result which we will need in the sequel is then

Theorem 3.6. *Let X, Y be connected CW-complexes such that*

$$\Pi_r(X) = 0, r = 1, \ldots, p-1, \Pi_s(Y) = 0, s = 1, \ldots, q-1 .$$

Then we have an exact sequence

$$\ldots \Gamma_n(X \times Y, X \vee Y) \xrightarrow{\lambda} \Pi_n(X \times Y, X \vee Y) \xrightarrow{\varrho} H_n(X \times Y, X \vee Y)$$
$$\xrightarrow{\mu} \Gamma_{n-1}(X \times Y, X \vee Y) \rightarrow \ldots \rightarrow H_{p+q+2}(X \times Y, X \vee Y) \xrightarrow{\mu} \Gamma_{p+q+1}$$
$$(X \times Y, X \vee Y) \xrightarrow{\lambda} \Pi_{p+q+1}(X \times Y, X \vee Y) \xrightarrow{\varrho} H_{p+q+1}(X \times Y, X \vee Y) \rightarrow 0 .$$

This follows from the arguments above and (2.10).

4. The Calculation of Γ_{p+q+1}

In this section and for the rest of this paper we take $p \geqslant 3$, $q \geqslant 3$. If p or q is 2, the problems discussed involve additional points of interest and the results depend on an unpublished theorem due to M. G. Barratt and J.H.C. Whitehead. It is hoped to publish a note later showing the modifications in the statements of our results which are necessary when p or q is 2.

Let X, Y be connected CW-complexes such that $\Pi_r(X) = 0$, $r = 1, \ldots p-1$, $\Pi_s(Y) = 0$, $s = 1, \ldots, q-1$, $p \geqslant 3$, $q \geqslant 3$. Write $K = X \times Y$, $L = X \vee Y$, $K_r = K^r \cup L$. Our object in this section is to calculate $\Gamma_{p+q+1}(K, L)$.

Now by a standard result on CW-complexes, X is of the same homotopy type as a CW-complex X_0 such that X_0^{p-1} is a single point; similarly, Y is of the same homotopy type as a CW-complex Y_0 such that Y_0^{q-1} is a single point. Then $X \vee Y$ is of the same homotopy type as $X_0 \vee Y_0$ (with the single vertex identified) and $X \times Y$ is of the same homotopy type as $X_0 \times Y_0$. Moreover, the pair $(X \times Y, X \vee Y)$ are of the same relative homotopy type as $(X_0 \times Y_0, X_0 \vee Y_0)$ so that

$$\Gamma_{p+q+1}(X \times Y, X \vee Y) \approx \Gamma_{p+q+1}(X_0 \times Y_0, X_0 \vee Y_0) \qquad (4.1)$$

We will therefore assume that X^{p-1} is a single point, x_0, and Y^{q-1} is a single point, y_0, but these assumptions will not appear in our results.

Consider the diagram

$$\begin{array}{ccccc}
\Pi_{p+q+1}(K_{p+q+1}, K_{p+q}) & \xrightarrow{d} & \Pi_{p+q}(K_{p+q}, L) & \xrightarrow{i} & \Pi_{p+q}(K_{p+q+1}, L) \\
\downarrow \zeta & & \downarrow \eta & & \downarrow \theta \\
\Pi_{p+q+2}(K_{p+q+1}, K_{p+q}) & \xrightarrow{d'} & \Pi_{p+q+1}(K_{p+q}, L) & \xrightarrow{i'} & \Pi_{p+q+1}(K_{p+q+1}, L)
\end{array} .$$

The horizontal lines are extracts from the exact homotopy sequence of the triple (K_{p+q+1}, K_{p+q}, L); ζ is induced by a map $I^{p+q+2}, \dot{I}^{p+q+2} \to I^{p+q+1}, \dot{I}^{p+q+1}$ representing a generator of $\Pi_{p+q+2}(I^{p+q+1}, \dot{I}^{p+q+1})$; and η, θ are induced by a map $I^{p+q+1}, \dot{I}^{p+q+1} \to I^{p+q}, \dot{I}^{p+q}$ representing a generator of $\Pi_{p+q+1}(I^{p+q}, \dot{I}^{p+q})$. It is clear that $\theta i = i'\eta$. We prove that $d'\zeta = \eta d$. For consider

$$\begin{array}{ccccc}
\Pi_{p+q+1}(K_{p+q+1}, K_{p+q}) & \xrightarrow{d} & \Pi_{p+q}(K_{p+q}) & \xrightarrow{j} & \Pi_{p+q}(K_{p+q}, L) \\
\downarrow \zeta & & \downarrow \nu & & \downarrow \eta \\
\Pi_{p+q+2}(K_{p+q+1}, K_{p+q}) & \xrightarrow{d'} & \Pi_{p+q+1}(K_{p+q}) & \xrightarrow{j'} & \Pi_{p+q+1}(K_{p+q}, L)
\end{array}$$

where d, d' are homotopy boundaries, j, j' are injections, and ν is induced by an essential map $I^{p+q+2} \to \dot{I}^{p+q+1}$. Clearly $d'\zeta = \nu d$. Also since $p + q > 2$, the same homomorphism ν is induced by the map I^{p+q+1}, $\dot{I}^{p+q+1} \to I^{p+q}$, $\dot{I}^{p+q}$ (recall that $E : \Pi_{p+q}(S^{p+q-1}) \to \Pi_{p+q+1}(S^{p+q})$ is onto if $p + q > 2$, and, in fact, isomorphic if $p + q > 3$; the homomorphism ν is simply $\alpha \to \alpha \circ \xi$, $\alpha \in \Pi_{p+q}(K_{p+q})$, ξ generates $\Pi_{p+q+1}(S^{p+q})$). Thus $j'\nu = \eta j$, whence $d'\zeta = j'd'\zeta = j'\nu d = \eta j d = \eta d$.

Now $\Pi_i(K_{p+q}) = 0$, $i = 1, 2, \ldots \min(p-1, q-1)$. It is thus an easy consequence of the Whitehead suspension theorem[8]) that ζ maps $\Pi_{p+q+1}(K_{p+q+1}, K_{p+q})$ onto $\Pi_{p+q+2}(K_{p+q+1}, K_{p+q})$ and simply reduces the free abelian group $\Pi_{p+q+1}(K_{p+q+1}, K_{p+q})$ mod 2. Because of our special choice of X and Y, we have $L = K_{p+q-1}$, and a further application of the suspension theorem shows that η is onto $\Pi_{p+q+1}(K_{p+q}, L)$ and reduces $\Pi_{p+q}(K_{p+q}, L)$ mod 2.

Lemma 4.2. *The homomorphism θ maps $\Pi_{p+q}(K_{p+q+1}, L)$ onto $\Gamma_{p+q+1}(K, L)$ and reduces $\Pi_{p+q}(K_{p+q+1}, L)$ mod 2.*

We note first that $\Pi_{p+q}(K_{p+q+1}, L) = i\Pi_{p+q}(K_{p+q}, L)$. Thus $\Gamma_{p+q+1}(K, L) = i'\Pi_{p+q+1}(K_{p+q}, L) = i'\eta\Pi_{p+q}(K_{p+q}, L)$ $= \theta i\, \Pi_{p+q}(K_{p+q}, L) = \theta \Pi_{p+q}(K_{p+q+1}, L)$. Thus θ maps $\Pi_{p+q}(K_{p+q+1}, L)$ onto $\Gamma_{p+q+1}(K, L)$.

Since $2\Pi_{p+q+1}(K_{p+q}, L) = 0$, it follows that $2\Gamma_{p+q+1}(K, L) = 0$, so that $\theta\,(2\Pi_{p+q}(K_{p+q+1}, L)) = 0$. Assume that $\theta w = 0$, $w \in \Pi_{p+q}(K_{p+q+1}, L)$. Then $w = ix$, $x \in \Pi_{p+q}(K_{p+q}, L)$ and $0 = \theta w = \theta i x = i'\eta x$. By the property of exactness, $\eta x = d'y$, $y \in \Pi_{p+q+2}(K_{p+q+1}, K_{p+q})$, so that

$$\eta x = d'y = d'\zeta z = \eta d z \ ,$$

for some $z \in \Pi_{p+q+1}(K_{p+q+1}, K_{p+q})$. Since η is, algebraically, reduction mod 2, we have $x - dz \in 2\Pi_{p+q}(K_{p+q}, L)$, so that $w = ix = i(x - dz)$ $\in 2\Pi_{p+q}(K_{p+q+1}, L)$. This proves the lemma.

Now $\Pi_{p+q}(K_{p+q+1}, L) = \Pi_{p+q}(K, L) = \Pi_{p+q}(X \times Y, X \vee Y)$. By (2.9) and (2.11) $\Pi_{p+q}(X \times Y, X \vee Y)$ is isomorphic to $H_p(X) \otimes H_q(Y)$. In fact the group $\Pi_{p+q}(X \times Y, X \vee Y)$ is generated by elements of the form $\alpha \cdot \beta$, $\alpha \in \Pi_p(X)$, $\beta \in \Pi_q(Y)$, and the isomorphism $\Pi_{p+q}(X \times Y, X \vee Y) \approx H_p(X) \otimes H_q(Y)$ is achieved by mapping[9]) $\alpha \cdot \beta$ on $\varrho_1\alpha \otimes \varrho_2\beta$. We have proved

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[8]) Theorem 1 of [13] strengthened as in [6].

[9]) We will use the notation $\alpha \cdot \beta$ throughout the paper for the product, in Π_{p+q} $(X \times Y, X \vee Y)$, of $\alpha \in \Pi_p(X)$, $\beta \in \Pi_q(Y)$, defined as in (2.12).

Theorem 4.3. *Let X be a connected CW-complex with $\Pi_r(Y) = 0$, $r = 1, \ldots, p-1$, and let Y be a connected CW-complex with $\Pi_r(Y) = 0$, $r = 1, \ldots, q-1$, where $p, q \geqslant 3$. then*

$$\Gamma_{p+q+1}(X \times Y, X \vee Y) \approx H_p(X) \otimes H_q(Y) \otimes Z_2 .$$

Let η_r generate $\Pi_r(S^{r-1})$, $r \geqslant 4$, and let $\bar{\eta}_r$ generate $\Pi_{r+1}(I^r, \dot{I}^r)$, $r \geqslant 4$. Then it is easy to see that [10])

$$[\alpha, \beta] \circ \eta_{p+q} = [\alpha \circ \eta_{p+1}, \beta] = [\alpha, \beta \circ \eta_{q+1}] , \tag{4.4}$$

and that

$$d((\alpha \cdot \beta) \circ \bar{\eta}_{p+q}) = [\alpha, \beta] \circ \eta_{p+q} . \tag{4.5}$$

Since d is univalent, we have

$$(\alpha \cdot \beta) \circ \bar{\eta}_{p+q} = (\alpha \circ \eta_{p+1}) \cdot \beta = \alpha \cdot (\beta \circ \eta_{q+1}) . \tag{4.6}$$

Corollary 4.7. *The isomorphism $\Gamma_{p+q+1}(X \times Y, X \vee Y) \approx H_p(X) \otimes H_q(Y) \otimes Z_2$ is achieved by mapping $(\alpha \cdot \beta) \circ \bar{\eta}_{p+q}$ as the residue class mod 2 of $\varrho_1\alpha \otimes \varrho_2\beta$.*

Corollary 4.8. *The subgroup $d\lambda\Gamma_{p+q+1}(X \times Y, X \vee Y)$ of $\Pi_{p+q}(X \vee Y)$ is generated by elements expressible in any of the equivalent forms* (4.4). *It is, algebraically, a homomorphic image of $H_p(X) \otimes H_q(Y) \otimes Z_2$.*

