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On the immersion of n-manifolds in $(n+1)$-space

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I. Introduction

Let M be a closed, orientable, differentiable n-manifold of class C^1, for which a fixed orientation has been chosen. Suppose that an immersion of M in euclidean space E^{n+1} is given (that is: a map $f: M \to E^{n+1}$ of class C^1 whose Jacobian matrix has rank n at all points). At each point x of M the given orientations of M and E^{n+1} determine a unique direction for the unit normal vector $N(x)$. Considering $N(x)$ as a point of the unit sphere S^n, this defines the *normal map* $N: M \to S^n$. The degree d of this map has been called, by H. Hopf, the *curvatura integra* of the immersion.

We will be principally concerned with the following question, suggested to the author by Hopf[1]). Consider all immersions of a given n-manifold M in E^{n+1}. What are the corresponding values of the degree d?

For n even this question has been answered by Hopf ([2], [4]). In fact the degree d is uniquely determined by the formula

$$d = \tfrac{1}{2}\chi(M) ,$$

where $\chi(M)$ denotes the Euler characteristic.

For n odd Hopf has shown (not published) that the sphere S^n can be immersed in E^{n+1} with arbitrary odd degree.

In fact he showed that, for n odd, if M can be immersed in E^{n+1} with some odd degree (some even degree), then it can be immersed with arbitrary odd degree (arbitrary even degree). New proofs of these results are included in this paper. For a manifold M which is not parallelizable, it is shown that only odd degrees are possible. If M is parallelizable, and certain other conditions are satisfied, then it is shown that only even degrees are possible.

In section III the corresponding question is studied for an imbedding

[1]) This question is also studied in a forthcoming paper by M. Kervaire [5].

of M in E^{n+1} (immersion without self-intersections). Rather strong restrictions on the degree d are obtained. In particular d must be even or odd according as the sum of the Betti numbers of M is congruent to 0 or 2 modulo 4.

In section IV an immersion of the real projective space P^3 in E^4 is constructed. As a consequence it is shown that the sphere S^3 can be immersed in E^4 with arbitrary normal degree d. In conclusion, several unsolved problems are mentioned.

II. The normal degree for an immersion

We first state two lemmas.

(1) *Let $f:\ M \to E^{n+1}$ be an immersion with normal degree d, and let $r: E^{n+1} \to E^{n+1}$ be relection in a hyperplane. Then the immersion rf has degree $(-1)^n d$.*

In fact if $N(x)$ is the original normal vector, then $-rN(x)$ is the new normal vector. (The minus sign occurs because r reverses the orientation of E^{n+1}.) But $r \mid S^n$ has degree -1, and the antipodal map $S^n \to S^n$ has degree $(-1)^{n-1}$.

By the sum $M_1 + M_2$ of two oriented n-manifolds we mean, following Seifert [7], the manifold obtained by removing a small n-cell from each of the two, and then piecing the two manifolds together along the resulting boundaries. A differentiable structure can be constructed for the sum of two differentiable manifolds.

(2) *If M_1 and M_2 can be immersed in E^{n+1} with degrees d_1, d_2 respectively, then $M_1 + M_2$ can be immersed with degree $d_1 + d_2 - 1$.*

Pick any two points $x_1 \in M_1$, $x_2 \in M_2$. Perform an euclidean motion of M_2 so that the unit normal vectors at these two points point towards

