On the chain algebra of a loop space.

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1. Introduction

An important concept in homotopy theory is that of the loop space of a given space. Given a CW-complex K, James has described in [4] a reduced product complex K_∞ which has the singular homotopy type of the space of loops on the suspension of K; and Toda has also introduced a standard path space (in [9]), performing essentially the same function[1]). In this paper, we consider the loop space of a CW-complex K which need not be a suspension but such that K^1 is a single point, the base-point[2]). We do not construct a combinatorial equivalent of ΩK, the loop space, but instead obtain a chain-equivalent of the cubical chain group of ΩK. Our method lends itself readily to the computation of the homology groups of ΩK.

There is a fibre-space (LK, p, K), where LK is the space of paths on K terminating in the base-point and p associates with every path its initial point. Then ΩK is the fibre. We will in fact construct a system of chain groups and maps equivalent to that given by the fibre-space.

In this paper we adopt J. C. Moore's definition of a path in a space X. In this definition a path is a pair (f, r) where r is a non-negative real number and f is a map of the closed interval $[0, r]$ into X. Paths (f, r), (g, s) such that $f(r) = g(0)$ are added by the rule $(f, r) + (g, s) = (h, r + s)$, where

$$h(t) = f(t) \,, \qquad 0 \leqslant t \leqslant r \,,$$
$$h(t) = g(t - r) \,, \quad r \leqslant t \leqslant r + s \,.$$

Let X^I be the space of maps of the unit interval I into X and let R be the set of non-negative real numbers with its usual topology. A function

1) We understand that J. C. Milnor has described a construction replacing the space of loops on a suitably restricted complex by an equivalent topological group.

2) This restriction could be avoided at the cost of an increase in complication in the proofs of our results (and a small modification in some statements). However, the restriction is not so serious in practice, since, for any CW-complex K, the universal cover of K is of the homotopy type of a CW-complex of the given kind.

$h : EX \to X^I \times R$, where EX is the set of paths on X, is given by $h(f, r) = (f', r)$ where $f'(t) = f(rt)$, $0 \leqslant t \leqslant 1$. Then EX is topologized by requiring h to be a homeomorphism onto its image. Let

$$\varrho_t : X^I \times R \to X^I \times R$$

be the deformation given by $\varrho_t(f, r) = (f, r(1-t)+t)$. Let LX, ΩX be the subsets of EX consisting of paths (f, r) such that $f(r) = x_*$, $f(0) = f(r) = x_*$ respectively, where x_* is the base-point in X. Then LX, ΩX are topologized as subsets of EX. Let $L'(X)$, $\Omega'(X)$ be the subspaces of X^I corresponding to LX, ΩX in the classical definition. Then $\varrho_1 h(LX) = L'(X) \times 1$, $\varrho_1 h(\Omega X) = \Omega'(X) \times 1$ and ϱ_t respects the subspaces $h(LX)$, $h(\Omega X)$. This shows that $LX \simeq L'(X)$, which is contractible, and $\Omega X \simeq \Omega'(X)$. Moreover a homotopy equivalence

$$(LX, \Omega X) \simeq (L'(X), \Omega'(X))$$

is given by $g(f, r) = f'$ where $f'(t) = f(rt)$.

The advantage of Moore's definition is that the pairing of LX and ΩX to LX, by composition of paths, is associative and ΩX possesses a unit. The chain groups $C_*(LX)$, $C_*(\Omega X)$ inherit these properties and the algebraical analogue we construct when X is a CW-complex will reproduce the multiplicative features of the chain groups of the fibre-space. In particular, we define in section 2 the notion of a chain algebra [3]) $A(K)$ which describes the additive and multiplicative structure of $C_*(\Omega K)$.

In section 2 we state and prove the main theorem. In section 3 we prove that our constructions behave properly with respect to maps (not necessarily cellular) of CW-complexes. In section 4 we consider the problem of the relation of $A(K_1 \times K_2)$ to $A(K_1)$ and $A(K_2)$. A generalization of Samelson's result (see [8]) on the relation between Whitehead and Pontryagin products is obtained by considering products of arbitrarily many spheres. We also study a product whose role in homotopy groups is closely related to that of the torsion product in homology groups and obtain an analogue of Samelson's result for this product.

It should be noted that the mapping $\Psi : \Omega(X_1 \times X_2) \to \Omega X_1 \times \Omega X_2$, given by $\Psi l = (p_1 l, p_2 l)$ where $p_i : X_1 \times X_2 \to X_i$, $i = 1, 2$, is the projection, is not a homeomorphism in Moore's definition. However it follows from the commutativity of the diagram

[3]) This will differ from a DGA-algebra over the integers, in the sense of Cartan ([2]), in not requiring that multiplication be anti-commutative.

$$\begin{array}{ccc} \Omega(X_1 \times X_2) & \xrightarrow{\Psi} & \Omega X_1 \times \Omega X_2 \\ \downarrow g & & \downarrow g_1 \times g_2 \\ \Omega'(X_1 \times X_2) & \xrightarrow{\Psi'} & \Omega'(X_1) \times \Omega'(X_2) \end{array}$$

that Ψ is a homotopy equivalence.

2. Chain-algebras and the main theorem

Let A be a differential graded free abelian group, $A = \sum_n A^n$ such that $A^n = 0$, $n < 0$, and $dA^n \subseteq A^{n-1}$. Then A will be called a chain algebra if a product is defined in A such that

(i) A is a ring with unit element;

(ii) $A^p A^q \subseteq A^{p+q}$;

(iii) $d(xy) = (dx)y + (-1)^p x(dy), \quad x \in A^p$.

We write 1 for the unit element; condition (ii) implies that $1 \in A^0$. A function φ from the chain algebra A to the chain algebra A' will be called a map if it is a chain mapping and a ring homomorphism [4]). An augmentation $\alpha : A \to A$ is a map whose image is the ring generated by 1. A map φ of augmented chain algebras is required to commute with α. Henceforth it will be understood that a chain algebra is augmented. The homology group $H_*(A)$ is an augmented graded ring with unit element and a map $\varphi : A \to A'$ induces a homomorphism

$$\varphi_* : H_*(A) \to H_*(A') .$$

Let $Q(\Omega K)$ be the group generated by the singular cubes of ΩK. Then the multiplication in ΩK induces a ring structure in $Q(\Omega K)$ in the usual way. Moreover the subgroup $D(\Omega K)$ generated by the degenerate singular cubes of ΩK (with respect to any co-ordinate) is an ideal in $Q(\Omega K)$. Let $C_*(\Omega K)$ be the quotient ring $Q(\Omega K)/D(\Omega K)$. Then $C_*(\Omega K)$ is a chain algebra with respect to the boundary operator induced by that in $Q(\Omega K)$; the unit element is the 0-cube at the unit element of ΩK and $C_*(\Omega K)$ is augmented by requiring α to be 1 on every 0-cube. The homology ring of $C_*(\Omega K)$ is the (singular) Pontryagin homology ring of ΩK. Our object is to use the structure of K as a CW-complex to construct a chain algebra A and a map $\theta : A \to C_*(\Omega K)$ such that θ_* is an isomorph-

[4]) We require a ring-homomorphism to have the property $\varphi(1) = 1$.

ism. With this end in view we write A' for $C_*(\Omega K)$. We recall that K is being restricted to having one 0-cell (the base-point) and no 1-cells.

Let $\{e_i^n\}$, $n = 0, 2, 3, \ldots$, $i \in$ indexing set T_n, be the cells of K and to each e_i^n except the vertex choose a generator $a_i = a_i^{n-1}$ of dimension $(n - 1)$. Let $A = A(K)$ be the ring with unit element freely generated by the elements a_i, and augmented by $\alpha(1) = 1$, $\alpha(a_i) = 0$, all i. Then A, provided with a suitable differential, will turn out to be the appropriate chain algebra.

Let LK be the space of paths on K terminating at the base-point and let $p : LK \to K$ associate with every path its initial point. Then $(LK, p; K)$ is a fibre-space with ΩK as fibre. Let $C_*(LK)$ be the group generated by the non-degenerate singular cubes of LK whose vertices lie in ΩK. Then $C_*(LK)$, given a graduation, differential and augmentation in the usual way, is the singular chain group of LK, which is, of course, acyclic. The pairing $LK \times \Omega K \to LK$, given by composition of paths, induces a pairing $(C_*(LK) \times C_*(\Omega K) \to C_*(LK)$ which is associative with a unit [in $C_*(\Omega K)$]. $C_*(LK)$ contains $C_*(\Omega K)$ and the pairing, restricted to $C_*(\Omega K) \times C_*(\Omega K)$, induces the ring structure in $C_*(\Omega K)$.

Let $C_*(K)$ be the singular chain group of K generated by the non-degenerate cubes of K all of whose vertices are at the base-point. Then the projection $p : LK \to K$ induces a chain mapping[5] $p : C_*(LK) \to C_*(K)$. We proceed to construct a system of chain groups and maps equivalent to that given by the fibre-space.

To this end, we introduce a free graded abelian group $B = B(K)$, freely generated by elements $b_i = b_i^n$ in $(1 - 1)$ dimension-preserving correspondence with the cells of K. The element b^0 will be written 1. B is augmented by $\alpha(1) = 1$, $\alpha(b_i^n) = 0$, $n > 0$. Then B is intended to play the role of $C_*(K)$; the latter will therefore be called B'. Define $C = C(K)$ as the tensor product $B \otimes A$, graded and augmented by the usual rules. Then A, B may be embedded in C by identifying y with $1 \otimes y$, x with $x \otimes 1$, $y \in A$, $x \in B$. There is a pairing $C \times A \to C$ given by $(x \otimes y, y') \to x \otimes yy'$; restricted to $A \times A$, this pairing induces the multiplication in A. It is clearly legitimate to write a typical generator of C as xy; this will be done when convenient. A projection[6] $\pi : C \to B$ is given by $\pi(xy) = \alpha(y)x$. Since C is to play the role of $C_*(LK)$, the latter will be called C'. We may now state the main theorem.