We recall that, as applications of the exact sequence (3.1) we have the exact sequences

$$\begin{aligned} \ldots \Gamma_n(X) \xrightarrow{\lambda_1} \Pi_n(X) \xrightarrow{\varrho_1} H_n(X) \xrightarrow{\mu_1} \Gamma_{n-1}(X) \to \ldots \to H_{p+2}(X) \xrightarrow{\mu_1} \Gamma_{p+1}(X) \\ \xrightarrow{\lambda_1} \Pi_{p+1}(X) \xrightarrow{\varrho_1} H_{p+1}(X) \to 0 , \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} \ldots \Gamma_n(Y) \xrightarrow{\lambda_2} \Pi_n(Y) \xrightarrow{\varrho_2} H_n(Y) \xrightarrow{\mu_2} \Gamma_{n-1}(Y) \to \ldots \to H_{q+2}(Y) \xrightarrow{\mu_2} \Gamma_{q+1}(Y) \\ \xrightarrow{\lambda_2} \Pi_{q+1}(Y) \xrightarrow{\varrho_2} H_{q+1}(Y) \to 0 . \end{aligned} \tag{4.10}$$

Moreover, composition with η_{p+1} induces an isomorphism

$$\Pi_p(X) \otimes Z_2 \approx H_p(X) \otimes Z_2 \approx \Gamma_{p+1}(X) \tag{4.11}$$

and composition with η_{q+1} induces an isomorphism

$$\Pi_q(Y) \otimes Z_2 \approx H_q(Y) \otimes Z_2 \approx \Gamma_{q+1}(Y) \tag{4.12}$$

We may thus re-express (4.3) as

$$\begin{aligned} \Gamma_{p+q+1}(X \times Y, X \vee Y) \approx \Gamma_{p+1}(X) \otimes H_q(Y) \approx H_p(X) \otimes \Gamma_{q+1}(Y) \\ \approx \Gamma_{p+1}(X) \otimes \Gamma_{q+1}(Y) \end{aligned} \tag{4.13}$$

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[10]) Relations (4.4) are special cases of (3.59) of [9].

Relation (4.6) may then be expressed in the form[11])

$$\Gamma_{p+q+1}(X \times Y, X \vee Y) = \Gamma_{p+1}(X) \cdot \Pi_q(Y) = \Pi_p(X) \cdot \Gamma_{q+1}(Y) \qquad (4.14)$$

5. Calculation of $\Pi_{p+q+1}(K_{p+q+1}, L)$ and the homomorphism μ.

By (2.6) we have

$$H_{p+q+1}(X \times Y, X \vee Y) \approx H_{p+1}(X) \otimes H_q(Y) + H_p(X) \otimes H_{q+1}(Y) + \operatorname{Tor}(H_p(X), H_q(Y)) \qquad (5.1)$$

and $H_{p+q+2}(X \times Y, X \vee Y) \approx H_{p+2}(X) \otimes H_q(Y) + H_{p+1}(X) \otimes H_{q+1}(Y) + H_p(X) \otimes H_{q+2}(Y) + \operatorname{Tor}(H_{p+1}(X), H_q(Y)) + \operatorname{Tor}(H_p(X), H_{q+1}(Y))$ (5.2)

By (3.6), we know that $\Pi_{p+q+1}(X \times Y, X \vee Y)$ is an extension of $\Gamma_{p+q+1}(X \times Y, X \vee Y) - \mu H_{p+q+2}(X \times Y, X \vee Y)$ by $H_{p+q+1}(X \times Y, X \vee Y)$. In view of (4.3), (5.1), and (5.2), it remains to express μ and the extension class in terms of known invariants of homotopy type. However, we will argue from the special choice of complexes X, Y made in the previous section and we therefore devote this section to a discussion of the non-invariant group $\Pi_{p+q+1}(K_{p+q+1}, L)$, under the assumptions $X^{p-1} = Y^{q-1} = x_0$. Consider the exact sequence

$$\ldots \to \Pi_{p+q+1}(K_{p+q}, L) \xrightarrow{i} \Pi_{p+q+1}(K_{p+q+1}, L) \xrightarrow{j} \Pi_{p+q+1}(K_{p+q+1}, K_{p+q}) \xrightarrow{d} \Pi_{p+q}(K_{p+q}, L) \to \ldots \qquad (5.3)$$

Since $L = K_{p+q-1}$, d is the homology boundary operator. Thus $d^{-1}(0) = Z_{p+q+1}(K, L)$, the group of $(p + q + 1)$-dimensional relative cycles of K mod L. Thus j maps $\Pi_{p+q+1}(K_{p+q+1}, L)$ onto $Z_{p+q+1}(K, L)$ with kernel $i\Pi_{p+q+1}(K_{p+q}, L) = \Gamma_{p+q+1}(K, L)$. Since $Z_{p+q+1}(K, L)$ is free abelian we may choose any (univalent) homomorphism $\Theta : Z_{p+q+1}(K, L) \to \Pi_{p+q+1}(K_{p+q+1}, L)$ such that $j\Theta = 1$ and obtain

$$\Pi_{p+q+1}(K_{p+q+1}, L) = \Gamma_{p+q+1}(K, L) + \Theta Z_{p+q+1}(K, L) \ . \qquad (5.4)$$

We now make a special choice of Θ. To this end, consider the exact sequences

$$\ldots \to \Pi_{p+1}(X^p) \xrightarrow{i_1} \Pi_{p+1}(X^{p+1}) \xrightarrow{j_1} \Pi_{p+1}(X^{p+1}, X^p) \xrightarrow{d_1} \Pi_p(X^p) \to \ldots \qquad (5.5)$$

[11]) If $\alpha \in \Gamma_{p+1}(X)$ is represented by $f: I^{p+1} \to X^p$, and $\beta \in \Pi_q(Y)$ is represented by $g: I^q \to Y^q$, then the map $h: I^{p+q+1} \to X \times Y$, given by $h(a, b) = (f(a), g(b))$, $a \in I^{p+1}$, $b \in I^q$, is of the form $h: I^{p+q+1}, \dot{I}^{p+q+1} \to K_{p+q}, L$. Thus it may be seen that the product of (2.12) induces a product of elements in $\Gamma_{p+1}(X)$ and $\Pi_q(Y)$ with values in $\Gamma_{p+q+1}(X \times Y, X \vee Y)$.

and

$$\ldots \to \Pi_{q+1}(Y^q) \xrightarrow{i_2} \Pi_{q+1}(Y^{q+1}) \xrightarrow{j_2} \Pi_{q+1}(Y^{q+1}, Y^q) \xrightarrow{d_2} \Pi_q(Y^q) \to \ldots \quad (5.6)$$

there are (univalent) homomorphisms $\Theta_1 : Z_{p+1}(X) \to \Pi_{p+1}(X^{p+1})$, $\Theta_2 : Z_{q+1}(Y) \to \Pi_{q+1}(Y^{q+1})$ such that

$$\Pi_{p+1}(X^{p+1}) = \Gamma_{p+1}(X) + \Theta_1 Z_{p+1}(X) \ , \quad (5.7)$$

$$\Pi_{q+1}(Y_{q+1}) = \Gamma_{q+1}(Y) + \Theta_2 Z_{q+1}(Y) \ , \quad (5.8)$$

and $\qquad j_1\Theta_1 = 1, \ j_2\Theta_2 = 1$.

Lemma 5.9. $Z_{p+1}(X) \otimes Z_q(Y) + Z_p(X) \otimes Z_{q+1}(Y)$ *is a direct summand in* $Z_{p+q+1}(K, L)$.

Since $C_{p+q+1}(K, L) = \sum\limits_{\substack{r+s=p+q+1 \\ r\not\approx 0, \ s\not\approx 0}} C_r(X) \otimes C_s(Y)$, it is clear that $Z_{p+1}(X) \otimes Z_q(Y)$ is a subgroup of $Z_{p+q+1}(K, L)$. Now we may express $C_r(X)$ as $Z_r(X) + D_r(X)$, $C_s(Y)$ as $Z_s(Y) + D_s(Y)$. Then

$$C_r(X) \otimes C_s(Y) = Z_r(X) \otimes Z_s(Y) + D_r(X) \otimes C_s(Y) + Z_r(X) \otimes D_s(Y) \ ,$$

by a standard theorem on tensor products. Writing $D_{rs}(X, Y)$ for $D_r(X) \otimes C_s(Y) + Z_r(X) \otimes D_s(Y)$, we have

$$C_{p+q+1}(K, L) = \sum_{\substack{r+s=p+q+1 \\ r\not\approx 0, \ s\not\approx 0}} Z_r(X) \otimes Z_s(Y) + \sum_{\substack{r+s=p+q+1 \\ r\not\approx 0, \ s\not\approx 0}} D_{rs}(X, Y), \ldots \quad (5.10)$$

and, restricting this to $Z_{p+q+1}(K, L)$, we have

$$Z_{p+q+1}(K,L) = \sum_{\substack{r+s=p+q+1 \\ r\not\approx 0, \ s\not\approx 0}} Z_r(X) \otimes Z_s(Y) + Z_{p+q+1}(K,L) \cap \sum_{\substack{r+s=p+q-1 \\ r\not\approx 0, \ s\not\approx 0}} D_{rs}(X, Y) \quad (5.11)$$

Relation (5.11) establishes the lemma. It should be noted that (5.11) is quite independent of the special choice of X^{p-1} and Y^{q-1}, indeed of any special properties of X, Y at all.

We now define Θ on $Z_{p+1}(X) \otimes Z_q(Y) + Z_p(X) \otimes Z_{q+1}(Y)$. Let $\alpha \in Z_{p+1}(X)$, $\beta \in Z_q(Y)$ and let ϱ_2 be the natural isomorphism $\varrho_2 : \Pi_q(Y^q) \approx H_q(Y^q) = Z_q(Y)$. Let $f : I^{p+1}, \dot{I}^{p+1} \to X^{p+1}, x_0$ represent $\Theta_1 \alpha$, let $g : I^q, \dot{I}^q \to Y^q, y_0$ represent $\varrho_2^{-1}\beta$. Then $h : I^{p+q+1}, \dot{I}^{p+q+1} \to K_{p+q+1}, L$, given by

$$h(a, b) = (f(a), g(b)) \, , \ a \in I^{p+1}, b \in I^q \ ,$$

represents an element of $\Pi_{p+q+1}(K_{p+q+1}, L)$ which we may designate $\Theta_1 \alpha \cdot \varrho_2^{-1}\beta$. We define

$$\Theta^*(\alpha \otimes \beta) = \Theta_1 \alpha \cdot \varrho_2^{-1}\beta \ . \quad (5.12)$$

The map f, regarded as a map of $(I^{p+1}, \dot{I}^{p+1})$ into (X^{p+1}, X^p) represents $j_1 \Theta_1 \alpha = \alpha$. The map g, regarded as a map of $(I^q, \dot{I}^q)$ into (Y^q, Y^{q-1}) represents $\beta \in Z_q(Y)$. Finally, the map h, regarded as a map of $(I^{p+q+1}, \dot{I}^{p+q+1})$ into (K_{p+q+1}, K_{p+q}) represents $j(\Theta_1 \alpha \cdot \varrho_2^{-1} \beta)$. Thus

$$j\Theta^*(\alpha \otimes \beta) = \alpha \otimes \beta .$$

Similarly we define Θ^* on $Z_p(X) \otimes Z_{q+1}(Y)$ by

$$\Theta^*(\gamma \otimes \delta) = \varrho_1^{-1}\gamma \cdot \Theta_2 \delta , \tag{5.13}$$

$\gamma \in Z_p(X)$, $\delta \in Z_{q+1}(Y)$, and we have $j\Theta^*(\gamma \otimes \delta) = \gamma \otimes \delta$.