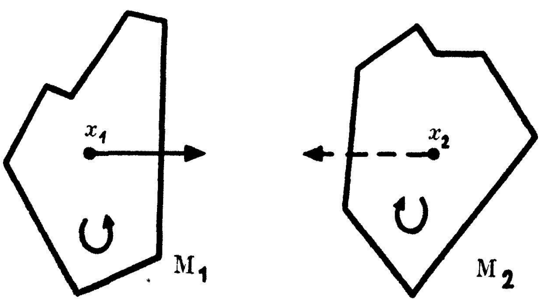


Fig. 1

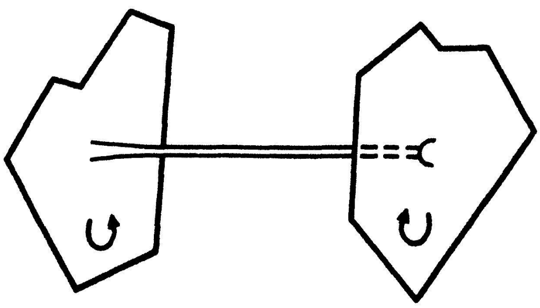


Fig. 2

each other. (See figure 1. The orientation is important.) Now remove a small neighborhood of each of these points and insert a tube, as in figure 2. The result is clearly an immersion of $M_1 + M_2$. The degree

d_3 for this immersion can be computed as follows. To determine d_3 (respectively d_1, d_2) it is sufficient to consider those points x of $M_1 + M_2$ (respectively M_1, M_2) for which the normal vector lies in a neighborhood of some fixed vector N_0. Selecting $N_0 = N(x_1)$ we may assume that the point x_1 makes a contribution of $+1$ to the degree of the map $N : M_1 \to S^n$. (It may be necessary to first make a bulge in M_1 for this to be true.) Now in the immersion of $M_1 + M_2$, the contribution from the point x_1 has disappeared, but the contributions from all other points of M_1 and M_2 remain unchanged. Therefore

$$d_3 = d_1 + d_2 - 1 \ .$$

Propositions (1) and (2) can be used as follows. Let n be odd. According to (1) the sphere S^n can be immersed in E^{n+1} with normal degree -1. Therefore by (2) the sum $S^n + S^n$ can be immersed with degree $(-1) + (-1) - 1 = -3$. By an obvious induction we find that the k-fold sum $S^n + \cdots + S^n$ can be immersed with degree

$$(1 - 2(k-1)) + (-1) - 1 = 1 - 2k \ .$$

Since $S^n + \cdots + S^n$ is homeomorphic to S^n, this implies that S^n can be immersed with any negative, odd degree. Making use of proposition (1) again, this implies Hopf's result:

(3) *For n odd the sphere S^n can be immersed in E^{n+1} with arbitrary odd degree.*

Any n-manifold M is homeomorphic to $M + S^n$. Therefore (2) and (3) imply the following.

(4) **Theorem.** *For n odd, if M can be immersed in E^{n+1} with degree d, then it can also be immersed with any degree d' which is congruent to d modulo 2.*

The possible normal maps N are strongly restricted by the following lemma.

(5) *The normal map $N : M \to S^n$ is covered by a bundle map* (see for example Steenrod [8]) *of the tangent bundle of M into the tangent bundle of S^n.*

In fact for each tangent vector at a point x of M there corresponds the parallel vector with the same length at the point $N(x)$ of S^n.

This proposition can be restated as follows: The tangent bundle of M is the bundle which is induced by the map $N : M \to S^n$. Since a map $M \to S^n$ of degree zero is inessential, and since an inessential map induces a trivial bundle[2]), we have:

[2]) See Hopf [3] and Steenrod [8], 10.3, 11.5.

(6) *If M can be immersed in E^{n+1} with normal degree zero, then M is parallelizable.*

This gives an answer to our problem for any manifold M which is not parallelizable.

(7) **Theorem.** *Let M be an n-manifold which is not parallelizable, n odd. If M can be immersed in E^{n+1} at all, then it can be immersed with arbitrary odd degree, but cannot be immersed with even degree.*

This follows immediately from (4) and (6). As an example let M be the sphere S^n. Steenrod and J. H. C. Whitehead have proved [9] that S^n can only be parallelizable when n has the form $2^k - 1$. Thus S^n *cannot be immersed in E^{n+1} with even degree unless n has the form $2^k - 1$.* It will be shown in section IV that S^3 can actually be immersed in E^4 with even degree. The first unsettled case is therefore S^7.

Another consequence of (5) is the following. *Let n be a dimension for which S^n is parallelizable* (say $n = 7$). *Then every orientable n-manifold immersed in E^{n+1} is parallelizable.* This follows from the fact that any bundle induced from a trivial bundle is trivial.