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[5]) Where no confusion will arise, we will use the same symbol for a map and the induced chain mapping.

[6]) We may regard the augmentation of an element in A, B or C as an ordinary integer.

Theorem 2.1. *Differentials $d : C, A \to C, A$, $\bar{d} : B \to B$, and chain maps $\theta : C, A \to C', A'$, $\bar{\theta} : B \to B'$ may be defined such that*

(i) *A is a chain algebra with respect to $d \mid A$;*

(ii) *$\theta \mid A$ is a map of chain algebras and θ is product-preserving*[7]*;*

(iii) *$\bar{\theta}\pi = p\theta$, $\pi d = \bar{d}\pi$;*

(iv) $\theta_* : H_*(A) \cong H_*(A') = H_*(\Omega K)$.
$\bar{\theta}_* : H_*(B) \cong H_*(B') = H_*(K)$.
$\theta_* : H_*(C) \cong H_*(C') = H_*(LK)$.

Notice that since π maps C onto B, $\bar{d}$ and $\bar{\theta}$ are determined by d and θ.

The differential d and the map θ will be defined inductively on the sections of K. Let K^n be the n-section of K and let ${}^nA$, ${}^nB$, ${}^nC$ be $A(K^n)$, $B(K^n)$, $C(K^n)$ respectively; we regard them as embedded in A, B, C. Similarly we define ${}^nA'$, ${}^nB'$, ${}^nC'$ and embed them in A', B', C'.

For $n = 1$, define $d(1) = 0$, $\theta(1) = 1$; the theorem is trivially verified. Suppose now that d and θ have been determined on ${}^nC$, ${}^nA$ so that the theorem is verified. We proceed to determine d and θ on ${}^{n+1}C$, ${}^{n+1}A$. To determine d on ${}^{n+1}A$ it is sufficient to determine it on the generators. On the generators of dimension $<n$ we determine it by the embeddings ${}^nA \subseteq {}^{n+1}A$, ${}^nA' \subseteq {}^{n+1}A'$. Let a be a generator of dimension n, corresponding to a cell e^{n+1} in K^{n+1}. Let $f : E^{n+1}, S^n \to K^{n+1}, K^n$ be the characteristic map for this cell, inducing $f' : LS^n, \Omega S^n \to LK^n, \Omega K^n$, $f'' : LE^{n+1}, \Omega E^{n+1} \to LK^{n+1}, \Omega K^{n+1}$; and let $\beta \in H_{n-1}(\Omega S^n)$, with $\alpha(\beta) = 0$ if $n = 1$, be such that the suspension of β generates[8] $H_n(S^n)$. Choose an $(n-1)$ cycle z in ${}^nA$ such that $\theta_*\{z\} = f'_*\beta$ – this is possible by the inductive hypothesis – and define $da = z$. Then $d^2 = 0$ on all cells and, hence, by the product rule, d^2 is zero on ${}^{n+1}A$. If $n = 1$, we must take $da = 0$, since $\alpha(\beta) = 0$, so that α is obviously an augmentation of A with respect to the differential being defined on A.

We next define a retraction $s : {}^{n+1}C \to {}^{n+1}C$, raising dimension by 1, by

$$(R1)\ s(1) = 0, \quad s a_i^{r-1} = b_i^r, \quad s b_i^r = 0, \quad r > 1 ,$$
$$(R2)\ s(xy) = (sx)y + (\alpha x)sy, \quad x \in {}^{n+1}C, \quad y \in {}^{n+1}A$$

and extend the differential to a differential d on ${}^{n+1}C$ by defining[9]

$$(D1)\ db_i^r = (1 - sd)a_i^{r-1}, \quad r > 1 ,$$
$$(D2)\ d(xy) = (dx)y + (-1)^p x dy, \quad x \in {}^{n+1}C^p, \quad y \in {}^{n+1}A .$$

[7]) In the sense of the pairings $C \times A \to C$, $C' \times A' \to C'$.

[8]) If $n > 1$, β generates $H_{n-1}(\Omega S^n)$.

[9]) Notice that the chain group B has the differential $\bar{d}$. B is only embedded in C as a subgroup.

Then s is clearly consistent with the two distributive laws; it is also consistent with the associative law of multiplication since

$$s(x(yz)) = (sx)(yz) + (\alpha x)s(yz) = (sx)(yz) + \alpha(x)(sy)z + \alpha(x)\alpha(y)sz\,,$$

while

$$s((xy)z) = (s(xy))z + \alpha(xy)sz = (sx)yz + \alpha(x)(sy)z + \alpha(x)\alpha(y)sz\,.$$

Similarly d is consistent with the two distributive laws and the associative law of multiplication.

We now prove

Lemma 2.1. *For* $x \in {}^{n+1}C$, $(ds + sd)x = (1 - \alpha)x$.

If $x = 1$, this is trivial. Thus it holds for $x \in {}^{n+1}C^0$. Now let $x = a$, a generator of ${}^{n+1}A$ with $sa = b$. Then $\alpha(a) = 0$ and $(ds + sd)(a) = db + sda = a$ by $(D1)$. Next let $x = b$, a generator of ${}^{n+1}B$ with $sa = b$, then $\alpha(b) = 0$ and

$$(ds + sd)b = sdb = s(1 - sd)a = sa - s^2da\,.$$

Now by $(R1)$ s^2 is zero on the generators of ${}^{n+1}B$ and of ${}^{n+1}A$; thus by $(R2)$ s^2 is zero on ${}^{n+1}C$. It follows that $(ds + sd)b = sa = b$, so that the lemma is verified on the generators of ${}^{n+1}B$ and of ${}^{n+1}A$.

Now suppose that $x \in {}^{n+1}C^p$, $y \in {}^{n+1}A$ and the lemma is verified for x and y. Then, using $(R2)$ and $(D2)$ we have

$$\begin{aligned}(ds + sd)(xy) &= d((sx)y + (\alpha x)sy) + s((dx)y + (-1)^p xdy)\\ &= (dsx)y + (-1)^{p+1}(sx)(dy) + (\alpha x)dsy\\ &\quad + (sdx)y + (-1)^p(sx)(dy) + (-1)^p(\alpha x)sdy\\ &= (dsx + sdx)y + (\alpha x)(dsy + (-1)^p sdy)\,.\end{aligned}$$

Now if $p > 0$, $\alpha x = 0$ and $(ds + sd)(xy) = xy = (1 - \alpha)(xy)$. If $p = 0$, then

$$(ds + sd)(xy) = xy - \alpha x\cdot y + \alpha x(y - \alpha y) = xy - \alpha x\cdot\alpha y = (1 - \alpha)xy.$$

Thus the lemma is completely established.

Lemma 2.2. *d is a differential on* ${}^{n+1}C$.

The only assertion to be proved is that $d^2 = 0$. This certainly holds on ${}^{n+1}A$ and so, in the light of $(D2)$ it is sufficient to verify it on a generator of ${}^{n+1}B$. Let b be a generator with $sa = b$. Then $d^2b = d(1 - sd)a = (d - dsd)a = (1 - ds)da$. Now $(ds + sd)da = (1 - \alpha)da$. Thus $dsda = da$ since $d^2a = 0$, $\alpha da = 0$. This implies $d^2b = 0$ and hence the lemma.

Lemma 2.3. *${}^{n+1}C$ is acyclic.*

For, by lemma 2.1, s is a chain-homotopy between α and the identity.

Lemma 2.4. *The kernel of π, restricted to ${}^{n+1}C$, is stable under d.*

For an arbitrary element of ${}^{n+1}C$ is expressible as $x_0 \otimes 1 + \sum_{i>0} x_i \otimes y_i$, where $x_i \in {}^{n+1}B$ and $y_i \in {}^{n+1}A^{n_i}$, $n_i > 0$. The π-image of this is x_0, so that the kernel of π, restricted to ${}^{n+1}C$, consists of elements of the form

$$\sum_{i>0} x_i \otimes y_i \ , \qquad \text{or} \qquad \sum_{i>0} x_i y_i \ .$$

The set of such expressions is obviously stable under d since $d({}^{n+1}A^1) = 0$.

It follows that d induces a differential $\bar{d}$ on ${}^{n+1}B$; it is given by

$$\bar{d}b = -\pi s d a \ .$$

Notice also that the definitions of s and d respect the embedding of ${}^{n}C$, ${}^{n}A$ in ${}^{n+1}C$, ${}^{n+1}A$.

We next define θ; we recall that θ is to be a product-preserving map ${}^{n+1}C$, ${}^{n+1}A \to C_*(LK^{n+1})$, $C_*(\Omega K^{n+1})$. It is sufficient to define θ on the generators of ${}^{n+1}B$, ${}^{n+1}A$ and, as above, we determine it on the generators of ${}^{n+1}B$ of dimension $< n+1$ and on those of ${}^{n+1}A$ of dimension $< n$ by means of the embeddings ${}^{n}C$, ${}^{n}A \subseteq {}^{n+1}C$, ${}^{n+1}A$; $C_*(LK^n)$, $C_*(\Omega K^n) \subseteq C_*(LK^{n+1})$, $C_*(\Omega K^{n+1})$. We conserve the notation of this section and let $i: LS^n, \Omega S^n \to LE^{n+1}, \Omega E^{n+1}$, $j: LK^n, \ \Omega K^n \to LK^{n+1}, \Omega K^{n+1}$ be injections; then $jf' = f''i$ and $\theta = j\theta$ on ${}^{n}C$. Let ζ be a cycle in the class β and let $i\zeta = d\eta$, $\eta \in C_n(\Omega E^{n+1})$. Now $\theta z - f'\zeta = dx'$, $x' \in C_n(\Omega K^n)$. We define[10] $\theta a = jx' + f''\eta$. Then

$$d\theta a = djx' + df''\eta = j\theta z - jf'\zeta + f''i\zeta = j\theta z = \theta d a \ .$$

Now let b, as before, be the generator of B corresponding to e^{n+1} (and hence to a above). Since LS^n is acyclic, $\zeta = d\xi$, $\xi \in C_n(LS^n)$. Moreover, $p\xi$ is an n-cycle of S^n whose class generates $H_n(S^n)$ – by the definition of β. Since LE^{n+1} is acyclic and $i\xi - \eta$ is a cycle of LE^{n+1}, it follows that $i\xi - \eta = d\varkappa$, $\varkappa \in C_{n+1}(LE^{n+1})$. Moreover $p\varkappa$ is an $(n+1)$-relative cycle of E^{n+1} mod S^n whose class generates $H_{n+1}(E^{n+1}, S^n)$ – in fact, under $d: H_{n+1}(E^{n+1}, S^n) \to H_n(S^n)$, we have $d\{p\varkappa\} = \{p\xi\} = S\beta$. We now proceed to define θb. We have

$$d(f'\xi - \theta s z + x') = f'\zeta - \theta z + \theta z - f'\zeta = 0 \ ,$$

since $\alpha z = 0$, $dz = 0$. Thus $f'\xi - \theta s z + x'$ is a cycle in LK^n and so

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[10]) If $n = 1$, then $x' = 0$ and $\theta a = f''\eta$.