Now let Θ' be an arbitrary isomorphism of $Z_{p+q+1}(K, L)$ into

$$\Pi_{p+q+1}(K_{p+q+1}, L)$$

such that $j\Theta' = 1$; such exists as shewn in obtaining (5.4). Now

$$Z_{p+q+1}(K, L) = Z_{p+1}(X) \otimes Z_q(Y) + Z_p(X) \otimes Z_{q+1}(Y) + R ,$$

where R is some free abelian group. We define

$$\Theta : Z_{p+q+1}(K, L) \to \Pi_{p+q+1}(K_{p+q+1}, L) \text{ by}$$

$$\Theta \mid Z_{p+1}(X) \otimes Z_q(Y) + Z_p(X) \otimes Z_{q+1}(Y) = \Theta^*, \Theta \mid R = \Theta' | R .$$

Since $j\Theta^* = 1$ and $j\Theta' = 1$, it follows that $j\Theta = 1$. Then Θ is univalent and will be used in the sequel as a specific isomorphism of $Z_{p+q+1}(K, L)$ into $\Pi_{p+q+1}(K_{p+q+1}, L)$ verifying (5.4).

Let $$f : I^{p+2}, \dot{I}^{p+2} \to X^{p+2}, X^{p+1}$$

be a characteristic map for a $(p+2)$ cell, e^{p+2}, in X and let $g : I^q, \dot{I}^q \to Y^q, y_0$ be a characteristic map for a q-cell, e^q in Y. Let h: $I^{p+q+2}, \dot{I}^{p+q+2} \to K_{p+q+2}, K_{p+q+1}$ be given by

$$h(a, b) = (f(a), g(b)), \ a \in I^{p+2}, b \in I^q .$$

Then h is a characteristic map for $e^{p+2} \times e^q$. Let $f \mid \dot{I}^{p+2}$ represent $\alpha + \Theta_1 \beta$, $\alpha \in \Gamma_{p+1}(X)$, $\beta \in Z_{p+1}(X)$; let g represent $\iota \in Z_q(Y)$ and $\bar{\iota} \in \Pi_q(Y)$; let h represent $\gamma \in \Pi_{p+q+2}(K_{p+q+2}, K_{p+q+1})$; and let

$$\Pi_{p+q+2}(K_{p+q+2}, K_{p+q+1}) \xrightarrow{d} \Pi_{p+q+1}(K_{p+q+1}, L) \xrightarrow{i} \Pi_{p+q+1}(K_{p+q+2}, L)$$

be an extract of the exact sequence of the triple $(K_{p+q+2}, K_{p+q+1}, L)$. Now it is clear that i is onto and, moreover, that $\Pi_{p+q+1}(K_{p+q+2}, L)$ is isomorphic with $\Pi_{p+q+1}(K, L)$. Thus

$$\Pi_{p+q+1}(K, L) \approx \Pi_{p+q+1}(K_{p+q+1}, L) - d\Pi_{p+q+2}(K_{p+q+2}, K_{p+q+1}) .$$

Theorem 5.14. $d\gamma = \alpha \cdot \bar{\iota} + \Theta(\beta \otimes \iota)$.

Put $X_0 = X^{p+1} \cup e^{p+2}$, $Y_0 = S^q = g(I^q)$, $M = X_0 \times Y_0$, $N = X_0 \vee Y_0$, and let h represent $\gamma_0 \in \Pi_{p+q+2}(M, M^{p+q+1})$. Then $\Pi_{p+q+1}(M, N)$ is obtained from $\Pi_{p+q+1}(M^{p+q+1}, N)$ by adding the relation $d_0\gamma_0 = 0$, d_0 being the boundary homomorphism

$$d_0 : \Pi_{p+q+2}(M, M^{p+q+1}) \to \Pi_{p+q+1}(M^{p+q+1}, N) .$$

Let g represent $\iota_0 \in Z_q(Y_0)$, $\bar{\iota}_0 \in \Pi_q(Y_0)$. We will prove that

$$d_0\gamma_0 = \alpha \cdot \bar{\iota}_0 + \Theta(\beta \otimes \iota_0) , \tag{5.15}$$

and the theorem will follow by embedding M, N in K, L. First let $\alpha = 0$, $\beta = 0$; then e^{p+2} is attached inessentially to X^{p+1}, and the homomorphism d_0 induces the invariant homomorphism

$$\mu : H_{p+q+2}(M, N) \to \Gamma_{p+q+1}(M, N) .$$

In fact, if $\bar{\gamma}_0$ is the homology class of the cycle γ_0, then $d_0\gamma_0 = \mu\bar{\gamma}_0$. Now γ_0 is in $\lambda\Pi_{p+q+2}(M, N)$; this is easily seen if we replace X_0 by $X^{p+1} \cup S^{p+2}$, as we may do since λ is a homotopy invariant. Thus $d_0\gamma_0 = \mu\bar{\gamma}_0 = 0$ and (5.15) is established in this special case. We assume henceforth that at least one of α, β is non-zero.

In general, $\Pi_{p+1}(X_0)$ is obtained from $\Pi_{p+1}(X^{p+1})$ by adding the relation $\alpha + \Theta_1\beta = 0$. Thus, in $\Pi_{p+q}(N)$, the relation $[k(\alpha + \Theta_1\beta), \bar{\iota}_0] = 0$ holds, where k is the injection $\Pi_{p+1}(X^{p+1}) \to \Pi_{p+1}(X_0)$. Now the boundary homomorphism $d : \Pi_{p+q+1}(M, N) \to \Pi_{p+q}(N)$ is univalent and $d(k\alpha \cdot \bar{\iota}_0) = [k\alpha, \bar{\iota}_0]$, $d(k\Theta_1\beta \cdot \bar{\iota}_0) = [k\Theta_1\beta, \bar{\iota}_0]$. Moreover, in Y_0, $\varrho_2^{-1}\iota_0 = \bar{\iota}_0$, since $Y_0 = Y_0^q$. Thus, in $\Pi_{p+q+1}(M, N)$, we have the relation

$$k\alpha \cdot \bar{\iota}_0 + k\Theta_1\beta \cdot \varrho_2^{-1}\iota_0 = 0 .$$

The left-hand side is just the injection in $\Pi_{p+q+1}(M, N)$ of

$$\alpha \cdot \bar{\iota}_0 + \Theta_1\beta \cdot \varrho_2^{-1}\iota_0 \in \Pi_{p+q+1}(M^{p+q+1}, N) ,$$

and $\Theta_1\beta \cdot \varrho_2^{-1}\iota_0 = \Theta(\beta \otimes \iota_0)$, by (5.12). Moreover, $\alpha \cdot \bar{\iota}_0 + \Theta(\beta \otimes \iota_0) \neq 0$ unless $\alpha = 0$, $\beta = 0$. Thus there must exist an integer τ such that $\tau d_0\gamma_0 = \alpha \cdot \bar{\iota}_0 + \Theta(\beta \otimes \iota_0)$. Suppose first that $\beta = 0$; then $\alpha \neq 0$, so that $\alpha \cdot \bar{\iota}_0 \neq 0$ and $\tau d_0\gamma_0 = \alpha \cdot \bar{\iota}_0$. Since $Z_{p+q+1}(M, N)$ is free abelian, this shows that $d_0\gamma_0 \in \Gamma_{p+q+1}(M, N)$, so that τ is odd and $\tau d_0\gamma_0 = d_0\gamma_0 = \alpha \cdot \bar{\iota}_0$. Now suppose that $\beta \neq 0$. We then show that $\tau = 1$ by studying the homology boundary.

For consider the diagram

$$\begin{array}{ccc} \Pi_{p+q+2}(M, M^{p+q+1}) & \overset{\varrho}{\approx} & H_{p+q+2}(M, M^{p+q+1}) \\ \downarrow d_0 & & \downarrow \partial \\ \Pi_{p+q+1}(M^{p+q+1}, N) & \overset{j}{\rightarrow} & Z_{p+q+1}(M, N) \end{array}$$

Clearly $jd_0 = \partial\varrho$. Now $\tau j d_0 \gamma_0 = j(\alpha \cdot \bar{\iota}_0 + \Theta(\beta \otimes \iota_0)) = \beta \otimes \iota_0$. On the other hand $\partial\varrho\gamma_0$ is the homology boundary of the cell $e^{p+2} \times e^q$, which is just $\beta \otimes \iota_0$. Thus $\beta \otimes \iota_0 = \tau(\beta \otimes \iota_0)$, and $\tau = 1$ since $\beta \otimes \iota_0 \neq 0$ and $Z_{p+q+1}(M, N)$ is free abelian. Thus (5.15) holds in all cases and the theorem follows.

Since $\mu: H_{p+q+2}(K, L) \rightarrow \Gamma_{p+q+1}(K, L)$ is induced by d:

$$\Pi_{p+q+2}(K_{p+q+2}, K_{p+q+1}) \rightarrow \Pi_{p+q+1}(K_{p+q+1}, L),$$

theorem 5.14 tells us how μ operates on $H_{p+2}(X) \otimes H_q(Y)$ and similarly, how μ operates[12] on $H_p(X) \otimes H_{q+2}(Y)$. In fact, if $\mu_1: H_{p+2}(X) \rightarrow \Gamma_{p+1}(X)$ is defined as in (4.9), we have the obvious

Corollary 5.16 *Let* $\kappa \in H_{p+2}(X)$, $\xi \in \Pi_q(Y)$. *Then* $\mu(\kappa \otimes \varrho_2 \xi) = \mu_1 \kappa \cdot \xi$. A similar formula holds, of course, for $\mu \mid H_p(X) \otimes H_{q+2}(Y)$.

Theorem 5.17 *The homomorphism* μ *is zero on* $H_{p+1}(X) \otimes H_{q+1}(Y)$.

Now any element of $H_{p+1}(X)$ may be represented by a map f: $I^{p+1}, \dot{I}^{p+1} \rightarrow X, x_0$; and any element of $H_{q+1}(Y)$ may be represented by a map $g: I^{q+1}, \dot{I}^{q+1} \rightarrow Y, y_0$. Thus any element of $H_{p+1}(X) \otimes H_{q+1}(Y)$ may be represented by a map f: $I^{p+q+2}, \dot{I}^{p+q+2} \rightarrow K, L$. This shows that $\mu \mid (H_{p+1}(X) \otimes H_{q+1}(Y)) = 0$.

We have now[13] described μ on $H_{p+2}(X) \otimes H_q(Y) + H_{p+1}(X) \otimes H_{q+1}(Y) + H_p(X) \otimes H_{q+2}(Y)$. We do not describe μ explicitly on Tor $(H_p(X), H_{q+1}(Y))$ + Tor $(H_{p+1}(X), H_q(Y))$ but calculate it in a simple but typical case.

Theorem 5.18. *Let* $X = S^p \cup e^{p+1}$, $Y = S^q \cup S^{q+1} \cup e^{q+2}$, *where* e^{p+1} *is attached to* S^p *by a map of degree* σ, *and* e^{q+2} *is attached to* $S^q \cup S^{q+1}$, $(S^q \cap S^{q+1} = y_0)$, *by a map which is essential over* S^q *and of degree* τ *over*

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[12]) Starting from a p-cell of X and a $(q+2)$-cell of Y, the formula corresponding to (5.14) would be, with the obvious notation, $d\gamma = \bar{\iota} \cdot \alpha + (-1)^p \Theta(\iota \otimes \beta)$. See theorem 5.18.

[13]) Note that $\Gamma_{p+q+1}(K, L) - \mu(H_{p+2}(X) \otimes H_q(Y) + H_{p+1}(X) \otimes H_{q+1}(Y) + H_p(X) \otimes H_{q+1}(Y)) \approx (\Gamma_{p+1}(X) - \mu_1 H_{p+2}(X)) \otimes (\Gamma_{q+1}(Y) - \mu_2 H_{q+2}(Y))$.