In order to complete the results of this section, it would be natural to ask the following question. Let n be a dimension for which S^n is not parallelizable, and let M be an n-manifold which is parallelizable. Can M be immersed in E^{n+1} with odd degree?

The answer to this question is negative, at least for the special class of manifolds which are "sphere-like" in the sense of Puppe [6]. Let $\eta : M \to S^n$ denote the (unique up to homotopy) map of degree 1. Puppe calls the manifold M *sphere-like* if for any space X and any essential map $\xi : S^n \to X$, the composition $\xi\eta : M \to X$ is also essential. He proves that any n-manifold which can be imbedded in E^{n+1} is sphere-like. (Immersion in E^{n+1} would not be sufficient: a counter-example is provided by the projective space P^3).

(8) *Let M be an n dimensional manifold which is sphere-like and parallelizable, n being an odd dimension for which S^n is not parallelizable. If M can be immersed in E^{n+1} at all, then it can be immersed with arbitrary even degree, but cannot be immersed with odd degree.*

In fact let X be a classifying space for the general linear group GL_n. Then the tangent bundle of S^n is induced by a essential map $\xi : S^n \to X$. If M could be immersed in E^{n+1} with odd degree, then it could be immersed with degree $+1$. But this would imply, by the definition of sphere-like, that the composition $\xi N : M \to X$ was essential, and hence that M was not parallelizable.

III. The normal degree for an imbedding

Let $\Sigma(M) = \beta_0(M) + \beta_1(M) + \cdots + \beta_n(M)$ denote the sum of the Betti numbers of the manifold M. (Any field may be used as coefficient group.)

(9) **Theorem.** *For any imbedding of M in E^{n+1} the degree d of the normal map satisfies*

$$d \equiv \tfrac{1}{2}\Sigma(M) \qquad (\text{mod. } 2), \qquad |d| \leq \tfrac{1}{2}\Sigma(M) .$$

We may assume that M is oriented so that the normal vectors point away from the bounded component A of $E^{n+1} - M$. Under this additional hypothesis we will prove the stronger inequality

$$2 - \tfrac{1}{2}\Sigma(M) \leq d \leq \tfrac{1}{2}\Sigma(M) .$$

A theorem of Hopf ([4] Satz VI) asserts that under the above assumptions, the normal degree d is equal to the Euler characteristic of the manifold $\bar{A}$ bounded by M. Thus we have

$$d = \chi(\bar{A}) . \tag{a}$$

For any space $\bar{A}$ the definitions $\Sigma = \Sigma\, \beta_i$, $\chi = \Sigma\,(-1)^i\, \beta_i$ and the inequalities $\beta_i \geq 0$, $\beta_0 \geq 1$ imply that

$$\left.\begin{array}{l} \chi(\bar{A}) \equiv \Sigma(\bar{A}) \qquad (\text{mod } 2), \text{ and} \\ 2 - \Sigma(\bar{A}) \leq \chi(\bar{A}) \leq \Sigma(\bar{A}) . \end{array}\right\} \tag{b}$$

Let S^{n+1} be the one point compactification of E^{n+1}, and define $B = S^{n+1} - \bar{A}$. From the Alexander duality theorem it follows that

$$\Sigma(M) = \Sigma(S^{n+1} - M) = \Sigma(A) + \Sigma(B) .$$

On the other hand the Alexander duality theorem applied to $\bar{A}$ implies that

$$\Sigma(\bar{A}) = \Sigma(S^{n+1} - \bar{A}) = \Sigma(B) .$$

Since $\Sigma(A) = \Sigma(\bar{A})$, it follows that

$$\Sigma(\bar{A}) = \tfrac{1}{2}\Sigma(M) . \tag{c}$$

Now combining formulas (a), (b), and (c), this completes the proof of theorem (9).