$f'\xi - \theta sz + x' = dx''$, $x'' \in C_{n+1}(LK^n)$. We define[11] $\theta b = jx'' - f''\varkappa$. Then $\theta db = \theta(1 - sd)a = jx' + f''\eta - \theta sz$, and $d\theta b = djx'' - df''\varkappa = jf'\xi - \theta sz + jx' - f''i\xi + f''\eta$, so that $\theta db = d\theta b$. Thus θ is defined on ${}^{n+1}C$.

We next show that a map $\bar\theta : {}^{n+1}B \to {}^{n+1}B'$ is defined by $\bar\theta\pi = p\theta$; it is sufficient to show that $\bar\theta$ is single-valued. As above, let $\Sigma x_i y_i$ be a typical element of the kernel of π, $x_i \in {}^{n+1}B$, $y_i \in {}^{n+1}A^{n_i}$, $n_i > 0$. Then $\theta(x_i y_i) = \theta x_i \theta y_i$; but $\theta y_i \in C_{n_i}(\Omega K^{n+1})$ so that $p\theta(x_i y_i)$ is a sum of degenerate cubes and so is zero in $C_*(K)$. Thus $p\theta$ is zero on the kernel of π so that $\bar\theta$ is single-valued.

The inductive definition of d and θ will be established when we have shown that

$$\theta_* : H_*({}^{n+1}A) \cong H_*(\Omega K^{n+1}) \tag{2.1}$$

$$\bar\theta_* : H_*({}^{n+1}B) \cong H_*(K^{n+1}) \tag{2.2}$$

$$\theta_* : H_*({}^{n+1}C) \cong H_*(LK^{n+1}) \tag{2.3}$$

(2.3) is trivial since ${}^{n+1}C$, LK^{n+1} are acyclic and $\theta(1) = 1$. To prove (2.2), observe that $\bar\theta b = p\theta b = pjx'' - pf''\varkappa = pjx'' - fp\varkappa$. Thus $\bar\theta b$ is a relative cycle of K^{n+1} mod K^n whose class generates

$$H_{n+1}(K^n \cup e^{n+1}, K^n)\ .$$

Thus $\bar\theta_* : H_{n+1}({}^{n+1}B, {}^nB) \cong H_{n+1}(K^{n+1}, K^n)$ and (2.2) follows from the inductive hypothesis and the 5-lemma.

To establish (2.1) we introduce a filtration into ${}^{n+1}C$. Then θ will be a filtration-preserving map from ${}^{n+1}C$ to $C_*(LK^{n+1})$, filtered by the Serre filtration, and we will be able to apply a theorem due to J. C. Moore (see [6]) which asserts that, since the first terms of the spectral sequence are well behaved[12], and since the map induces isomorphisms of the homology groups of the fibre-spaces and of the bases, it must therefore induce isomorphisms of the homology groups of the fibres. To avoid an undue proliferation of superscripts and subscripts, we will permit ourselves in this part of the argument to write A, B, C for ${}^{n+1}A$, ${}^{n+1}B$, ${}^{n+1}C$.

We filter C by putting $C_p = \sum_{q \leqslant p} B^q \otimes A$; equivalently if $x \in B^p$, $y \in A$, then $w(xy) = p$. Moreover if b is a q-dimensional generator of B and $y \in A$ then $d(by) = (db)y + (-1)^q b dy = ay - (sz)y + (-1)^q b dy$

[11]) If $n = 1$, then $x'' = 0$ and $\theta b = -f''\varkappa$. Note that, in defining θa, θb, we have used $\zeta, \eta, \xi, \varkappa$ for fixed chains of standard spaces and x', x'' depend on f.

[12]) We make the notion of 'good behaviour' precise in our application below.

and so clearly belongs to C_q. Thus $dC_p \subseteq C_p$ and C is a differential filtered group. Also $\theta(by) = \theta b \cdot \theta y$ and $\theta y \in C(\Omega K^{n+1})$. Thus $p\theta(by)$ is a sum of cubes only depending on their first q co-ordinates. It follows that $\theta(by) \in C'_q$, so that θ respects filtration. Let $E_r^{p,q}$, $E_r'^{p,q}$ be the terms of the spectral sequences associated with C, C' so that θ induces $\theta_* : E_r^{p,q} \to E_r'^{p,q}$.

Define $\psi : B^p \otimes A^q \to E_0^{p,q}$ by $\psi(x \otimes y) = \{xy\}$. Then ψ is an isomorphism and $\psi d_F = d_0 \psi$ where $d_F(x \otimes y) = (-1)^p x \otimes dy$. Thus the induced map $\psi_* : B^p \otimes H_q(A) \to E_1^{p,q}$ is an isomorphism. Define $d_B : B^p \otimes H_q(A) \to B^{p-1} \otimes H_q(A)$ by $d_B(x \otimes \{y\}) = \bar{d}x \otimes \{y\}$. We will show that $\psi_* d_B = d_1 \psi_*$.

Now $d_1 \psi_*(x \otimes \{y\}) = d_1\{xy\} = \{(dx)y\}$, while $\psi_* d_B(x \otimes \{y\}) = \psi_*(\bar{d}x \otimes \{y\}) = \{(\bar{d}x)y\}$. Suppose $x \in B^p$; then $dx = x_0 + \sum_{i>0} x_i y_i$, $y_i \in A^{n_i}$, $x_i \in B^{p-1-n_i}$, where $n_i > 0$ if $i > 0$, and $\bar{d}x = x_0$. Thus $(\bar{d}x)y - (dx)y = \sum_{i>0} x_i y_i y \in C_{p-2}$, whence $\{(dx)y\} = \{(\bar{d}x)y\}$. It follows that ψ_* induces an isomorphism $\psi_{**} : H_p(B ; H_q(A)) \cong E_2^{p,q}$.

Let φ be the map $E_0'^{p,q} \to B'^p \otimes A'^q$ introduced by Serre. Then since K is simply-connected we know that φ induces isomorphisms

$$\varphi_* : E_1'^{p,q} \cong B'^p \otimes H_q(A') \ , \qquad \varphi_{**} : E_2'^{p,q} \cong H_p(B' ; H_q(A')) \ .$$

Consider the diagram

$$\begin{tikzcd} B^p \otimes H_q(A) \arrow[r, "\theta"] \arrow[d, "\psi_*"] & B'^p \otimes H_q(A') \\ E_1^{p,q} \arrow[r, "\theta_*"] & E_1'^{p,q} \arrow[u, "\varphi_*"] \end{tikzcd}$$

where $\Theta(y \otimes \{x\}) = \bar{\theta} y \otimes \{\theta x\}$. Then $\Theta = \varphi_* \theta_* \psi_*$. For

$$\theta_* \psi_*(y \otimes \{x\}) = \theta_*\{yx\} = \{\theta y x\} \ .$$

Now if u is a p-cube of LK^{n+1}, v a q-cube of ΩK^{n+1}, then $\varphi(uv) = pu \otimes v$. Thus $\varphi\theta(yx) = \varphi(\theta y \cdot \theta x) = p\theta y \otimes \theta x = \bar{\theta} y \otimes \theta x$ and so $\varphi_*\{\theta y x\} = \bar{\theta} y \otimes \{\theta x\} = \Theta(y \otimes \{x\})$.

We have now verified the conditions of validity of Moore's theorem [13]). The proof of this theorem sets up and filters the chain mapping-cylinder of $\theta : C \to C'$. It then follows from the diagram above that the first terms of the spectral sequence of this filtration also are properly related to the appropriate tensor products, and then an inductive argument

[13]) Théorème B, p. 3–04, of [6]. The fact that ψ_* goes in the opposite direction in the statement of the theorem is, of course, of no consequence.

shows that the spectral sequence is trivial. This leads immediately to the conclusion that

$$\theta_* : H_q(A) \cong H_q(A') \ .$$

The proof of Theorem 2.1 is now practically complete. We have shown that differentials d, $\bar{d}$ and maps θ, $\bar{\theta}$ may be defined verifying (i), (ii) and (iii) and such that

$$\theta_* : H_*({}^nA) \cong H_*(\Omega K^n)$$
$$\bar{\theta}_* : H_*({}^nB) \cong H_*(K^n)$$
$$\theta_* : H_*({}^nC) \cong H_*(LK^n)$$

for all n. It follows immediately that $\bar{\theta}_* : H_*(B) \cong H_*(K)$. Since the retraction s may be defined over all C, it follows that C is acyclic so that $\theta_* : H_*(C) \cong H_*(LK)$. We again apply the spectral sequence argument to deduce that $\theta_* : H_*(A) \cong H_*(\Omega K)$ and the proof is complete.