S^{q+1}. *Let* $\bar{\varrho} = (\sigma, \tau)$, *the h.c.f. of* σ *and* τ, *and let* $\sigma = \bar{\varrho}\sigma'$. *Then, if* $\bar{\varrho}$ *is even,*

$$\mu Tor\,(H_p(X), H_{q+1}(Y)) = 0, \sigma' \text{ even}, = \Gamma_{p+q+1}(X \times Y, X \vee Y), \sigma' \text{ odd}. \tag{5.19}$$

$$\textit{Moreover } \Pi_{p+q+1}(X \times Y, X \vee Y) \approx \Pi_p(X) \otimes \Pi_{q+1}(Y) = \begin{cases} Z_{2\bar{\varrho}}, \sigma' \text{ even} \\ Z_{\bar{\varrho}}, \sigma' \text{ odd}. \end{cases} \tag{5.20}$$

Consider the exact sequence

$$H_{p+q+2}(X \times Y,\ X \vee Y) \xrightarrow{\mu} \Gamma_{p+q+1}(X \times Y, X \vee Y) \xrightarrow{\lambda} \Pi_{p+q+1}(X \times Y,\ X \vee Y)$$
$$\xrightarrow{\varrho} H_{p+q+1}(X \times Y,\ X \vee Y) \to 0\,.$$

In our special case[14] this is

$$Z_{\bar{\varrho}} \to Z_2 \to \Pi_{p+q+1}(X \times Y,\ X \vee Y) \to Z_{\bar{\varrho}} \to 0 \tag{5.21}$$

Now Tor $(H_p(X), H_{q+1}(Y))$ is generated by $\tau'(e^{p+1} \times S^{q+1}) + (-1)^{p+1}$ $\sigma'(S^p \times e^{q+2})$, where $\tau = \bar{\varrho}\tau'$. For the boundary of this chain is $\tau'\sigma(S^p \times S^{q+1}) - \sigma'\tau(S^p \times S^{q+1}) = 0$ and $\bar{\varrho}(\tau'(e^{q+1} \times S^{q+1}) + (-1)^{p+1}$ $\sigma'(S^p \times e^{q+2})) = \tau(e^{p+1} \times S^{q+1}) + (-1)^{p+1}\sigma(S^p \times e^{q+2})$, which is the boundary of $(-1)^{p+1}(e^{p+1} \times e^{q+2})$. We study the behaviour of $e^{p+1} \times S^{q+1}$ under d: $\Pi_{p+q+2}(K^{p+q+2}, K^{p+q+1}) \to \Pi_{p+q+1}(K^{p+q+1}, L)$, where, as usual, $K = X \times Y, L = X \vee Y$. It is clear, in fact, that d coincides with the homology boundary on $e^{p+1} \times S^{q+1}$ in the sense that

$$d(e^{p+1} \times S^{q+1}) = \Theta\sigma(S^p \times S^{q+1})\,.$$

On the other hand, if η_{q+1} generates $\Pi_{q+1}(S^q)$ and ι_p generates $\Pi_p(S^p)$ it follows from (5.14) that

$$d(S^p \times e^{q+2}) = \iota_p \cdot \eta_{q+1} + (-1)^p\Theta\tau(S^p \times S^{q+1})\,.$$

Recall that $\iota_p \cdot \eta_{q+1}$ generates $\Gamma_{p+q+1}(K, L)$. Then

$$d(\tau'(e^{p+1} \times S^{q+1}) + (-1)^{p+1}\sigma'(S^p \times e^{q+2})) = \sigma'(\iota_p \cdot \eta_{q+1})\,,$$

and this establishes (5.19).

Let σ' be odd. Then λ maps to zero and $\varrho : \Pi_{p+q+1}(X \times Y, X \vee Y) \approx Z_{\bar{\varrho}}$. Moreover, ϱ maps $\iota_p \cdot \iota_{q+1}$ onto the generator of $H_{p+q+1}(X \times Y, X \vee Y)$, where ι_{q+1} generates $\Pi_{q+1}(S^{q+1})$. This establishes part of (5.20) if σ' is odd and shows, moreover, that $\Pi_{p+q+1}(X \times Y, X \vee Y)$, is generated by $\iota_p \cdot \iota_{q+1}$. Let ι_p, ι_{q+1}, also stand for the injections of ι_p, ι_{q+1} in $\Pi_p(X), \Pi_{q+1}(Y)$ respectively. Then ι_p generates $\Pi_p(X) = Z_\sigma$ and ι_{q+1}

[14]) If A, B are finitely generated Abelian groups and A', B' their finite parts, then Tor $(A, B) \approx A' \otimes B'$.

generates[15] $\Pi_{q+1}(Y) = Z_{2\tau}$. Thus, since $(\sigma, 2\tau) = \bar{\varrho}$, σ' being odd, we have (5.20) and the isomorphism $\Pi_p(X) \otimes \Pi_{q+1}(Y) \approx \Pi_{p+q+1}(X \times Y, X \vee Y)$ is established by the mapping

$$\iota_p \otimes \iota_{q+1} \to \iota_p \cdot \iota_{q+1} \; . \tag{5.21}$$

We now take the case of σ' even. Then $\iota_p \cdot \eta_{q+1} \neq 0$ and $\iota_p \cdot (\eta_{q+1} + \tau\iota_{q+1}) = 0$, since $\eta_{q+1} + \tau\iota_{q+1} = 0$ in $\Pi_{q+1}(Y)$. Thus $\iota_p \cdot \iota_{q+1}$ is not of order $\bar{\varrho}$. Since, as before, $\iota_p \cdot \iota_{q+1}$ maps onto the generator of $H_{p+q+1}(X \times Y, X \vee Y)$ and since the kernel of λ is Z_2, it follows that $\iota_p \cdot \iota_{q+1}$ is of order $2\bar{\varrho}$ and generates $\Pi_{p+q+1}(X \times Y, X \vee Y)$. Since $(\sigma, 2\tau) = 2\bar{\varrho}$, (5.20) is verified if σ' is even and the isomorphism $\Pi_p(X) \otimes \Pi_{q+1}(Y) \approx \Pi_{p+q+1}(X \times Y, X \vee Y)$ is again achieved by (5.21).

6. The main theorem

Let $\alpha \in \Pi_p(X)$, $\beta \in \Pi_q(Y)$, where X is any connected CW-complex such that $\Pi_r(X) = 0$, $r = 1, 2, \ldots, p-1$, and Y is any connected CW-complex such that $\Pi_s(Y) = 0$, $s = 1, 2, \ldots, q-1$, $p \geqslant 3$, $q \geqslant 3$. Let $\bar{\alpha} \in \Gamma_{p+1}(X)$, $\bar{\beta} \in \Gamma_{q+1}(Y)$ be the images of α, β under the isomorphisms (4.11), (4.12) of $\Pi_p(X) \otimes Z_2$ with $\Gamma_{p+1}(X)$, $\Pi_q(Y) \otimes Z_2$ with $\Gamma_{q+1}(Y)$. Let λ_1, λ_2 mean the same as in (4.9), (4.10) and let G be the group obtained from

$$\Pi_p(X) \otimes \Pi_{q+1}(Y) + \Pi_{p+1}(X) \otimes \Pi_q(Y)$$

by identifying $\alpha \otimes \lambda_2\bar{\beta}$ with $\lambda_1\bar{\alpha} \otimes \beta$, all α, β. Our main theorem is

Theorem 6.1. $\varrho^{-1}(H_p(X) \otimes H_{q+1}(Y) + H_{p+1}(X) \otimes H_q(Y)) \approx G$. *Moreover, G is isomorphically embedded in $\Pi_{p+q+1}(X \times Y, X \vee Y)$ by the embeddings*

$$\alpha \otimes \gamma \to \alpha \cdot \gamma, \; \alpha \in \Pi_p(X), \gamma \in \Pi_{q+1}(Y) \; ,$$

$$\delta \otimes \beta \to \delta \cdot \beta, \; \delta \in \Pi_{p+1}(X), \beta \in \Pi_q(Y) \; .$$

Before proving the theorem, we note the following consequence. Identifying G with its image in $\Pi_{p+q+1}(X \times Y, X \vee Y)$, and letting w stand for the projection $H_{p+q+1}(X \times Y, X \vee Y) \to \mathrm{Tor}(H_p(X), H_q(Y))$, we have

Theorem 6.2. *$\Pi_{p+q+1}(X \times Y, X \vee Y)$ is an extension of G by Tor $(H_p(X), H_q(Y))$. Precisely, $w\varrho$ is a homomorphism of $\Pi_{p+q+1}(X \times Y, X \vee Y)$ onto Tor $(H_p(X), H_q(Y))$ with kernel G.* We now prove theorem 6.1.

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[15]) This follows easily from the Whitehead suspension theorem. See [2], [5].

Let G' be the subgroup of $\Pi_{p+q+1}(X \times Y, X \vee Y)$ generated by all products $\alpha \cdot \gamma, \delta \cdot \beta$, $\alpha \in \Pi_p(X)$, $\gamma \in \Pi_{q+1}(Y)$, $\delta \in \Pi_{p+1}(X)$, $\beta \in \Pi_q(Y)$. Then (i) ϱ maps G' onto $H_p(X) \otimes H_{q+1}(Y) + H_{p+1}(X) \otimes H_q(Y)$, which we will write $H(X, Y)$, (ii) G' contains $\lambda \Gamma_{p+q+1}(X \times Y, X \vee Y)$, (iii) $\alpha \cdot \lambda_2 \bar{\beta} = \lambda_1 \bar{\alpha} \cdot \beta$ (see 4.6, 4.14). It follows from (i) and (ii) that $G' = \varrho^{-1}(H(X, Y))$, and from (iii) that the mapping $\kappa : G \to G'$, given by $\kappa(\alpha \otimes \gamma) = \alpha \cdot \gamma, \kappa(\delta \otimes \beta) = \delta \cdot \beta$, is a homomorphism, κ, of G onto G'. It remains to prove that κ is univalent. Let us suppose this established for all *finite* complexes X, Y satisfying the conditions of the theorem. Then we show that it is true for all complexes X, Y. It is certainly sufficient to prove the result if $X^{p-1} = x_0$, $Y^{q-1} = y_0$, so we assume this. Let $\xi \in G(X, Y)$ and let ξ be the class containing

$$\sum_i \alpha_i \otimes \gamma_i + \sum_j \delta_j \otimes \beta_j \in \Pi_p(X) \otimes \Pi_{q+1}(Y) + \Pi_{p+1}(X) \otimes \Pi_q(Y), \alpha_i \in \Pi_p(X),$$

$\gamma_i \in \Pi_{q+1}(Y)$, $\delta_j \in \Pi_{p+1}$, $\beta_j \in \Pi_q(Y)$. Then $\kappa(\xi) = \sum_i \alpha_i \cdot \gamma_i + \sum_j \delta_j \cdot \beta_j$. If $\kappa(\xi) = 0$, there exist finite sub-complexes $X^* \subset X$, $Y^* \subset Y$ such that

$$\text{(i) } \alpha_i = \Phi \alpha_i^*, \alpha_i^* \in \Pi_p(X^*), \gamma_i = \Phi \gamma_i^*, \gamma_i^* \in \Pi_{q+1}(Y^*),$$
$$\delta_j = \Phi \delta_j^*, \delta_j^* \in \Pi_{p+1}(X^*), \quad \beta_j = \Phi \beta_j^*, \beta_j^* \in \Pi_q(Y^*),$$

where Φ stands for the relevant injection, and

$$\text{(ii) } \sum_i \alpha_i^* \cdot \gamma_i^* + \sum_j \delta_j^* \cdot \beta_j^* = 0 .$$

Let $\xi^* \in G(X^*, Y^*)$ be the class containing $\sum_i \alpha_i^* \otimes \gamma_i^* + \sum_j \delta_j^* \otimes \beta_j^*$ and let κ^* be the homomorphism $G(X^*, Y^*) \to G'(X^*, Y^*)$. Then $\kappa^*(\xi^*) = 0$ and, by the special choice of X, Y, X^*, Y^* satisfy the conditions of the theorem[16], so that, by our hypothesis, κ^* is univalent. Thus $\xi^* = 0$. On the other hand, the 'injection' homomorphism

$$\Pi_p(X^*) \otimes \Pi_{q+1}(Y^*) + \Pi_{p+1}(X^*) \otimes \Pi_q(Y^*) \to \Pi_p(X) \otimes \Pi_{q+1}(Y)$$
$$+ \Pi_{p+1}(X) \otimes \Pi_q(Y)$$

induces a homomorphism $\Psi: G(X^*, Y^*) \to G(X, Y)$, and it is clear that $\Psi \xi^* = \xi$. Thus $\xi = 0$ and κ is univalent.