Slightly stronger restrictions on the degree can be obtained by taking the ring structure of $H^*(M)$ into account. In fact (following Thom [10] chapter V) the inclusion maps $M \to \bar{A}$, $M \to \bar{B}$ determine a homomorphism $H^*(\bar{A}) + H^*(\bar{B}) \to H^*(M)$ which is an isomorphism except in the top and bottom dimensions. Thus for a number d to occur as the normal degree for an imbedding of M, the ring $H^*(M)$ must split into the sum of two subalgebras, which are admissible in the sense of Thom, and one of which has Euler characteristic d.

As an example consider the 3-dimensional torus $S^1 \times S^1 \times S^1$, with Betti numbers $(1, 3, 3, 1)$. According to theorem (9) the four values $d = -2, 0, 2, 4$ would all be possible. However the above technique can be used to show that only degrees 0 and 2 can occur. Both degrees do actually occur.

The 3-manifold $(S^1 \times S^2) + (S^1 \times S^2) + (S^1 \times S^2)$ has the same Betti numbers as the 3-torus, but a different ring structure. This manifold can actually be imbedded with all four degrees $-2, 0, 2, 4$. The usual imbedding of $S^1 \times S^2$ in S^4 splits S^4 into two components with Euler characteristics 0 and 2. Hence making use of lemma (2) (modified so as to apply to imbeddings), the two-fold sum can be imbedded with degrees $-1, 1, 3$; and the three-fold sum with degrees $-2, 0, 2, 4$.

IV. An immersion of P^3 in E^4

The object of this section will be to show that:

(10) *The projective space P^3 can be immersed in E^4.*

Before giving the proof we state two corollaries.

(11) *The sphere S^3 can be immersed in E^4 with even normal degree.*

Let d be the normal degree of the immersion $f : P^3 \to E^4$ of proposition (10). Since the covering map $p : S^3 \to P^3$ has degree 2, the composition $fp : S^3 \to E^4$ is an immersion with normal degree $2d$.

By (4) this implies that S^3 can be immersed in E^4 with arbitrary normal degree. In view of (2) this implies the following.

(12) *If the 3-manifold M can be immersed in E^4, then it can be immersed with arbitrary normal degree.*

The proof of (10) follows. We start from the known fact that the projective plane P^2 can be immersed in E^3. (This immersion is known as Boy's surface. See [1] pg. 280.) The projection $p : S^2 \to P^2$ carries a neighborhood of the equator into a Moebius band B^2. The immersion $g : P^2 \to E^3$ can be chosen so that B^2 is mapped homeomorphically, and so that its image $g(B^2) \subset E^3$ is a Moebius band with a single twist.

Let B^3 denote a smooth neighborhood of P^2, considered as a subspace of P^3. Thus the boundary of B^3 is a 2-sphere, and the space P^3 is obtained from B^3 by adjoining a 3-cell which is matched with B^3 along this boundary 2-sphere.

It will first be shown that the immersion g of P^2 in E^3 can be extended to an immersion of B^3 in E^3. Let $N : S^2 \to S^2$ be the normal map associated with the immersion $gp : S^2 \to E^3$. Define $g' : S^2 \times [-1, 1] \to E^3$ by

$$g'(x, t) = gp(x) + \varepsilon\, tN(x)$$

(where the two terms are added as vectors). For ε sufficiently small, this map g' is clearly an immersion.

Note that B^3 can be obtained from $S^2 \times [-1, 1]$ by identifying the points (x, t) with $(-x, -t)$. Since $g' : S^2 \times [-1, 1] \to E^3$ satisfies the identity

$$g'(-x, -t) = g'(x, t) \ ,$$

it follows that g' gives rise to an immersion of B^3 in E^3.

Now adding a fourth coordinate, we define an immersion g'' of B^3 in the half-space $E^3 \times (-\infty, 0)$ by the formula

$$g''(\pm(x, t)) = (g'(x, t), t^2 - 1) \ .$$

This immersion g'' maps the interior of B^3 in the open half-space, and maps the boundary 2-sphere into $E^3 \times [0]$. The immersion h of this boundary 2-sphere is defined by

$$h(x) = g''(\pm(x, 1)) = (gp(x) + \varepsilon N(x), 0) \ .$$

To complete the proof we must construct an immersion of a 3-cell in the half-space $E^3 \times [0, \infty]$ so that the immersion of its boundary 2-sphere in $E^3 \times [0]$ coincides with the given immersion h.