Corollary 2.1. *If K is a subcomplex of K^* and if d, θ are given on $C(K)$, $A(K)$ then d^*, θ^* may be chosen so that $d^* \mid C(K) = id$, $\theta^* \mid C(K) = j\theta$, where $i : C(K) \to C(K^*)$, $j : C_*(LK) \to C_*(LK^*)$ are injections.*

Corollary 2.2. *Let K be the union of subcomplexes K_i with a single common point, the single 0-cell of each K_i. Then $A(K)$ may be chosen as the free product of the $A(K_i)$, and θ may be given by $\theta b_i = \theta_i b_i$, $\theta a_i = \theta_i a_i$ where $\theta_i : C(K_i) \to C_*(LK_i)$.*

These two corollaries follow immediately from the definitions of d and θ. By a free product of chain-algebras A_i we understand the chain algebra which is, qua algebra, the free product of the algebras A_i and whose differential is given by

$$d(a_{i_1} \dots a_{i_k}) = \sum_{q=1}^{k} (-1)^{r_q} a_{i_1} \dots (da_{i_q}) \dots a_{i_k} \ , \qquad a_{i_q} \in A_{i_q}^{n_q} \ ,$$

where $r_q = \sum_{s=1}^{q-1} n_s$.

In the light of theorem 2.1, corollary 2.2 may be regarded as a generalization of the theorem due to Bott and Samelson (see [17]) when K is a wedge of spheres.

Before stating the next corollary, which is in the nature of an example, we draw attention to the fact that the map $\bar{\theta} : B \to C_*(K)$ reverses orientation, in the sense that the generator b^n corresponds to the negative of the class of the oriented n-cell e^n in $H_n(K^n, K^{n-1})$.

Corollary 2.3. *Let* $K = S^m \cup e^{m+1}$, $m \geqslant 2$, *where* e^{m+1} *is attached by a map of degree* r. *Then* $A(K)$ *is the chain algebra generated by* a, a', *with* $\dim a = m-1$, $\dim a' = m$ *and* $da' = -ra$.

For certainly $da' = ka$, for some integer k. Now let b, b' be the generators of B. Then since the attaching map is of degree r, we have $\bar{d}b' = rb$. Thus $\pi db' = rb$; but $\pi db' = \pi(a' - sda') = \pi(a' - ksa) = \pi(a' - kb) = -kb$, whence[14] $k = -r$. We note that the differential in C is given by $db = a$, $db' = a' + rb$.

Corollary 2.4. Let $K = S^m \times S^n$, $m, n \geqslant 2$. Then we may take for $A(K)$ the chain algebra (a_1, a_2, a) with $\dim a_1 = m-1$, $\dim a_2 = n-1$, $\dim a = m+n-1$ and

$$da = \varepsilon(a_1 a_2 - (-1)^{(m-1)(n-1)} a_2 a_1) \ , \qquad \varepsilon = \pm 1 \ .$$

Let $K_0 = S^m \vee S^n$. Then $A(K_0) = (a_1, a_2)$ and θa_1 belongs to a generator g_1 of $H_{m-1}(\Omega S^m)$, θa_2 belongs to a generator g_2 of $H_{n-1}(\Omega S^n)$. Now e^{m+n} is attached to K_0 by a map, f, in the class $[\iota_m, \iota_n]$ and, by Samelson's theorem (see [8], $f'_* \beta = \varepsilon(g_1 g_2 - (-1)^{(m-1)(n-1)} g_2 g_1)$. It follows therefore that we may choose $da = \varepsilon(a_1 a_2 - (-1)^{(m-1)(n-1)} a_2 a_1)$. We note that the differential in C is given by $db_1 = a_1$, $db_2 = a_2$, $db = (1 - sd)a = a - \varepsilon(b_1 a_2 - (-1)^{(m-1)(n-1)} b_2 a_1)$. We note also that θa is a relative cycle in the class generating $H_{m+n-1}(\Omega(S^m \times S^n), \Omega(S^m \vee S^n))$.

For further discussion of product complexes, see section 4.

3. Induced maps of chain-algebras

Let $f : K_1 \to K_2$ be a map[15] of CW-complexes, inducing $f' : LK_1, \Omega K_1 \to LK_2, \Omega K_2$. Our main object in this section is to realize the induced homology homomorphism f'_* by an appropriate $\varphi_* : H_*(A(K_1)) \to H_*(A(K_2))$, induced by a map $\varphi : C(K_1), A(K_1) \to C(K_2), A(K_2)$. Although $A(K)$ is not uniquely determined by K, we may then think of the passage from the category of CW-complexes and maps to that of chain algebras and maps given by $(K, f) \to (A(K), \varphi)$ as a (multi-valued) covariant functor. We will prove

[14]) The minus sign can be avoided by replacing ($R1$), ($D1$) by $sa^{r-1} = (-1)^r b^r$, $db^r = (-1)^r (1 - sd) a^{r-1}$. Of course, to compute $H_*(\Omega K)$ one may take a chain algebra generated by a, a' with $da' = ra$.

[15]) Recall that all complexes considered in this paper have one 0-cell and no 1-cells. A map is required to send 0-cell to 0-cell.

Theorem 3.1. *There are chain-maps* $\varphi : C(K_1), A(K_1) \to C(K_2), A(K_2)$, $\bar{\varphi} : B(K_1) \to B(K_2)$ *such that*

(*i*) φ *is product-preserving and* $\varphi s = s\varphi$;

(*ii*) $\bar{\varphi}\pi_1 = \pi_2\varphi$;

(*iii*) *the diagrams*

$$\begin{array}{ccc} C(K_1), A(K_1) & \xrightarrow{\theta_1} & C'(K_1), A'(K_1) \\ \downarrow \varphi & & \downarrow f' \\ C(K_2), A(K_2) & \xrightarrow{\theta_2} & C'(K_2), A'(K_2) \end{array} \qquad \begin{array}{ccc} B(K_1) & \xrightarrow{\bar{\theta}_1} & B'(K_1) \\ \downarrow \bar{\varphi} & & \downarrow f \\ B(K_2) & \xrightarrow{\bar{\theta}_2} & B'(K_2) \end{array}$$

are commutative to within chain homotopy.

We will define φ and a chain homotopy $\psi : C(K_1), A(K_1) \to C'(K_2), A'(K_2)$, such that $d\psi + \psi d = f'\theta_1 - \theta_2\varphi$, inductively on the sections of K_1. Define $\varphi(1) = 1$, $\psi(1) = 0$. Suppose φ, ψ defined on $C(K_1^n)$, and let a be the generator of $A(K_1)$ corresponding to the cell e^{n-1} in K_1. Then φ, ψ are defined on da and

$$\theta_2\varphi da = f'\theta_1 da - d\psi da = d(f'\theta_1 a - \psi da) .$$

Since θ_{2*} is $(1-1)$, there is an element $g_2 \in A(K_2)$ with $dg_2 = \varphi da$. Now $f'\theta_1 a - \psi da - \theta_2 g_2$ is a cycle in $A'(K_2)$; since θ_{2*} is onto, there is a cycle z_2 in $A(K_2)$ and an element g_2' in $A'(K_2)$ such that

$$\theta_2 z_2 + dg_2' = f'\theta_1 a - \psi da - \theta_2 g_2 .$$

We put $\varphi a = g_2 + z_2$, $\psi a = g_2'$. Then $d\varphi a = dg_2 = \varphi da$ and

$$f'\theta_1 a - \theta_2\varphi a = f'\theta_1 a - \theta_2 g_2 - \theta_2 z_2 = dg_2' + \psi da = (d\psi + \psi d)a ,$$

as required. Extend φ to a map of $A(K_1^{n+1})$ into $A(K_2)$; direct computation shows that ψ is extended to $A(K_1^{n+1})$ by the formula
$\psi(xy) = (\psi x)(f'\theta_1 y) + (-1)^p(\theta_2\varphi x)(\psi y)$, for $x \in {}^{n+1}A^p$, $y \in {}^{n+1}A$, $A = A(K_1)$. Extend φ to $C(K_1^{n+1})$ by putting $\varphi b = s\varphi a$. Then $s\varphi = \varphi s$ on the generators of $A(K_1^{n+1})$, $B(K_1^{n+1})$ and hence on the whole of $C(K_1^{n+1})$. Certainly ψb may be defined since $C'(K_2)$ is acyclic and ψ is extended to the whole of $C(K_1^{n+1})$ by the same formula as above, where now $x \in {}^{n+1}C^p$ and $y \in {}^{n+1}A$. The inductive definitions of φ and ψ are complete.

Now $\bar{\varphi}$ is defined by (ii), provided we can show that $\pi_2\varphi$ is zero on the kernel of π_1. A typical element of the kernel is $\Sigma x_i y_i$, $y_i \in A^{n_i}(K_1)$, $n_i > 0$. Then $\varphi(x_i y_i) = \varphi(x_i)\varphi(y_i)$, $\varphi(y_i) \in A^{n_i}(K_2)$, so that

$$\pi_2(\varphi(x_i)\varphi(y_i)) = 0 .$$

Thus $\bar{\varphi}$ is defined. A chain homotopy $\bar{\psi}: B(K_1) \to B'(K_2)$ such that $\bar{d}\bar{\psi} + \bar{\psi}\bar{d} = f\bar{\theta}_1 - \bar{\theta}_2\bar{\varphi}$ is defined by $\bar{\psi}\pi_1 = p_2\psi$, provided $p_2\psi$ is zero on the kernel of π_1. Now if $y_i \in A^{n_i}(K_1)$, $n_i > 0$, then

$$f'\theta_1 y_i \in A'^{n_i}(K_2) \ , \quad \psi y_i \in A'^{n_i+1}(K_2) \ .$$

It follows from the product formula for ψ that $p_2\psi(x_i y_i) = 0$ in $B'(K_2)$, $x_i \in C^{p_i}(K_1)$, so that $\bar{\psi}$ is defined as required. This completes the proof of the theorem.