We may now assume that X and Y are finite complexes. Also, we continue to assume that $X^{p-1} = x_0$, $Y^{q-1} = y_0$. It follows from (2.6)

[16]) If we take X, Y arbitrary, subject only to the hypotheses of the theorem, then we need to show at this stage that X^*, Y^* may be embedded in finite subcomplexes of X, Y whose first p–1 (resp. q–1) homotopy groups vanish.

and (4.3) that the inclusion $X^{p+2} \times Y^{q+2}, X^{p+2} \vee Y^{q+2} \subset X \times Y, X \vee Y$ induces a homomorphism of $H_{p+q+2}(X^{p+2} \times Y^{q+2}, X^{p+2} \vee Y^{q+2})$ onto H_{p+q+2} $(X \times Y, X \vee Y)$ and isomorphisms of $\Gamma_{p+q+1}(X^{p+2} \times Y^{q+2}, X^{p+2} \vee Y^{q+2})$, $H_{p+q+1}(X^{p+2} \times Y^{q+2}, X^{p+2} \vee Y^{q+2})$ onto $\Gamma_{p+q+1}(X \times Y, X \vee Y)$, $H_{p+q+1}(X \times Y$ $X \vee Y)$. It therefore follows from the 'five lemma' (lemma 4.3, p. 16, of [4]) that the inclusion $X^{p+2} \times Y^{q+2}, X^{p+2} \vee Y^{q+2} \subset X \times Y, X \vee Y$ induces an isomorphism of the sequence

$$\Gamma_{p+q+1}(X^{p+2} \times Y^{q+2}, X^{p+2} \vee Y^{q+2}) \to \Pi_{p+q+1}(X^{p+2} \times Y^{q+2}, X^{p+2} \vee Y^{q+2}) \to H_{p+q+1}(X^{p+2} \times Y^{q+2}, X^{p+2} \vee Y^{q+2}) \to 0,$$

onto the sequence

$$\Gamma_{p+q+1}(X \times Y, X \vee Y) \to \Pi_{p+q+1}(X \times Y, X \vee Y) \to H_{p+q+1}(X \times Y, X \vee Y) \to 0.$$

Since, also, it induces an isomorphism of $G(X^{p+2}, Y^{q+2})$ onto $G(X, Y)$, it is sufficient to establish (6.1) if $X = X^{p+2}$, $Y = Y^{q+2}$. We assume then that $X = X^{p+2}$, $Y = Y^{q+2}$. Finally we choose X_0, Y_0 of the same homotopy type as X, Y and in the Chang normal form (see [2]). That is to say, X_0 is the union of a finite number of *elementary* complexes with a single common vertex and Y_0 is similarly defined. We refer to [2] or [6], p. 481, for a description of the elementary complexes. We write $X_0 = \bigcup_i X_i$, $Y_0 = \bigcup_j Y_j$, where the X_i, Y_j are elementary complexes. Then

$$\Pi_p(X_0) = \sum_i \Pi_p(X_i),\ \Pi_{p+1}(X_0) = \sum_i \Pi_{p+1}(X_i),\ \Pi_q(Y_0) = \sum_i \Pi_q(Y_i) \text{ and}$$
$$\Pi_{q+1}(Y_0) = \sum_i \Pi_{q+1}(Y_i); \text{ and similarly for homology groups.} \qquad (6.3)$$

We omit the verification of (6.1) when X, Y are elementary complexes; this follows readily from (3.6), the only case of any difficulty having been dealt with in (5.18). Theorem (6.1) will then have been proved when we have shown that

$$G'(X_0, Y_0) = \sum_{i,j} G'(X_i, Y_j). \qquad (6.4)$$

Now let $\Psi_{ij} : X_0 \times Y_0, X_0 \vee Y_0 \to X_i \times Y_j, X_i \vee Y_j$ be the projection and let $\Phi_{ij} : X_i \times Y_j, X_i \vee Y_j \to X_0 \times Y_0, X_0 \vee Y_0$ be the inclusion map. Then it is clear that, under the induced homomorphism $(\Psi_{ij})_* : \Pi_{p+q+1}(X_0 \times Y_0, X_0 \vee Y_0) \to \Pi_{p+q+1}(X_i \times Y_j, X_i \vee Y_j)$, $G'(X_0, Y_0)$ is mapped into $G'(X_i, Y_j)$ and that, under the induced homomorphism $(\Phi_{ij})_* : \Pi_{p+q+1}(X_i \times Y_j, X_i \vee Y_j) \to \Pi_{p+q+1}(X_0 \times Y_0, X_0 \vee Y_0)$, $G'(X_i, Y_j)$

is mapped into $G'(X_0, Y_0)$. Now $\Psi_{ij}\Phi_{ij} = 1$, and, writing $(\mu_{ij})_*$ for $(\Phi_{ij})_*(\Psi_{ij})_*$ we have

$$(\mu_{ij})_*(\mu_{kl})_* = (\mu_{ij})_* \quad \text{if } i = k,\ j = l\,, \\ = 0\,, \quad \text{otherwise.}$$

It is now a standard algebraic result that, under these circumstances, the $(\Phi_{ij})_*$ are univalent and, identifying elements of $G'(X_i, Y_j)$ with their images under $(\Phi_{ij})_*$, we have

$$G'(X_0, Y_0) = \sum_{i,j} G'(X_i, Y_j) + R\,,$$

where R is the intersection of $G'(X_0, Y_0)$ and the kernels of the $(\mu_{ij})_*$. Formula (6.4) follows if we show that any element of $G'(X_0, Y_0)$ is expressible as $\sum_{i,j} \gamma_{ij}, \gamma_{ij} \in G'(X_i, Y_j)$. Let κ_{ij} be the isomorphism $G(X_i, Y_j) \to G'(X_i, Y_j)$. Then if $\gamma \in G'(X_0, Y_0), \gamma = \kappa\bar{\gamma}, \bar{\gamma} \in G(X_0, Y_0)$, $\bar{\gamma} = \sum_{i,j} \bar{\gamma}_{ij}, \bar{\gamma}_{ij} \in G(X_i, Y_j)$, and $\gamma = \kappa\bar{\gamma} = \sum_{i,j} \kappa_{ij}\bar{\gamma}_{ij}$. This proves (6.4) and hence completes the proof of the theorem.

Geometrically, the subgroup, G' of $\Pi_{p+q+1}(X \times Y, X \vee Y)$ consists of those elements expressible as 'Whitehead' products. More precisely, if $d: \Pi_{p+q+1}(X \times Y, X \vee Y) \to \Pi_{p+q}(X \vee Y)$ is the (univalent) boundary, then dG' is generated by those elements expressible as Whitehead products $[\alpha, \beta], \alpha \in \Pi_r(X), \beta \in \Pi_s(Y)$, where $r = p$, $s = q + 1$ or $r = p + 1$, $s = q$. We have shewn the algebraic structure of this group and of the difference group $\Pi_{p+q+1}(X \times Y, X \vee Y) - G'$. In the next section we will discuss the nature of the group extension. Meanwhile we note, summing up,

Theorem 6.5. $\Pi_{p+q}(X \vee Y) = \iota_1 \Pi_{p+q}(X) + \iota_2 \Pi_{p+q}(Y) + d\Pi_{p+q+1}(X \times Y, X \vee Y)$; *if X, Y are connected CW-complexes such that $\Pi_r(X) = 0$, $r = 1, \ldots, p-1$, $\Pi_s(Y) = 0$, $s = 1, \ldots, q-1$, $p \geqslant 3$, $q \geqslant 3$, then*

$$d\Pi_{p+q+1}(X \times Y, X \vee Y)$$

contains a subgroup dG' generated by all Whitehead products $[\alpha, \gamma]$, $\alpha \in \Pi_p(X)$, $\gamma \in \Pi_{q+1}(Y)$, and $[\delta, \beta], \delta \in \Pi_{p+1}(X)$, $\beta \in \Pi_q(Y)$; and G' is isomorphic to the group obtained from $\Pi_p(X) \otimes \Pi_{q+1}(Y) + \Pi_{p+1}(X) \otimes \Pi_q(Y)$ by identifying $\alpha \otimes \lambda_2\bar{\beta}$ with $\lambda_1\bar{\alpha} \otimes \bar{\beta}$, where $\lambda_2\beta$ is the image of β in the homomorphism $\Pi_q(Y) \to \Pi_{q+1}(Y)$ induced by composition with $\eta_{q+1}, \neq 0, \in \Pi_{q+1}(S^q)$ and $\lambda_1\bar{\alpha}$ is similarly defined. Calling this group G, the isomorphism $\kappa : G \approx G'$ is induced by

$$\kappa(\alpha \otimes \gamma) = \alpha \cdot \gamma, \kappa(\delta \otimes \beta) = \delta \cdot \beta\,.$$

The factor group of $d\Pi_{p+q+1}(X \times Y, X \vee Y)$ *by* dG' *is isomorphic to* $Tor(H_p(X), H_q(Y))$.

7. Calculation of $\varrho^{-1}(\mathrm{Tor}(H_p(X), H_q(Y)))$

We discuss in this section the nature of the group extension of G' by $\mathrm{Tor}(H_p(X), H_q(Y))$, and we give another interpretation of those elements of $\Pi_{p+q+1}(X \times Y, X \vee Y)$ which are mapped by ϱ onto $\mathrm{Tor}(H_p(X), H_q(Y))$. However, we limit the discussion to the simple yet typical case in which

$$X = S^p \cup e^{p+1}, \quad Y = S^q \cup e^{q+1},$$

e^{p+1} being attached by a map of degree σ, and e^{q+1} being attached by a map of degree τ. Let $k = (\sigma, \tau)$, the h.c.f. of σ, τ. Then, if k is odd, $\Gamma_{p+q+1}(X \times Y, X \vee Y) = 0$ and $\mathrm{Tor}(H_p(X), H_q(Y)) = Z_k$, so that $\Pi_{p+q+1}(X \times Y, X \vee Y) = Z_k$. Let us suppose k even. Then

$$\Gamma_{p+q+1}(X \times Y, X \vee Y) = Z_2$$

and $\Pi_{p+q+1}(X \times Y, X \vee Y)$ is an extension of Z_2 by Z_k.