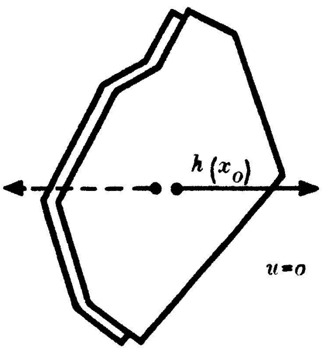
This immersion will be described by giving the intersections of its image with the variable hyperplane $E^3 \times [u]$ as u varies from 0 to 4. Any immersion f' of a 3-manifold M in $E^4 = E^3 \times (-\infty, \infty)$ can be described by this method. In general the intersections $f'(M) \cap (E^3 \times [u])$ are immersions of 2-manifolds which vary continuously with u. However critical points in this picture appear whenever the tangent plane of M coincides with $E^3 \times [u_0]$. In the neighborhood of such a point we may take the coordinates ξ_1, ξ_2, ξ_3 of E^3 as parameters for M, so that the immersion f' is given by the equation

$$u = u_0 + \Sigma\, a_{ij} \xi_i \xi_j + \text{(remainder term)}.$$

Assuming that the matrix (a_{ij}) has rank 3, there are four cases, according as the signature of this matrix is $3, 1, -1$, or -3. Geometrically these four cases are distinguished as follows. As u increases past u_0, the quadratic surface $\Sigma\, a_{ij}\, \xi_i\, \xi_j = u - u_0$ either

(a) changes from the vacuous set to an ellipsoid,

(b) changes from a hyperboloid of two sheets to a hyperboloid of one sheet (see the transition from figure 3 to figure 4),

Fig. 3


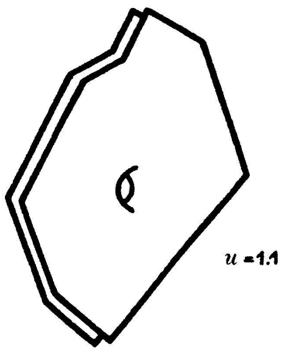


Fig. 4

(c) changes from a hyperboloid of one sheet to a hyperboloid of two sheets, or

(d) changes from an ellipsoid to the vacuous set.

If remainder terms are added, the situation in the large will change, but the local situation will not. These four types of singularities will be used in the following proof.

Let us start out at $u = 0$ with the immersion h of S^2 in $E^3 \times [0]$ which was described above. Each sheet of the original image $g(P^2) \subset E^3$ is sandwiched between two sheets of the image $h(S^2)$. (See figure 3.) Let x_0 be the north pole of S^2. As u varies from 0 to 1 let the surface in $E^3 \times [u]$ change as follows. Two bulges appear in $h(S^2)$ at the points $h(x_0)$, $h(-x_0)$. These expand towards each other and, at $u = 1$, meet each other in a critical point of type (b). Thus as u increases past 1 the figure takes the form illustrated in figure 4. In the large, the surface immersed in $E^3 \times [u]$ is no longer a 2-sphere, but rather a torus.

Now as u varies from 1 to 2 let the tube joining the two sheets of $h(S^2)$ spead out (figure 5) until, for $u = 2$, nothing is left of the surface except a small tubular neighborhood (figure 6) of the Moebius band $g(B^2)$.

As u varies from 2 to 3, smooth this torus out (figure 7) and shrink its inner circle β to a point. Thus for $u = 3$ there is a critical point of type (c). As u increases past 3 the surface, in the large, changes back from a torus to a 2-sphere.

As u varies from 3 to 4, smooth this 2-sphere out, and shrink it to a point. For $u = 4$ there is a critical point of type (d), and the surface disappears.