We make some remarks about this theorem. First we note that if φ' is any other suitable chain map $C(K_1)$, $A(K_1) \to C(K_2)$, $A(K_2)$, then $\theta_2\varphi' \simeq f'\theta_1 \simeq \theta_2\varphi$; since θ_2 is a chain equivalence, it follows that any two choices of φ are chain homotopic. Similarly any two choices of $\bar{\varphi}$ are chain homotopic. Let us write $\varphi(f)$ for φ; then we see that if $f: K_1 \to K_2$, $g: K_2 \to K_3$ are maps we may choose $\varphi(gf)$ to be $\varphi(g)\varphi(f)$. We also note the trivial fact that if f is an injection and if d, θ have been chosen on K_2 consistently with their values on K_1, then $\varphi, \bar{\varphi}$ may be taken as injections. Finally, we remark that if f is a map $K_1, L_1 \to K_2, L_2$ where L_i is a subcomplex of K_i, $i = 1, 2$, then φ, ψ may be chosen so that

$$\varphi(C(L_1), A(L_1)) \subseteq C(L_2), A(L_2), \quad \psi(C(L_1), A(L_1)) \subseteq C'(L_2), A'(L_2).$$

Now let $f_0: L_1 \to L_2$ be a map of CW-complexes and let $K_i = L_i \cup e_i^{n+1}$, where $g_i: E^{n+1}, S^n \to K_i, L_i$ is a characteristic map for e_i^{n+1}, $i = 1, 2$. Suppose $f_0 g_1 \mid S^n \simeq g_2 \mid S^n$. Then we may extend f_0 to a map $f: K_1 \to K_2$ with $fg_1 \simeq g_2$. Now $A(K_i)$ is formed from $A(L_i)$ by adjoining a new generator a_i. We prove

Theorem 3.2. *If we*[16]) *have chosen d and θ on K_1 and L_2 and φ on L_1, then we may choose d and θ on a_2 and φ, ψ on a_1 so that $\varphi a_1 = a_2$.*

We first choose da_2. Adopting the notation of the previous section, we have only to choose da_2 so that $\theta_2 da_2 \sim g_2'\zeta$. Now

$$\theta_2\varphi da_1 = f'\theta_1 da_1 - d\psi da_1 = f'g_1'\zeta + f'dx' - d\psi da_1 \ .$$

Now $f'g_1' \simeq g_2'$; there is a chain homotopy $\omega: C_*(LE^{n+1})$, $C_*(\Omega E^{n+1})$, $C_*(LS^n)$, $C_*(\Omega S^n) \to C'(K_2)$, $A'(K_2)$, $C'(L_2)$, $A'(L_2)$ with[17])

$$f'g_1' - g_2' = d\omega + \omega d \ , \quad f'g_1'' - g_2'' = d\omega + \omega d \ .$$

It follows that $\theta_2\varphi da_1 = g_2'\zeta + d(\omega\zeta + f'x' - \psi da_1)$. We may, and

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[16]) We will say that d and θ are chosen on K if they are chosen on $A(K)$.

[17]) In the argument which follows it is cumbersome and unnecessary always to distinguish g_1', g_2' from g_1'', g_2''; however, we simply copy the notation of theorem 2.1.

do, choose $da_2 = \varphi da_1$. Then we may take $x_2' = \omega\zeta + f'x_1' - \psi da_1$ and $\theta_2 a_2 = j_2 x_2' + g_2''\eta$. Now

$$f'\theta_1 a_1 - \psi da_1 - \theta_2 a_2 = f'j_1 x_1' + f'g_1''\eta - \psi da_1 - j_2\omega\zeta - j_2 f'x_1' + j_2\psi da_1 - g_2''\eta = f'g_1''\eta - g_2''\eta - j_2\omega\zeta = d\omega\eta + \omega d\eta - j_2\omega\zeta = d\omega\eta + \omega i\zeta - j_2\omega\zeta = d\omega\eta \,.$$

Thus we may choose $\varphi a_1 = a_2$, $\psi a_1 = \omega\eta$, and the theorem is proved.

Suppose f_0 is a homotopy equivalence. Then $\varphi_* : H_*(A(L_1)) \cong H_*(A(L_2))$. Also f is a homotopy equivalence so that $\varphi_* : H_*(A(K_1)) \cong H_*(A(K_2))$. This is the topological analogue of the following purely algebraic theorem.

Theorem 3.3. *Let $\varphi : A \to A'$ be a map of chain algebras inducing an isomorphism $\varphi_* : H_*(A) \to H_*(A')$. Let $\overline{A}$ be defined by adjoining a generator a to A and let $\overline{A}'$ be defined by adjoining a generator a' to A' of the same dimension, n, as a. Let $\varphi da = da'$. Then the map $\overline{\varphi} : \overline{A} \to \overline{A}'$ given by $\overline{\varphi} \mid A = \varphi$, $\varphi a = a'$, induces an isomorphism $\overline{\varphi}_* : H_*(\overline{A}) \cong H_*(\overline{A}')$.*

Filter $\overline{A}$ by the rule $\omega(x_0 a x_1 a \ldots a x_p) = p$, $x_i \in A$ and filter $\overline{A}'$ similarly. Let the associated groups of the spectral sequence be $E_r^{p,q}$, $E_r'^{p,q}$. Then $E_r'^{p,q} = E_r'^{p,q} = 0$ if $q < pn - p$. Now $\overline{\varphi}$ is filtration-preserving and $d\overline{A}_p \subseteq \overline{A}_p$, $d\overline{A}_p' \subseteq \overline{A}_p'$, where $(\overline{A}_p)$, $(\overline{A}_p')$ are the filtering subgroups. Thus $\overline{\varphi}$ is a map of differential filtered groups.

Let $A^{(p)}$ be the tensor product of p copies of A and define $A'^{(p)}$ similarly. Then φ induces $\tilde{\varphi} : A^{(p)} \to A'^{(p)}$ which is a chain equivalence since φ is a chain equivalence. Let $A^{(p)q}$, $q \geqslant pn - p$, be the homogeneous component of $A^{(p)}$ of dimension $p + q - pn$ and let $\psi : A^{(p)q} \to E_0^{p,q}$ be defined by $\psi(x_0 \otimes \ldots \otimes x_p) = (-1)^\sigma x_0 a x_1 \ldots a x_p$, where $x_i \in A^{n_i}$ and $\sigma = n \sum_{i=0}^{p} i n_i$. Then ψ is an isomorphism and $\psi d = d_0 \psi$ so that ψ induces $\psi_* : H_{p+q-pn}(A^{(p)}) \cong E_1^{p,q}$. Similarly $\psi' : A'^{(p)q} \to E_0'^{p,q}$ induces $\psi_*' : H_{p+q-pn}(A'^{(p)}) \cong E_1'^{p,q}$. Also $\psi'\tilde{\varphi} = \overline{\varphi}\psi : A^{(p)q} \to E_0'^{p,q}$, so that $\overline{\varphi}$ induces $\overline{\varphi}_* : E_1^{p,q} \cong E_1'^{p,q}$. It follows that $\overline{\varphi}$ induces $\overline{\varphi}_* : E_\infty^{p,q} \cong E_\infty'^{p,q}$ and hence $\overline{\varphi}_* : H_*(\overline{A}) \cong H_*(\overline{A}')$.

The spectral sequence $E_r^{p,q}$ seems the appropriate tool for studying the effect on $H_*(\Omega K)$ of adding a cell to K, since $\overline{A}_0 = A$.

Our next result is in the nature of an example.

Theorem 3.4. *Let $K = S^n \cup e^{2n}$, $n \geqslant 2$, and suppose $A(K)$ is the chain algebra generated by a_1, a_2 with $da_2 = pa_1^2$. Then p is the Hopf invariant of the attaching map for e^{2n}.*

Let K', K'' be copies of K and let $K' \times K''$ be decomposed into cells in the obvious way. We write a_1, a_2 for the generators of $A(K)$ corresponding to the cells e^n, e^{2n} of K and $a_1', a_2', a_1'', a_2'', a_{11}, a_{12}, a_{21}, a_{22}$ for the generators of $A(K' \times K'')$ corresponding to the cells e'^n, e'^{2n}, e''^n, e''^{2n}, $e'^n \times e''^n$, $e'^n \times e''^{2n}$, $e'^{2n} \times e''^n$, $e'^{2n} \times e''^{2n}$ of $K' \times K''$. Let $f : K \to K' \times K''$ be the diagonal map. Suppose d, θ chosen on $A(K' \times K'')$ consistent with the embedding of $K' \vee K''$ in $K' \times K''$. Let $\varphi : A(K) \to A(K' \times K'')$, $\overline{\varphi} : B(K) \to B(K' \times K'')$ be associated with f, let the cells of $B(K)$, $B(K' \times K'')$ be symbolized similarly to the generators of $A(K)$, $A(K' \times K'')$ and let the Hopf invariant of the attaching map be q. Then

$$\overline{\varphi} b_1 = b_1' + b_1'', \quad \overline{\varphi} b_2 = b_2' + q b_{11} + b_2'' .$$

A dimensionality argument[18]) shows that $\varphi a_1 = \varrho a_1' + \sigma a_1''$, $\varphi a_2 = \lambda a_2' + \mu a_{11} + \nu a_2''$. Applying s and comparing with the formulae for $\overline{\varphi}$, we find $\varrho = \sigma = \lambda = \nu = 1$, $\mu = q$. Now $d\varphi a_2 = \varphi d a_2 = p \varphi a_1^2 = p a_1'^2 + p a_1' a_1'' + p a_1'' a_1' + p a_1''^2$, while $d(a_2' + q a_{11} + a_2'') = p a_1'^2 + (-1)^n q a_1' a_1'' + q a_1'' a_1' + p a_1''^2$ (we orient the cell a_{11}, or a in corollary 2.4, so that $\varepsilon = (-1)^m$). Comparing coefficients, $p = (-1)^n q$, $p = q$. This proves the theorem and also shows that the Hopf invariant is zero if n is odd.