M. G. Barratt has shewn[17] that, if k is even,

$$\Pi_{p+q+1}(X \times Y, X \vee Y) = Z_{2k}, \quad \text{if } \sigma = 4m+2, \ \tau = 4n+2 \qquad (7.1)$$

$$= Z_2 + Z_k, \quad \text{otherwise.} \qquad (7.2).$$

We will give here a proof of (7.2) which is independent of Barratt's work but which does not give the result (7.1). Let us assume without loss of generality that σ is divisible by 4. Let $\sigma = 2\sigma'$ and let $k' = (\sigma', \tau)$. Let $X' = S^p \cup e^{p+1}$, where e^{p+1} is attached by a map of degree σ'. A map $S^p \to S^p$ of degree 2 may be extended to a map of X' into X and thence to a map $f\colon X' \times Y, X' \vee Y \to X \times Y, X \vee Y$. Consider the diagram

$$\begin{array}{ccccc}
Z_2 = \Gamma_{p+q+1}(X' \times Y, X' \vee Y) & \xrightarrow{\lambda'} & \Pi_{p+q+1}(X' \times Y, X' \vee Y) & \xrightarrow{\varrho'} & H_{p+q+1}(X' \times Y, X' \vee Y) = Z_{k'} \\
\downarrow \zeta & & \downarrow \eta & & \downarrow \theta \\
Z_2 = \Gamma_{p+q+1}(X \times Y, X \vee Y) & \xrightarrow{\lambda} & \Pi_{p+q+1}(X \times Y, X \vee Y) & \xrightarrow{\varrho} & H_{p+q+1}(X \times Y, X \vee Y) = Z_k
\end{array}$$

where ζ, η, θ are induced by f. Then $\theta\varrho' = \varrho\eta$ and $\eta\lambda' = \lambda\zeta$; λ and λ' are univalent. Moreover $\zeta\Gamma_{p+q+1}(X' \times Y, X' \vee Y) = 0$. For $\Gamma_{p+q+1}(X' \times Y, X' \vee Y)$ is generated by $\eta_{p+1} \cdot \iota_q$, in the usual notation, and $\zeta(\eta_{p+1} \cdot \iota_q) = 2\,\eta_{p+1} \cdot \iota_q = 0$. $\iota_q = 0$.

[17]) This result is to appear in a forthcoming paper.

We now distinguish two cases

Case 1: $k = k'$. Put $\sigma' = \sigma_0 k$, $\tau = \tau_0 k$. Then $H_{p+q+1}(X' \times Y, X' \vee Y)$ is generated by $\bar{\gamma}' = \tau_0(e^{p+1} \times e^q) + (-1)^{p+1}\sigma_0(e^p \times e^{q+1})$ and

$$\theta(\tau_0(e^{p+1} \times e^q) + (-1)^{p+1}\sigma_0(e^p \times e^{q+1})) = \tau_0(e^{p+1} \times e^q) + (-1)^{p+1} 2\sigma_0(e^p \times e^{q+1}), = \bar{\gamma} ,$$

which is a generator of $H_{p+q+1}(X \times Y, X \vee Y)$. Let β' generate

$$\Gamma_{p+q+1}(X' \times Y, X' \vee Y) ,$$

let $\alpha' \in \varrho'^{-1}\bar{\gamma}'$ and let $\alpha \in \varrho^{-1}\bar{\gamma}$. Then we have shewn that $\theta\bar{\gamma}' = \bar{\gamma}$. Thus $\varrho\eta\alpha' = \theta\varrho'\alpha' = \theta\bar{\gamma}' = \bar{\gamma}$ so that $\alpha - \eta\alpha' \in \varrho^{-1}(0) = Z_2$ and $k\alpha = k \cdot \eta\alpha'$, since k is even. Now $k\alpha' = k'\alpha' = 0$ or $\lambda'\beta'$. Since $\eta\lambda'\beta' = \lambda\zeta\beta' = 0$, we have $\eta k\alpha' = 0$, and $k\alpha = 0$. This shows that $\Pi_{p+q+1}(X \times Y, X \vee Y) = Z_2 + Z_k$.

Case 2: $k = 2k'$. Put $\sigma' = \sigma_0 k'$, $\tau = 2\tau_0 k'$. Then $H_{p+q+1}(X' \times Y, X' \vee Y)$ is generated by $\bar{\gamma}' = 2\tau_0(e^{p+1} \times e^q) + (-1)^{p+1}\sigma_0(e^p \times e^{q+1})$ and

$$\theta(2\tau_0(e^{p+1} \times e^q) + (-1)^{p+1}\sigma_0(e^p \times e^{q+1})) = 2\tau_0(e^{p+1} \times e^q) + (-1)^{p+1} 2\sigma_0(e^p \times e^{q+1}) = 2\bar{\gamma} ,$$

where $\bar{\gamma}$ is a generator of $H_{p+q+1}(X \times Y, X \vee Y)$. Define β', α', α as before. Then $\theta\bar{\gamma}' = 2\bar{\gamma}$ so that $\varrho\eta\alpha' = \theta\varrho'\alpha' = \theta\bar{\gamma}' = 2\bar{\gamma} = 2\varrho\alpha$ and $2\alpha - \eta\alpha' \in \varrho^{-1}(0) = Z_2$. Multiplying by k', which is even, we have $k\alpha = \eta k'\alpha'$ and, as before $k'\alpha' = 0$ or $\lambda'\beta'$ so that $\eta k'\alpha' = 0$. We have $k\alpha = 0$, so that, again, $\Pi_{p+q+1}(X \times Y, X \vee Y) = Z_2 + Z_k$.

We now give a different interpretation of an element $\gamma \in \Pi_{p+q+1}(X \times Y, X \vee Y)$ which is mapped by ϱ onto a generator of $H_{p+q+1}(X \times Y, X \vee Y)$. Here we make no assumptions on the parities of σ, τ, but we assume, without real loss of generality, that $p \leqslant q$. We write $k = (\sigma, \tau)$. Consider the exact homotopy sequence

$$\ldots \to \Pi_{p+q}(X \vee S^q) \xrightarrow{i} \Pi_{p+q}(X \vee Y) \xrightarrow{j} \Pi_{p+q}(X \vee Y, X \vee S^q) \xrightarrow{d} \Pi_{p+q-1}(X \vee S^q) .$$

Let $g_r : \Pi_r(I^{q+1}, \dot{I}^{q+1}) \to \Pi_r(X \vee Y, X \vee S^q)$ be the homomorphism induced by the characteristic map for $e^{q+1} \subset Y$. Then $\Pi_{p+q}(X \vee Y, X \vee S^q)$ contains a subgroup (actually a direct summand) generated by the generalized Whitehead product[18] $[g_{q+1}\alpha, \iota_p]$, where α generates $\Pi_{q+1}(I^{q+1}, \dot{I}^{q+1})$ and ι_p generates $\Pi_p(X)$. Moreover $d[g_{q+1}\alpha, \iota_p] = \pm\tau[\iota_p, \iota_q]$, where ι_q generates $\Pi_q(S^q)$. It may be shown that $[g_{q+1}\alpha, \iota_p]$ is of order σ, whereas $d[g_{q+1}\alpha, \iota_p]$ is of order $\sigma_0 = \frac{\sigma}{k}$. Thus $\sigma_0[g_{q+1}\alpha, \iota_p]$ is of order k and is mapped by d to zero. Then if d' is the univalent homomorphism $d' : \Pi_{p+q+1}(X \times Y, X \vee Y) \to \Pi_{p+q}(X \vee Y)$, we may suppose that, for some $\eta \in \Pi_{p+q}(X)$, $\beta \in \Pi_{p+q}(Y)$, $\gamma \in \Pi_{p+q+1}(X \times Y, X \vee Y)$,

$$j(\iota_1\eta + \iota_2\beta + d'\gamma) = \sigma_0[g_{q+1}\alpha, \iota_p] = w, \text{ say.}$$

Clearly $j\iota_1\Pi_{p+q}(X) = 0$, so that $j(\iota_2\beta + d'\gamma) = w$. Consider the diagram

$$\begin{array}{ccc} \Pi_{p+q}(X \vee Y) & \xrightarrow{j} & \Pi_{p+q}(X \vee Y, X \vee S^q) \\ \mu_2 \downarrow \uparrow \iota_2 & & \mu_2^* \downarrow \uparrow \iota_2^* \\ \Pi_{p+q}(Y) & \xrightarrow{j'} & \Pi_{p+q}(Y, S^q), \end{array}$$

where $j, j', \iota_2, \iota_2^*$ are injections and μ_2, μ_2^* are homomorphisms induced by the projection $X \vee Y \to Y$. Then $\mu_2^* w = 0$, so that

$$j'\mu_2(\iota_2\beta + d'\gamma) = \mu_2^* j(\iota_2\beta + d'\gamma) = 0;$$

but $\mu_2 d' = 0$, $\mu_2\iota_2 = 1$, so that $j'\beta = 0$. Thus $j\iota_2\beta = \iota_2^* j'\beta = 0$, and $jd'\gamma = w$.

Now w is of order k; thus $md'\gamma \notin i\Pi_{p+q}(X \vee S^q)$ if $0 < m < k$; it follows immediately, using (4.8), that $m\gamma \notin \lambda\Gamma_{p+q+1}(X \times Y, X \vee Y)$ if $0 < m < k$, so that, as required, γ is mapped by ϱ onto a generator of $H_{p+q+1}(X \times Y, X \vee Y)$.

8. A generalization

We consider in this section a CW-complex L which is the union of a finite number of CW-complexes X_i, $i = 1, \ldots, k$, with a single common point, and we assume for simplicity that[19]

$$\Pi_r(X_i) = 0, r = 0, 1, \ldots, p-1, \tag{8.1}$$

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[18]) It may be shown that $\Pi_{p+q}(X \vee Y, X \vee S^q) \approx \Pi_{p+q}(I^{q+1}, \dot{I}^{q+1}) + \Pi_p(X \vee S^q)$, where the first factor is embedded by g_{p+q} and the second by $\beta \to [g_{q+1}\alpha, \beta]$, $\beta \in \Pi_p(X \vee S^q)$.

We give here an alternative formulation, due to *M. G. Barratt*, of the generalized Whitehead product (originally due to *W. S. Massey*). Let $\varrho \in \Pi_{p+q}(E^{p+1} \vee S^q, S^p \vee S^q)$ be the element whose boundary is $[\iota_p, \iota_q]$. Elements $\xi \in \Pi_{p+1}(Z, Z_0)$, $\eta \in \Pi_q(Z_0)$ determine a class of maps $f : E^{p+1} \vee S^q, S^p \vee S^q \to Z, Z_0$. Then $[\xi, \eta] \in \Pi_{p+q}(Z, Z_0)$ is the image of ϱ under the homomorphism induced by f.

[19]) We adopt the convention that '$\Pi_0(X) = 0$' means 'X is arcwise-connected'.

for all $i = 1, \ldots, k$. We write K for $X_1 \times X_2 \times \ldots \times X_k$, K_{ij} for $X_i \times X_j$, L_{ij} for $X_i \vee X_j$. We identify L with $X_1 \vee X_2 \vee \ldots \vee X_k$ in the natural way and embed K_{ij}, L_{ij} in K, L as subcomplexes. Thus, for example, if $\Phi_{ij} : K_{ij}, L_{ij} \to K, L$ is the embedding map, then

$$\Phi_{ij}(x_i, x_j) = (x_0, \ldots, x_0, x_i, x_0, \ldots, x_0, x_j, x_0, \ldots, x_0), \in K$$

where $x_i \in X_i$, $x_j \in X_j$ and x_0 is the common point of $X_1, X_2, \ldots X_k$. Suppose $p > 1$ and consider the exact sequences

$$\begin{aligned} \wedge^p_{ij} : H_{3p-1}(K_{ij}, L_{ij}) &\xrightarrow{\mu_{ij}} \Gamma_{3p-2}(K_{ij}, L_{ij}) \xrightarrow{\lambda_{ij}} \Pi_{3p-2}(K_{ij}, L_{ij}) \\ &\xrightarrow{\varrho_{ij}} H_{3p-2}(K_{ij}, L_{ij}) \to \ldots \to H_{2p}(K_{ij}, L_{ij}) \to 0 , \end{aligned} \tag{8.2}$$

$$\begin{aligned} \wedge^p : H_{3p-1}(K, L) \xrightarrow{\mu} \Gamma_{3p-2}(K, L) \xrightarrow{\lambda} \Pi_{3p-2}(K, L) \xrightarrow{\varrho} H_{3p-2}(K, L) \to \ldots \\ \to H_{2p}(K, L) \to 0 . \end{aligned} \tag{8.3}$$

Then Φ_{ij} induces a homomorphism of $\wedge^p_{ij}$ into $\wedge^p$ in the sense of [4, Ch. 1]. We call the homomorphism $(\Phi_{ij})_*$ and allow the same symbol to stand for the induced homomorphisms of the constituent groups of $\wedge^p_{ij}$ into those of $\wedge^p$. We prove

Theorem 8.4. $\wedge^p = \sum_{i<j} (\Phi_{ij})_* \wedge^p_{ij}$, *and each* $(\Phi_{ij})_*$ *is univalent.*

We prove first

Lemma 8.5. $H_r(K, L) = \sum_{i<j} (\Phi_{ij})_* H_r(K_{ij}, L_{ij})$ *and each* $(\Phi_{ij})_*$ *is univalent, if* $r < 3p$.