It is clear that the object described above for $0 \leq u \leq 4$ is an immersion of some 3-manifold in the half-space $E^3 \times (0, \infty)$, and that the

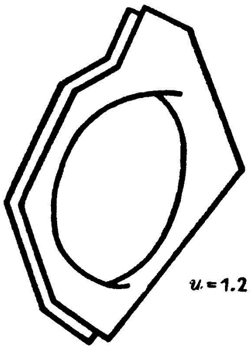


Fig. 5

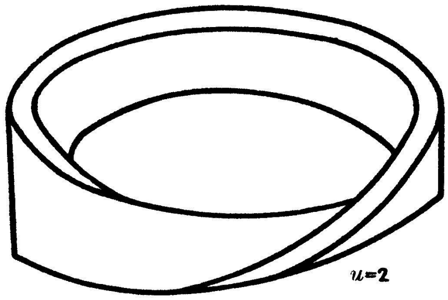


Fig. 6

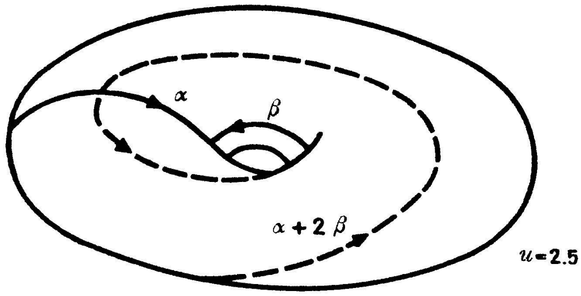


Fig. 7

boundary of this manifold is a 2-sphere, immersed in $E^3 \times [0]$ by the map h. Our problem is to prove that this manifold is actually a 3-cell.

It is not hard to show that the portion of this 3-manifold for which $2\frac{1}{2} \leq u \leq 4$ is a solid torus, whose boundary is the torus of figure 7. Similarly the portion for which $0 \leq u \leq 2\frac{1}{2}$ is a solid torus with an interior 3-cell removed; having as boundaries both the torus $u = 2\frac{1}{2}$ and the 2-sphere $u = 0$.

However the 3-manifolds which can be obtained by matching two solid toruses T_1, T_2 (i. e. manifolds possessing a Heegard diagram of genus 1) have been thoroughly studied (see Seifert [7], § 5). Every such manifold is either S^3, $S^1 \times S^2$, or a Lens space. Thus to prove that such a manifold $T_1 \cup T_2$ is a 3-sphere, it is sufficient to check that its first homology group is trivial. In particular it is sufficient to check that every 1-cycle in the torus $T_1 \cap T_2$ is the sum of a cycle which bounds in T_1 and a cycle which bounds in T_2.

In our case the first homology group of the torus $u = 2\frac{1}{2}$ is generated by the meridian α and the parallel β. But β bounds in the solid torus $2\frac{1}{2} \leq u \leq 4$, and the combination $\alpha + 2\beta$ (obtained by traversing the boundary of the Moebius band) bounds in $0 \leq u \leq 2\frac{1}{2}$. Since β and $\alpha + 2\beta$ generate the homology group of the torus, this completes the proof.

In conclusion we summarize several problems which have been suggested by the results of this paper.

(a) For what n can the projective space P^n be immersed in E^{n+1}? From arguments concerning the Stiefel-Whitney classes, one can show [3]) that n must have the form $2^k - 1$ or $2^k - 2$. For n odd it follows from (7) that if P^n can be immersed in E^{n+1} then S^n must be parallelizable (compare the proof of (11)). Since the immersion is known to be possible for $n = 0, 1, 2, 3$, the first unsolved cases are for dimensions $6, 7$.

(b) For what n can S^n be immersed in E^{n+1} so that the degree d is even? This is known to be possible for $n = 1, 3$. The sphere S^n must be parallelizable.

(c) Let n be a dimension for which S^n is not parallelizable. Can some parallelizable n-manifold be immersed in E^{n+1} with odd degree? Can some (necessarily parallelizable) n-manifold be immersed in E^{n+1} both with odd and with even degree?

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