If e^{2n} is attached by a map of Hopf invariant 1, then $H_*(\Omega K) \cong H_*(A)$ where A is the chain algebra generated by $a_1 = a_1^{n-1}$, $a_2 = a_2^{2n-1}$ with $da_2 = a_1^2$. It may be of interest to compute the ring $H_*(A)$. We prove

Theorem 3.5. *$H_{r(3n-2)}(A) = Z_\infty$, generated by $\{g\}^r$, $g = a_1 a_2 - (-1)^{n-1} a_2 a_1$, $H_{r(3n-2)+n-1}(A) = Z_\infty$, generated by $\{g^r a_1\}$, $H_m(A) = 0$, for other values of m. Moreover $\{a_1 g\} = (-1)^n \{g a_1\}$.*

We remark first that in the topological case n is even so that $g = a_1 a_2 + a_2 a_1$ and $\{a_1 g\} = \{g a_1\}$. Thus the theorem asserts that $H_*(A)$ is a commutative ring in this case, isomorphic with the tensor product of an exterior ring generated by $\{a_1\}$ and a polynomial ring generated by $\{g\}$. We now prove the theorem.

Consider the exact sequence $0 \to A_{k-n+1} \xrightarrow{i} A_k \xrightarrow{j} A_{k-2n+1} \to 0$, where $ix = xa_1$, $j(xa_1 + ya_2) = y$, $x \in A_{k-n+1}$, $y \in A_{k-2n+1}$. This induces the exact homology sequence

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18) This argument only holds if $n > 2$; if $n = 2$, the expression for φa_2 could, a priori, contain terms in a'^3 and a''^3. We may either eliminate this possibility by considering projections $K' \times K'' \to K'$, $K' \times K'' \to K''$, (whereby we may also deduce $\lambda = \nu = 1$) – or leave these terms in the expression until they are annihilated in the passage from φ to $\overline{\varphi}$.

$$\cdots \to H_{k-2n+2}(A) \xrightarrow{d_*} H_{k-n+1}(A) \xrightarrow{i_*} H_k(A) \xrightarrow{j_*} H_{k-2n+1}(A) \xrightarrow{d_*} H_{k-n}(A) \to \cdots,$$

where $i_*\{x\} = \{xa_1\}$, $j_*\{xa_1 + ya_2\} = \{y\}$, and $d_*\{y\} = (-1)^\sigma\{ya_1\}$, $\sigma = \dim y$. Now the homology groups of A are certainly as stated in dimensions $< 3n - 2$. Suppose inductively that they are as stated in dimensions $<(r + 1)(3n - 2)$, $r \geqslant 0$. The part of the homology sequence beginning $H_{(r+1)(3n-2)+2n-2}(A) \xrightarrow{i_*} \cdots$ and ending... $\xrightarrow{d_*} H_{r(3n-2)+2n-2}(A)$ will be trivial except for

$$\begin{aligned} j_* &: H_{(r+1)(3n-2)} \cong H_{r(3n-2)+n-1} \ , \\ i_* &: H_{(r+1)(3n-2)} \cong H_{(r+1)(3n-2)+n=1} \ , \\ d_* &: H_{(r+1)(3n-2)} \cong H_{(r+1)(3n-2)+n-1} \ , \end{aligned}$$

Thus $H_{(r+1)(3n-2)} = Z_\infty$ generated by $\{g^r a_1 a_2 + xa_1\}$, x being chosen arbitrarily, subject only to the condition that $g^r a_1 a_2 + xa_1$ be a cycle; since g is a cycle and g^{r+1} is of this form, it follows that $H_{(r+1)(3n-2)}$ is generated by $\{g\}^{r+1}$. It then follows that $H_{(r+1)(3n-2)+n-1} = Z_\infty$, generated by $\{g^{r+1}a_1\}$ and that H_* is zero in all other dimensions $<(r + 2)(3n - 2)$.

We complete the proof of the theorem by observing that $d(a_2^2) = a_1^2 a_2 - a_2 a_1^2 = a_1(a_1 a_2 - (-1)^{n-1} a_2 a_1) - (-1)^n (a_1 a_2 - (-1)^{n-1} a_2 a_1) a_1$.

4. Product complexes.

The main object of this section is to obtain a chain equivalence from $A(K_1 \times K_2)$ to $A(K_1) \otimes A(K_2)$. We first provide a universal example for the chain algebra of a product complex.

Theorem 4.1. *Let $E_1 = e^0 \cup e^p \cup e^{p+1}$ be a $(p + 1)$-element decomposed in the usual way into the cells e^0, e^p, e^{p+1} and let E_2 be a $(q + 1)$-element similarly decomposed, $p, q \geqslant 2$. Then $A(E_1 \times E_2)$ is freely generated by elements*[19] *a_1, a_2, c_1, c_2, b, e, $\bar{e}$, t, corresponding to the cells e^p, e^q, e^{p+1}, e^{q+1}, $e^p \times e^q$, $e^{p+1} \times e^q$, $e^p \times e^{q+1}$, $e^{p+1} \times e^{q+1}$ and d, θ may be chosen on $E_1 \times E_2$ to give $dc_1 = - a_1$, $dc_2 = - a_2$,*

$$\begin{aligned} db &= (-1)^p(a_1 a_2 - (-1)^{(p-1)(q-1)} a_2 a_1) \\ de &= -b + (-1)^{p+1}(c_1 a_2 - (-1)^{p(q-1)} a_2 c_1) \ , \\ d\bar{e} &= (-1)^{p-1} b + (-1)^p (a_1 c_2 - (-1)^{(p-1)q} c_2 a_1) \ , \\ dt &= (-1)^p e - \bar{e} + (-1)^{p+1}(c_1 c_2 - (-1)^{pq} c_2 c_1) \ . \end{aligned}$$

The first two boundary formulae are given by corollaries 2.2 and 2.3.

[19]) The notations used for generators of $A(E_1 \times E_2)$ are chosen for their convenience in studying product complexes and are not related to previous notation.

The formula for db is given by corollary 2.4, the orientation being chosen so that, under the map $\varphi : A(E^{p+q}) \to A(S^p \times S^q)$ induced by the characteristic map E^{p+q}, $S^{p+q-1} \to S^p \times S^q$, $S^p \vee S^q$, the cell corresponding to S^{p+q-1} is mapped precisely by the Samelson formula (cf. Theorem 3.4).

Now consider $A(E_1 \times S_2^q) = \{a_1, a_2, c_1, b, e\}$. Since the injection $i_2 : S_2^q \to E_1 \times S_2^q$ and the projection $p_2 : E_1 \times S_2^q \to S_2^q$ are homotopy equivalences such that $p_2 i_2 = 1$, it follows readily that[20] we may choose $\varphi_2 = \varphi(p_2)$ such that $\varphi_2 a_2 = a_2$, $\varphi_2 a = 0$ for $a = a_1, c_1$, or b and φ_{2*} is an isomorphism. Now, if z is the element proposed for de, then z is a cycle and $\varphi_2 z = 0$. Thus z is a boundary; it follows that, for an arbitrary choice of de, there exist an integer k and an element $x \in \{a_1, a_2, c_1, b\}$ such that $d(ke + x) = z$ or $k(de) + dx = z$. It may be seen by inspection that no such equation can subsist in $\{a_1, a_2, c_1, b\}$ unless $k = \pm 1$. Thus, if e is suitably oriented, z is a proper choice for de. The orientation of e is chosen to give the correct boundary formula in $B(E_1 \times S_2^q)$, when the cells of $E_1 \times S_2^q$ are given the product orientation.

A similar argument establishes the formula of $d\bar{e}$; the orientation of $\bar{e}$ is chosen by the same considerations.

Finally the element z' proposed for dt is a cycle and therefore a boundary; it follows that, for an arbitrary choice of dt, there exist an integer k' and an element $x' \in \{a_1, a_2, c_1, c_2, b, e, \bar{e}\}$ such that $k'(dt) + dx' = z'$. It may be seen by inspection of $\{a_1, a_2, c_1, c_2, b, e, \bar{e}\}$ that this implies $k' = \pm 1$, so that z' is a proper choice for dt if t is suitably oriented; we choose the orientation for t as for e and $\bar{e}$ and the theorem is proved.

Now let $K_1 \times K_2$ be the topological product of two CW-complexes with its usual cellular decomposition[21]). Let

$$j : A(K_1 \times K_2) \to A(K_1) \otimes A(K_2)$$

be the ring homomorphism given by

$$\begin{aligned} ja &= a \otimes 1, \quad a \in A(K_1), \\ ja &= 1 \otimes a, \quad a \in A(K_2), \\ ja &= 0, \quad \text{for any other generator } a \text{ of } A(K_1 \times K_2). \end{aligned}$$

Let $\varphi_i : A(K_1 \times K_2) \to A(K_i)$, $i = 1, 2$, be the ring homomorphism given by

$$\begin{aligned} \varphi_i a &= a, \quad a \in A(K_i), \\ \varphi_i a &= 0, \quad \text{for any other generator } a \text{ of } A(K_1 \times K_2). \end{aligned}$$

---

[20]) Clearly a suitable φ for the projection $S_1^p \times S_2^q \to S_2^q$ is given by $\varphi(a_1) = 0$, $\varphi(a_2) = a_2$, $\varphi(b) = 0$, provided ∂b has been appropriately chosen.

[21]) We are not disturbed by the fact that $K_1 \times K_2$ need not be a CW-complex; theorem 2.1 holds for products of CW-complexes.

Let $\varrho : C_*X \times C_*Y \to C_*(X\times Y)$ be the standard chain equivalence of cubical homology theory.

The main theorem of this section is as follows.