We make the following inductive hypothesis. We assume that, for a particular value of k, and for $r < 3p$,

$$\begin{aligned} H_r(K) \approx \sum_{i<j\leqslant k} \Big(\sum_{\substack{l+m=r \\ l\neq 0, m\neq 0}} H_l(X_i) \otimes H_m(X_j) + \sum_{l+m=r-1} \mathrm{Tor}\,(H_l(X_i), H_m(X_j)) \Big) + H_r(L), \\ = H_r(K, L) + H_r(L) , \end{aligned}$$

where $H_r(L)$ is embedded isomorphically in $H_r(K)$ by injection, and

$$H_r(K_{ij}, L_{ij}), = \sum_{\substack{l+m=r \\ l\neq 0, m\neq 0}} H_l(X_i) \otimes H_m(X_j) + \sum_{l+m=r-1} \mathrm{Tor}\,(H_l(X_i), H_m(X_j)) ,$$

is imbedded isomorphically in $H_r(K, L)$ by injection. This assumption is trivial if $k = 1$, and follows from (2.5) and (2.6) if $k = 2$. For convenience let us actually identify $H_r(K_{ij}, L_{ij})$ with $(\Phi_{ij})_* H_r(K_{ij}, L_{ij})$. Let $\Pi_r(X_{k+1}) = 0$, $r = 0, 1, \ldots, p-1$, and let $K' = K \times X_{k+1}$, $L' = L \vee X_{k+1}$. Then

$$H_r(K') = \sum_{s+t=r} H_s(K) \otimes H_t(X_{k+1}) + \sum_{s+t=r-1} \mathrm{Tor}\,(H_s(K), H_t(X_{k+1})) . \tag{8.6}$$

We may identify $H_r(L)$, $(r > 0)$, and its injection in $H_r(K)$, with $\sum_{i=1}^{k} H_r(X_i)$ and $H_r(L')$ with $\sum_{i=1}^{k+1} H_r(X_i)$. Also, since each X_i, $i = 1, \ldots, k+1$, is connected,

$$H_0(K) = H_0(L) = H_0(X_i) = H_0(L_{ij}) = H_0(K_{ij}) = Z_\infty \; .$$

We apply the inductive hypothesis to (8.6). Then, if $r < 3p$,

$$H_r(K') = \sum_{i<j\leqslant k} \Big(\sum_{\substack{l+m+n=r \\ l\neq 0, m\neq 0}} H_l(X_i) \otimes H_m(X_j) \otimes H_n(X_{k+1}) + \sum_{l+m+n=r-1} \operatorname{Tor}(H_l(X_i), H_m(X_j)) \otimes H_n(X_{k+1}) + \sum_{\substack{l+m+n=r-1 \\ l\neq 0, m\neq 0}} \operatorname{Tor}(H_l(X_i) \otimes H_m(X_j), H_n(X_{k+1})) + \sum_{l+m+n=r-2} \operatorname{Tor}(\operatorname{Tor}[H_l(X_i), H_m(X_j)], H_n(X_{k+1})) \Big) + \sum_{i=1}^{k} \Big(\sum_{\substack{l+m=r \\ l\neq 0}} H_l(X_i) \otimes H_m(X_{k+1}) + \sum_{l+m=r-1} \operatorname{Tor}(H_l(X_i), H_m(X_{k+1})) \Big) + H_r(X_{k+1}) \; . \tag{8.7}$$

This formula reduces to the trivial equality $H_0(K') = H_0(X_{k+1})$ if $r = 0$. Assume $r > 0$.

We now apply the hypothesis (8.1) to the right hand side of (8.7). Since $\Pi_r(X_i) = 0$, $r < p$, $i = 1, \ldots, k+1$, it follows that $H_r(X_i) = 0$, $0 < r < p$. Thus we get non-zero terms from the first direct summand on the right of (8.7) only when $n = 0$. A similar remark applies to the second direct summand. The third and fourth summands provide no non-zero terms. Since $r > 0$, the fifth summand may be written

$$\sum_{i=1}^{k} \Big(\sum_{\substack{l+m=r \\ l\neq 0, m\neq 0}} H_l(X_i) \otimes H_m(X_{k+1}) + H_r(X_i) \Big) \; .$$

This combines with the sixth summand and $H_r(X_{k+1})$ to give

$$\sum_{i=1}^{k} \Big(\sum_{\substack{l+m=r \\ l\neq 0, m\neq 0}} H_l(X_i) \otimes H_m(X_{k+1}) \Big) + \sum_{l+m=r-1} \operatorname{Tor}(H_l(X_i), H_m(X_{k+1})) + H_r(L') \, .$$

All these remarks lead to the conclusion that

$$H_r(K') = \sum_{i<j\leqslant k+1} \Big(\sum_{\substack{l+m=r \\ l\neq 0, m\neq 0}} H_l(X_i) \otimes H_m(X_j) + \sum_{l+m=r-1} \operatorname{Tor}(H_l(X_i), H_m(X_j)) \Big) + H_r(L') \, , = \sum_{i<j\leqslant k+1} H_r(K_{ij}, L_{ij}) + H_r(L') \; ,$$

and $H_r(L')$ is embedded in $H_r(K')$ by (univalent) injection. Since the injection is univalent, the natural homomorphism $H_r(K') \to H_r(K', L')$ is onto $H_r(K', L')$ and its kernel is $H_r(L')$. Thus

$$H_r(K', L') \approx \sum_{i<j\leqslant k+1} H_r(K_{ij}, L_{ij}) \ ,$$

and it is clear that $H_r(K_{ij}, L_{ij})$ is embedded in $H_r(K', L')$ by injection. The full inductive hypothesis is thus verified, and the lemma is proved.

Lemma 8.8. *$\Pi_r(K, L) = \sum_{i<j} (\Phi_{ij})_* \Pi_r(K_{ij}, L_{ij})$ and each $(\Phi_{ij})_*$ is univalent, if $r < 3p - 1$.*

Let $\Psi_{ij} : K, L \to K_{ij}, L_{ij}$ be the obvious projection. Since we are taking K_{ij}, L_{ij} to be embedded in K, L (by Φ_{ij}), we may talk of the iterated projection $\Psi_{h,l}\Psi_{i,j}$. Then $\Psi_{h,l}\Psi_{i,j} K \subset L$ if $(h, l) \neq (i, j)$. It is a consequence of a standard theorem on abelian groups[20] that

$$\Pi_r(K, L) = \sum_{i<j} (\Phi_{ij})_* \Pi_r(K_{ij}, L_{ij}) + R \ ,$$

and each $(\Phi_{ij})_*$ is univalent. The group R is the intersection of the kernels of the $(\Psi_{ij})_*$ and we will show that $R = 0$, if $r < 3p - 1$. In fact we will show that every $\alpha \in \Pi_r(K, L)$ is expressible as

$$\sum_{i<j} \alpha_{ij}, \alpha_{ij} \in (\Phi_{ij})_* \Pi_r(K_{ij}, L_{ij}) \ .$$

To prove this it is convenient to assume, as we may, that $X_i^{p-1} = x_0$, $i = 1, \ldots, k$. Then $L = K^{2p-1} \cup L$, $L_{ij} = K_{ij}^{2p-1} \cup L_{ij}$. Use $(\Phi_{ij})_*$ for the homomorphism

$$(\Phi_{ij})_* : \Pi_r(K_{ij}^n \cup L_{ij}, L_{ij}) \to \Pi_r(K^n \cup L, L)$$

induced by the embedding, $K_{ij}^n \cup L_{ij}, L_{ij} \to K^n \cup L, L$.
We will show that $(\Phi_{ij})_*$ is univalent and

$$\Pi_r(K^n \cup L, L) = \sum_{i<j} (\Phi_{ij})_* \Pi_r(K_{ij}^n \cup L_{ij}) \ , \tag{8.9}$$

if $r < 3p - 1$ and $2p \leqslant n$. The lemma will then be proved by taking $n = 3p - 1$. Now (8.9) holds if $n = 2p$. For let Q be the union of a set of $2p$-elements in $(1 - 1)$ correspondence with the $2p$-cells of K mod L, and having a single point on the boundary of each element in common. Let P be the boundary of Q, let Q_{ij} be the subset of Q corresponding to the $2p$-cells of K_{ij} mod L_{ij} and let P_{ij} be the boundary of Q_{ij}. Since $2p < 3p, Q = \bigcup Q_{ij}$, $P = \bigcup P_{ij}$. Let us use Φ_{ij} for the embedding $Q_{ij}, P_{ij} \to Q, P$, and consider the diagram

[20]) Note that $\Pi_2(K, L), \Pi_2(K_{ij}, L_{ij})$ are abelian because $\Pi_1(L) = \Pi_1(L_{ij}) = 0$.

$$\begin{array}{ccc} \Pi_r(Q_{ij}, P_{ij}) & \rightarrow & \Pi_r(Q, P) \\ g_{ij} \downarrow & (\Phi_{ij})_* & \downarrow g \\ \Pi_r(K^{2p}_{ij} \cup L_{ij}, L_{ij}) & \rightarrow & \Pi_r(K^{2p} \cup L, L) \end{array} ,$$

where g_{ij}, g are induced by the characteristic maps for the $2p$-cells in K_{ij}, K. Then g_{ij}, g are isomorphisms[21], $(\Phi_{ij})_* \, g_{ij} = g(\Phi_{ij})_*$, and certainly

$$\Pi_r(Q, P) = \sum_{i<j} \Pi_r(Q_{ij}, P_{ij}), \text{ since } r < 4p - 2 .$$

(8.9), with $n = 2p$, is now an immediate consequence. Notice that $(\Phi_{ij})_*$ is univalent. Suppose (8.9) true for $n = m < 3p - 1$ and consider the diagram

$$\begin{array}{ccccccc} \Sigma_{r+1}(K^{m+1} \cup L, K^m \cup L) & \xrightarrow{d} & \Sigma_r(K^m \cup L, L) & \xrightarrow{i} & \Sigma_r(K^{m+1} \cup L, L) & \xrightarrow{j} & \Sigma_r(K^{m+1} \cup L, K^m \cup L) \\ \downarrow \Phi_1 & & \downarrow \Phi_2 & & \downarrow \Phi_3 & & \downarrow \Phi_4 \\ \Pi_{r+1}(K^{m+1} \cup L, K^m \cup L) & \xrightarrow{d} & \Pi_r(K^m \cup L, L) & \xrightarrow{i} & \Pi_r(K^{m+1} \cup L, L) & \xrightarrow{j} & \Pi_r(K^{m+1} \cup L, K^m \cup L) \end{array}$$

$$\begin{array}{c} \xrightarrow{d} \Sigma_{r-1}(K^m \cup L, L) \\ \downarrow \Phi_5 \\ \xrightarrow{d} \Pi_{r-1}(K^m \cup L, L) \end{array} .$$

Here $\Sigma_s(K^n \cup L, K^{n-1} \cup L) = \sum_{i<j} \Pi_s(K^n_{ij} \cup L_{ij}, K^{n-1}_{ij} \cup L_{ij})$,

$$\Sigma_s(K^n \cup L, L) = \sum_{i<j} \Pi_s(K^n_{ij} \cup L_{ij}, L_{ij}) ;$$

Φ_1 is defined by $\Phi_1 | \Pi_{r+1}(K^{m+1}_{ij} \cup L_{ij}, K^m_{ij} \cup L_{ij}) = (\Phi_{ij})_*$, Φ_2, Φ_3, Φ_4, Φ_5 are similarly defined, the lower horizontal line is the exact sequence of the triple $(K^{m+1} \cup L, K^m \cup L, L)$ and the upper horizontal line is the direct sum of the exact sequences of the triples $(K^{m+1}_{ij} \cup L_{ij}, K^m_{ij} \cup L_{ij}, L_{ij})$. The upper horizontal line is exact, commutativity holds round each square, Φ_2 and Φ_5 are isomorphisms (onto) by our inductive hypothesis and Φ_1 and Φ_4 are isomorphisms (onto) by an argument similar to that used to prove (8.9) in the case $n = 2p$. Thus, by the 'lemma of five homomorphisms'[22], Φ_3 is an isomorphism onto, and this proves (8.9) in the case $n = m + 1$. Thus (8.9) is proved for $n \leqslant 3p - 1$, and hence, since $r < 3p - 1$, for all $n \geqslant 2p$.