Theorem 4.2. *We may choose d and θ on $K_1\times K_2$ so that j is a chain mapping; with this choice φ_i is a φ-map*[22]) *associated with the projection $p_i : K_1\times K_2 \to K_i$, and the diagram*

$$\begin{array}{ccc} A(K_1\times K_2) & \xrightarrow{\ j\ } & A(K_1)\otimes A(K_2) \\ \downarrow{\scriptstyle\theta} & & \\ A'(K_1\times K_2) & & \downarrow{\scriptstyle\theta_1\otimes\theta_2} \\ \downarrow{\scriptstyle\Psi} & & \\ C_*(\Omega K_1\times\Omega K_2) & \xrightarrow{\ \varrho\ } & A'(K_1)\otimes A'(K_2) \end{array}$$

is homotopy-commutative and leads to a commutative diagram of isomorphisms of homology rings.

Suppose that d and θ have been chosen so that j is a chain mapping. Let $i:\ A(K_1)\otimes A(K_2) \to A(K_1\times K_2)$ be the chain mapping of chain groups given by $i(x\otimes y) = xy,\ x \in A(K_1),\ y\in A(K_2)$. Then $ji = 1$. Let ΩK_1, ΩK_2 be embedded in $\Omega(K_1\times K_2)$ and let

$$\eta : \Omega K_1\times\Omega K_2 \to \Omega(K_1\times K_2)$$

be the map given by $\eta(l_1, l_2) = l_1 l_2$ (composition of loops). Then[23]) $\theta i = \eta\varrho(\theta_1\otimes\theta_2)$. We next show that η is a homotopy inverse of Ψ. Since Ψ is a homotopy equivalence it is sufficient to show that $\Psi\eta \simeq 1$. Now $\Psi\eta(l_1, l_2) = (l_1\omega_s, \omega_r l_2)$ where l_1 is a loop of 'duration' r, l_2 is a loop of 'duration' s and ω_r, ω_s are constant loops of duration r, s. Thus a homotopy of the identity to $\Psi\eta$ is given by $h_t(l_1, l_2) = (l_1\omega_{st}, \omega_{rt}l_2)$. Then $\Psi\theta i \simeq \varrho(\theta_1\otimes\theta_2)$. Since Ψ, θ, ϱ and $\theta_1\otimes\theta_2$ are chain equivalences it follows that i is a chain equivalence. Since $ji = 1$, it follows that j is a chain inverse of i so that $\Psi\theta \simeq \varrho(\theta_1\otimes\theta_2)j$ and j_* is an isomorphism.

We still assume that j is a chain mapping and next prove that if p_1': $A'(K_1\times K_2) \to A'(K_1)$ is induced by p_1, then $p_1'\theta \simeq \theta_1\varphi_1$. Let $p_1'' : C_*(\Omega K_1\times\Omega K_2) \to A'(K_1)$ be induced by the projection

$$\Omega K_1\times\Omega K_2 \to \Omega K_1\ ;$$

22) In the sense of theorem 3.1.

23) We always suppose d and θ chosen consistently with the embedding

$$K_1 \vee K_2 \subseteq K_1\times K_2\ .$$

let $\tilde{p}_1 : A(K_1) \otimes A(K_2) \to A(K_1)$ be the map given by $\tilde{p}_1(x \otimes 1) = x$, $\tilde{p}_1(1 \otimes y) = 0$ and let $\tilde{p}_1' : A'(K_1) \otimes A'(K_2) \to A'(K_1)$ be defined similarly. Then the relations

$$p_1''\Psi = p_1' \,, \qquad \theta_1 \tilde{p}_1 = \tilde{p}_1'(\theta_1 \otimes \theta_2) \,, \qquad \tilde{p}_1' = p_1'' \varrho \,, \qquad \varphi_1 = \tilde{p}_1 j \,,$$

are obvious.

We have proved that $\theta \simeq \eta\varrho(\theta_1 \otimes \theta_2) j$ and $p_1'\eta \simeq p_1''$ since $\Psi\eta \simeq 1$. Thus

$$p_1'\theta \simeq p_1'\eta\varrho(\theta_1 \otimes \theta_2) j \simeq p_1''\varrho(\theta_1 \otimes \theta_2) j = \tilde{p}_1'(\theta_1 \otimes \theta_2) j = \theta_1 \tilde{p}_1 j = \theta_1 \varphi_1 \,.$$

Thus φ_1 is a suitable choice for $\varphi(p_1)$ and a similar argument shows that φ_2 is a suitable choice for $\varphi(p_2)$.

It remains to show that d and θ may be chosen so that j is a chain mapping. We observe first that j is a chain mapping on $A(K_1 \vee K_2)$, embedded in $A(K_1 \times K_2)$, and second that j is a chain mapping on the universal example $A(E_1 \times E_2)$.

We now prove that d and θ may be chosen on $E_1 \times K_2$, $E_1 = E_1^{p+1}$, $p \geqslant 2$, so that j is a chain mapping. The argument proceeds by induction on the sections of K_2. It is trivial for $E_1 \times K_2^0$ and follows easily for $E_1 \times K_2^2$ from theorem 4.1. Suppose inductively that d and θ have been chosen on $E_1 \times K_2^q$, $q \geqslant 2$, so that j is a chain mapping and let e be *a* $(q+1)$-cell attached to K_2^q, the characteristic map being $f : E_2^{q+1}$, $S_2^q \to K_2^q \cup e, K_2^q$. Let $\varphi_2 : A(E_2^{q+1}) \to A(K_2^q \cup e)$ be associated with f. We proceed to define a map $\varphi : A(E_1 \times S_2^q) \to A(E_1 \times K_2^q)$. In the notation of theorem 4.1, we put $\varphi(a_1) = a_1$, $\varphi(a_2) = \varphi_2(a_2)$, $\varphi(c_1) = c_1$. Then, so far as φ is defined, the diagram

$$\begin{array}{ccc} A(E_1 \times S_2^q) & \xrightarrow{j} & A(E_1) \otimes A(S_2^q) \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle 1 \otimes \varphi_2} \\ A(E_1 \times K_2^q) & \xrightarrow{j} & A(E_1) \otimes A(K_2^q) \end{array} \tag{4.3}$$

is commutative.

Consider the element $b \in A(E_1 \times S_2^q)$. Then $j\varphi db = 0$; since j is a chain equivalence onto $A(E_1) \otimes A(K_2^q)$, it follows that the kernel of j is acyclic, so that there exists an element $x \in A(E_1 \times K_2^q)$ with $dx = \varphi db$ and $jx = 0$. Define $\varphi b = x$. Then $\varphi d = d\varphi$ on b and the commutativity of (4.3) is preserved. Then $j\varphi de = 0$ and the same argument shows that there exists an element $y \in A(E_1 \times K_2^q)$ with $dy = \varphi de$, $jy = 0$; we take $\varphi e = y$. Thus we have defined a map φ making (4.3) a commutative diagram.

We now assert that φ is associated with the map

$$1\times f : E_1\times S_2^q \to E_1\times K_2^q \ .$$

To establish this, we consider the diagram

$$\begin{tikzcd}
A(E_1\times S_2^q) \arrow[rrr, "j"] \arrow[dddd, "\varphi"'] \arrow[dr, "\theta"] & & & A(E_1)\otimes A(S_2^q) \arrow[dddd, "1\otimes\varphi"] \arrow[dl, "\theta_1\otimes\theta_2"'] \\
& A'(E_1\times S_2^q) \arrow[r, "\eta\varrho"] \arrow[d, "(1\times f)'"] & A'(E_1)\otimes A'(S_2^q) \arrow[d, "1\otimes f'"] & \\
& A'(E_1\times K_2^q) & A'(E_1)\otimes A'(K_2^q) \arrow[l, "\eta\varrho"'] & \\
& & & \\
A(E_1\times K_2^q) \arrow[rrr, "j"'] \arrow[uur, "\theta"'] & & & A(E_1)\otimes A(K_2^q) \arrow[uul, "\theta_1\otimes\theta_2"]
\end{tikzcd}$$

We wish to show that $\theta\varphi \simeq (1\times f)'\theta$; but this follows from the commutativity properties of the diagram. Now $E_1\times E_2$ is obtained from $E_1\times S_2^q$ by attaching cells e^{q+1}, $e^p\times e^{q+1}$, $e^{p+1}\times e^{q+1}$ and each of these cells is mapped homeomorphically onto a cell of $E_1\times(K_2^q \cup e)$. The generators of $A(E_1\times E_2)$ corresponding to these cells are $c_2, \bar{e}, t$; let the generators of $A(E_1\times(K_2^q\cup e)$ corresponding to the cells $(1\times f)(e^{q+1})$, $(1\times f)(e^p\times e^{q+1})$, $(1\times f)(e^{p+1}\times e^{q+1})$ be called c_2, $\bar{e}^*$, t^*. Then by theorem 3.2, we may define d, θ on $\bar{e}^*$, t^* and extend φ to a map associated with $1\times f : E_1\times E_2 \to E_1\times(K_2^q\cup e)$ by putting $\varphi\bar{e} = \bar{e}^*$, $\varphi t = t^*$; but then $jd\bar{e}^* = jd\varphi\bar{e} = j\varphi d\bar{e} = (1\otimes\varphi_2)jde = 0$ and $j\bar{e}^* = 0$ so that j is a chain mapping on $\bar{e}^*$; and $jt^* = 0$, $jdt^* = jd\varphi t = j\varphi dt = (1\otimes\varphi_2)jdt$ (since, by definition, $j\varphi\bar{e} = (1\otimes\varphi_2)j\bar{e}\ (=0)) = 0$, so that j is a chain mapping on t^* and hence on the whole of $A(E_1\times(K_2^q\cup e))$. We proceed in this way over all the $(q+1)$-cells of K_2 and so define d and θ on $A(E_1\times K_2^{q+1})$ so that j is a chain mapping. This establishes the induction and hence the result when $K_1 = E_1$.