To complete the proof of 8.4, we require

[21]) The Whitehead suspension theorem may be used because $L = K^{2p-1} \cup L$, $L_{ij} = K^{2p-1}_{ij} \cup L_{ij}$.

[22]) Lemma 4.3, p. 16 of [4].

Lemma 8.10. *$\Gamma_r(K, L) = \sum_{i<j} (\Phi_{ij})_* \Gamma_r(K_{ij}, L_{ij})$ and $(\Phi_{ij})_*$ is univalent if $2p \leqslant r < 3p - 1$.*

This is trivial if $r=2p$, since both sides are then zero; so we assume $r > 2p$. We again assume that $X_i^{p-1} = x_0$, $i = 1, \ldots, k$; the lemma is then an easy consequence of the properties of the diagram above with $m = r - 1$, since $\Gamma_r(K, L) = i\Pi_r(K^{r-1} \cup L, L)$, $\sum_{i<j} (\Phi_{ij})_* \Gamma_r(K_{ij}, L_{ij})$ $= \Phi_3 i\, \Sigma_r(K^{r-1} \cup L, L)$ and $(\Phi_{ij})_* = \Phi_3 \mid i\Pi_r(K_{ij}^{r-1} \cup L_{ij}, L_{ij})$.

Thus theorem 8.4 is completely proved. It enables problems about the homotopy groups of unions of spaces to be referred back, under stated conditions, to problems about the homotopy groups of the union of two spaces. In particular, lemma 8.8 may be applied to extending theorem 6.5 in the case $p = q \geqslant 3$, to the union of more than two spaces. Theorem 8.4 is best possible in the sense that, in general, $\Pi_{3p-1}(K, L)$ $= \sum_{i<j} \Pi_{3p-1}(K_{ij}, L_{ij}) + R$, where R is non-zero. For example, if $p = 2$, $K = S_1^2 \times S_2^2 \times S_3^2$, $L = S_1^2 \vee S_2^2 \vee S_3^2$, then $R = Z_\infty + Z_\infty$, corresponding to the triple Whitehead products $([\iota_1, \iota_2], \iota_3)$, $([\iota_2, \iota_3], \iota_1)$ in $\Pi_4(L)$. Also, of course, $H_6(K, L)$ is, in this case, cyclic infinite whereas $H_6(K_{ij}, L_{ij}) = 0$. The univalence of the $(\Phi_{ij})_*$ does not, of course, depend on dimensions nor on special properties of the X_i.

Suppose that $\Pi_r(X_i) = 0$, $r = 1, \ldots, p_i - 1, i = 1, \ldots, k$, where $p_1 \leqslant p_2 \leqslant \ldots \leqslant p_k$. Then it is clear that, by formal changes in the arguments of this section we may show that the sequence beginning $H_{2p_1+p_2-1}(K, L) \ldots$ is the direct sum of the sequences beginning $H_{2p_1+p_2-1}(K_{ij}, L_{ij}) \ldots$ However, it is also clear, modifying slightly the argument of 8.5, that

$$H_r(K, L) = \sum_{i<j} H_r(K_{ij}, L_{ij}) ,$$

if $r < p_1 + p_2 + p_3$. We are thus led to attempt to extend theorem 8.4 back to $H_{p_1+p_2+p_3-1}(K, L)$. We will not prove the full result here but will content ourselves with the following preliminary result.[23]

Theorem 8.11. *$\Pi_n(S^p \times S^q \times S^r, S^p \vee S^q \vee S^r) = \Pi_n(S^p \times S^q, S^p \vee S^q) + \Pi_n(S^p \times S^r, S^p \vee S^r) + \Pi_n(S^q \times S^r, S^q \vee S^r)$, if $n < p + q + r - 1$.*

The asserted equality is, of course, to be understood in the sense of univalent injection.

Now $\Pi_n(S^p \vee S^q \vee S^r) = \Pi_n(S^p) + \Pi_n(S^q) + \Pi_n(S^r) + d\Pi_{n+1}(S^p \times S^q \times S^r, S^p \vee S^q \vee S^r)$, where d is univalent. It will thus be sufficient to show that, if $n < p + q + r - 2$,

[23]) I have not found theorem 8.11 in the literature, but I believe it to be well-known.

$$\Pi_n(S^p \vee S^q \vee S^r) = \Pi_n(S^p) + \Pi_n(S^q) + \Pi_n(S^r) + d\Pi_{n+1}(S^p \times S^q, S^p \vee S^q)$$
$$+ d\Pi_{n+1}(S^p \times S^r, S^p \vee S^r) + d\Pi_{n+1}(S^q \times S^r, S^q \times S^r) \ , \qquad (8.12)$$

where, for example, $d : \Pi_{n+1}(S^p \times S^q, S^p \vee S^q) \to \Pi_n(S^p \vee S^q \vee S^r)$ is the univalent homomorphism consisting of the homotopy boundary followed by the injection $\Pi_n(S^p \vee S^q) \to \Pi_n(S^p \vee S^q \vee S^r)$. Note that (8.12) is true for all n if, to the right hand side, we add the term dR, where

$$\Pi_{n+1}(S^p \times S^q \times S^r, S^p \vee S^q \vee S^r) = \Pi_{n+1}(S^p \times S^q, S^p \vee S^q) + \Pi_{n+1}(S^p \times S^r, S^p \vee S^r)$$
$$+ \Pi_{n+1}(S^q \times S^r, S^q \vee S^r) + R \ . \qquad (8.13)$$

Let us assume, without loss of generality, that $p \leqslant q \leqslant r$. Then

$$\Pi_n(S^p \vee S^q \vee S^r) = \Pi_n(S^p \vee S^q) + \Pi_n(S^r) + d\Pi_{n+1}((S^p \vee S^q) \times S^r, S^p \vee S^q \vee S^r) \ .$$

It is therefore sufficient to show that

$$d\Pi_{n+1}((S^p \vee S^q) \times S^r, S^p \vee S^q \vee S^r) = d\Pi_{n+1}(S^p \times S^r, S^p \vee S^r)$$
$$+ d\Pi_{n+1}(S^q \times S^r, S^q \vee S^r) \ . \qquad (8.14)$$

Now $\Pi_{n+1}((S^p \vee S^q) \times S^r, S^p \vee S^q \vee S^r) = \Pi_{n+1}(S^p \vee S^q \vee S^r \cup e^{p+r} \cup e^{q+r}, S^p \vee S^q \vee S)^r$.

Put $L = S^p \vee S^q \vee S^r$ and consider the sequence

$$\Pi_{n+1}(L \cup e^{p+r}, L) \xrightarrow{i} \Pi_{n+1}(L \cup e^{p+r} \cup e^{q+r}, L) \xrightarrow{j} \Pi_{n+1}(L \cup e^{p+r} \cup e^{q+r}, L \cup e^{p+r}) \ .$$

It follows from Theorem 1 of [1], since $\Pi_s(L, S^p \vee S^r) = 0, s < q$, that the injection

$$\Pi_{n+1}(S^p \times S^r, S^p \vee S^r) \to \Pi_{n+1}(L \cup e^{p+r}, L)$$

is onto, if $n < p + q + r - 2$. From the same theorem, or by the Whitehead suspension theorem, it follows that the injection

$$\Pi_{n+1}(S^q \times S^r, S^q \vee S^r) \to \Pi_{n+1}(L \cup e^{p+r} \cup e^{q+r}, L \cup e^{p+r})$$

is onto, if $n < p + q + r - 2$.

We note from the remark leading to (8.13) that we may easily prove that the right hand side of (8.14) appears as a direct factor in the left hand side. It is thus sufficient to show that, if $x \in d\Pi_{n+1}(L \cup e^{p+r} \cup e^{q+r}, L)$, then there exist $y \in d\Pi_{n+1}(S^p \times S^r, S^p \vee S^r)$, $z \in d\Pi_{n+1}(S^q \times S^r, S^q \vee S^r)$, such that

$$x = y + z$$

We will be more explicit about the relevant injections; let $\iota_{p,r}$ be the injection $\iota_{p,r} : \Pi_n(S^p \vee S^r) \to \Pi_n(L)$ and let $\iota_{q,r}$ be the injection

$\iota_{q,r} : \Pi_n(S^q \vee S^r) \to \Pi_n(L)$. Then we must show that y, z exist such that

$$x = \iota_{p,r} y + \iota_{q,r} z\,. \tag{8.15}$$

Consider the diagram

$$\begin{tikzcd}
\Pi_{n+1}(S^p \times S^r, S^p \vee S^r) \arrow[r, "\lambda"] \arrow[d, "d"'] & \Pi_{n+1}(L \cup e^{p+r}, L) \arrow[r, "i"] \arrow[d, "d"] & \Pi_{n+1}(L \cup e^{p+r} \cup e^{q+r}, L) \arrow[r, "j"] \arrow[dl, "d"] & \Pi_{n+1}(L \cup e^{p+r} \cup e^{q+r}, L \cup e^{p+r}) \arrow[dl, "d"] \\
\Pi_n(S^p \vee S^r) \arrow[r, "\iota_{p,r}"] & \Pi_n(L) \arrow[r, "k"] & \Pi_n(L \cup e^{p+r}) & \\
 & \Pi_n(S^q \vee S^r) \arrow[u, "\iota_{q,r}"'] & & \Pi_{n+1}(S^q \times S^r, S^q \vee S^r) \arrow[ll, "d"'] \arrow[uu, "\mu"']
\end{tikzcd}$$

The homomorphisms marked 'd' are homotopy boundaries, the rest injections, and all commutativity relations hold. Recall that, since $n < p + q + r - 2$, λ and μ are onto. Let $x = dx'$, $x' \in \Pi_{n+1}(L \cup e^{p+r} \cup e^{q+r}, L)$. Then $kx = kdx' = djx' = d\mu z'$, say, $= k\iota_{q,r} dz' = k\iota_{q,r} z$, $z \in d\Pi_{n+1}(S^q \times S^r, S^q \vee S^r)$. Thus $x - \iota_{q,r} z \in k^{-1}(0) = d\Pi_{n+1}(L \cup e^{p+r}, L)$ $= d\lambda\Pi_{n+1}(S^p \times S^r, S^p \vee S^r) = \iota_{p,r} d\Pi_{n+1}(S^p \times S^r, S^p \vee S^r)$, so that $x - \iota_{q,r} z = \iota_{p,r} y$, $y \in d\Pi_{n+1}(S^p \times S^r, S^p \vee S^r)$, and the theorem is proved.

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