Finally we consider the general case, and proceed by induction over the sections of K_1. It is an immediate consequence of the argument above that we may choose d and θ on $K_1^2\times K_2$ so that j is a chain mapping. Suppose inductively that d and θ have been chosen on $K_1^p\times K_2$ so that j is a chain mapping and let e be a $(p+1)$-cell attached to K_1^p, the characteristic map being $f : E_1^{p+1}, S_1^p \to K_1^p\cup e, K_1^p$. Let

$$\varphi_1 : A(E_1) \to A(K_1^p\cup e)$$

be associated with f. We assert that a map

$$\varphi : A(S_1^p \times K_2) \to A(K_1^p \times K_2)$$

may be defined so that the diagram

$$\begin{tikzcd} A(S_1^p \times K_2) \arrow[r, "j"] \arrow[d, "\varphi"'] & A(S_1^p) \otimes A(K_2) \arrow[d, "\varphi_1 \otimes 1"] \\ A(K_1^p \times K_2) \arrow[r, "j"] & A(K_1^p) \otimes A(K_2) \end{tikzcd}$$

is commutative[24]). We may define $\varphi a_1 = \varphi_1 a_1$, $a_1 \in A(S_1^p)$, $\varphi a_2 = a_2$, $a_2 \in A(K_2)$. We then define φa where a is a generator corresponding to a cell $e^p \times e^{n+1}$ in $S_1^p \times K_2$ inductively with respect to n. For if φ is defined on $A(S_1^p \times K_2^n \cup (S_1^p \vee K_2))$ so that $j\varphi = (\varphi_1 \otimes 1)j$ and if the generator a corresponds to a cell $e^p \times e^{n+1}$, then $j\varphi da = (\varphi_1 \otimes 1)jda = 0$ and so, as previously, there exists an element $x \in A(K_1^p \times K_2)$ such that $dx = \varphi da$ and $jx = 0$; we put $\varphi a = x$ and then $j\varphi a = (\varphi_1 \otimes 1)ja = 0$. This establishes that such a map φ may be defined. Arguing from a diagram analogous to (4.4) shows that φ is associated with the map $f \times 1 : S_1^p \times K_2 \to K_1^p \times K_2$.

Let e^n be an arbitrary cell of K_2, let a be the generator of $A(E_1 \times K_2)$ corresponding to $e^{p+1} \times e^n$ and let a^* be the generator of $A((K_1^p \cup e) \times K_2)$, $e = e^{p+1}$, corresponding to $e \times e^n$. Again applying theorem 3.2, we deduce that d, θ may be chosen on $A((K_1^p \cup e) \times K_2)$ so that the map φ may be extended to a map associated with $f \times 1 : E_1 \times K_2 \to (K_1^p \cup e) \times K_2$ by defining $\varphi a = a^*$ for all e^n in K_2. Then we still have $j\varphi = (\varphi_1 \otimes 1)j$. It remains to show that $jda^* = 0$ for all a^*; but $jda^* = jd\varphi a = j\varphi da = (\varphi_1 \otimes 1)jda = 0$, since $ja = 0$ and j is a chain mapping on $A(E_1 \times K_2)$. We proceed in this way over all the $(p+1)$-cells of K_1 and so define d and θ on $A(K_1^{p+1} \times K_2)$ so that j is a chain mapping. This establishes the induction and completes the proof of the theorem.

Corollary 4.1. *Let* $\varphi_i : A(K_i) \to A(L_i)$ *be associated with maps* $f_i : K_i \to L_i$, $i = 1, 2$, *and let* d, θ *be chosen on* $K_1 \times K_2$, $L_1 \times L_2$ *so that j is a chain mapping. Then we may choose a map*

$$\varphi : A(K_1 \times K_2) \to A(L_1 \times L_2)$$

so that $j\varphi = (\varphi_1 \otimes \varphi_2)j$ *and any such φ is associated with the product map* $f_1 \times f_2$.

---

[24]) We suppose $A(E_1 \times K_2)$ furnished with suitable d, θ to make j a map.

We establish the existence of such a map φ by an inductive argument analogous to that following diagram (4.3) and the required property of φ by an argument based on a diagram analogous to (4.4).

Corollary 4.2. *If $L_i \subseteq K_i$, $i = 1, 2$, and if d, θ have been chosen on $L_1 \times L_2$ so that j is a chain mapping, then d, θ may be extended to $K_1 \times K_2$ so that j remains a chain mapping.*

For this is essentially the procedure in the last part of the proof of theorem 4.2.

Now let $K = S_1 \times \cdots \times S_t$, where S_i is an n_i-sphere, $n_i \geqslant 2$, $i = 1, \ldots, t$. Then K may be decomposed into cells in the usual way: for each non-empty subset D of $\{1, 2, \ldots, t\}$, let e_D be the cell $\prod_{i \in D} e_i$, and let a_D be the corresponding generator of $A(K)$. We prove

Theorem 4.3. *d and θ may be chosen on K so that*

$$d a_D = \sum_{A,B} (-1)^{\varepsilon(A,B)} a_A a_B$$

where the sum extends over all partitions of D into non-empty subsets A, B and

$$\varepsilon(A, B) = \sum_{a \in A} n_a + \sum_{\substack{a \in A,\, b \in B \\ a > b}} n_a n_b .$$

We prove this by induction on t; it is trivial if $t = 1$ and reduces to the Samelson formula if $t = 2$. Suppose the theorem established for products of $t - 1$ spheres, $t \geqslant 3$, and consider K. We propose to choose d and θ on K so that $j: A(K) \to A(S_1 \times \cdots \times S_{t-1}) \otimes A(S_t)$ is a map. For any a_D, $D \neq \{1, 2, \ldots, t\}$, choose the proposed formula for $d a_D$; the inductive hypothesis tells us this is possible, and we observe (by direct computation) that $j d a_D = d j a_D$. Now let $D = \{1, 2, \ldots, t\}$; by corollary 4.2, there exists a choice for the boundary of a_D, say $d' a_D$, such that j remains a map and therefore a chain equivalence. Now we observe (by direct computation) that $x = \sum_{A,B} (-1)^{\varepsilon(A,B)} a_A a_B$ is a cycle and $j x = 0$. If follows that x is a boundary, so that $x = dy + k d' a_D$, where $y \in A(K - e_D)$ and k is an integer. Now (arguing as in theorem 4.1) we observe that $a_A a_B$, for example, cannot appear in the boundary of an element of $A(K - e_D)$ with non-zero coefficient. Thus it must appear in the boundary of a_D and we must have $k = \pm 1$. Thus, reorienting e_D if necessary, we have proved that x is a legitimate choice for $d a_D$. We observe, of course, that j remains a map with this choice. In fact, partitioning the ordered array $\{1, 2, \ldots, t\}$ in any way we please as $A_1, \ldots, A_s$, where A_i is the array $\{n_i, n_i+1, \ldots, n_{i+1}-1\}$, $n_1 = 1$, $n_{s+1} = t+1$, we find that

$$j : A(K) \to A(K_1) \otimes \cdots \otimes A(K_s)$$

is a map, where the definitions of j is an obvious extension of that for $s = 2$ and $K_i = \prod_{r \in A_i} S_r$.

Theorem 4.3 constitutes a generalization of Samelson's formula; it is consistent with the formula contained in remark (i) on p. 5 of [7].

J. C. Moore considers in [5] spaces with a single non-vanishing homology group, in dimension p, say. If this group is finitely generated, then an appropriate space is a wedge of subspaces X_i, where each X_i is a p-sphere or a p-sphere with a $(p+1)$-cell attached by a map of non-zero degree. We study here the Pontryagin ring of the loop space of a Moore space. The method is exemplified by the case when the Moore space Z is the wedge of two such subspaces X_i, but we will generalize the problem slightly by allowing $Z = X_1 \vee X_2$, where X_1 is a p-sphere or a p-sphere with a $(p+1)$-cell attached by a map of non-zero degree and X_2 is a q-sphere or a q-sphere with a $(q+1)$-cell attached by a map of non-zero degree. We take p, $q \geqslant 2$. We will also consider $H_*(\Omega P)$, where $P = X_1 \times X_2$. We first observe that, for quite arbitrary spaces X_1, X_2, $H_*(\Omega Z)$ and $H_*(\Omega P)$ contain $H_*(\Omega X_1) + H_*(\Omega X_2)$ as a direct summand; we will use the congruence symbol to indicate that we are computing modulo this subgroup.

We prove

Theorem 4.4. *Let $P = X_1 \times X_2$, where $X_1 = S^p \cup e^{p+1}$, e^{p+1} being attached by a map of degree $m \neq 0$, and $X_2 = S^q \cup e^{q+1}$, e^{q+1} being attached by a map of degree $n \neq 0$, $p, q \geqslant 2$. Then $A(P)$ is generated by a_1, a_2, c_1, c_2, b, e, $\bar{e}$, t, corresponding to the cells e^p, e^{p+1}, e^q, e^{q+1}, $e^p \times e^q$, $e^{p+1} \times e^q$, $e^p \times e^{q+1}$, $e^{p+1} \times e^{q+1}$ and d, θ may be chosen on P to give*
$dc_1 = -ma_1$, $dc_2 = -na_2$, $db = (-1)^p(a_1a_2 - (-1)^{(p-1)(q-1)}a_2a_1)$,
$de = -mb + (-1)^{p+1}(c_1a_2 - (-1)^{p(q-1)}a_2c_1)$,
$d\bar{e} = (-1)^{p-1}nb + (-1)^p(a_1c_2 - (-1)^{(p-1)q}c_2a_1)$,
$dt = (-1)^p ne - m\bar{e} + (-1)^{p+1}(c_1c_2 - (-1)^{pq}c_2c_1)$.

Consider $E_1 \times E_2$ and use the same symbols for the generators of $A(E_1 \times E_2)$. Let $f_i : E_i \to X_i$ be characteristic maps, $i = 1, 2$. Then we may take $\varphi_1 a_1 = ma_1$, $\varphi_1 c_1 = c_1$, $\varphi_2 a_2 = na_2$, $\varphi_2 c_2 = c_2$. We will define d on $A(P)$ so that j is a chain mapping, and we will also define an appropriate $\varphi : A(E_1 \times E_2) \to A(P)$, in accordance with corollary 4.1.

The formula for db is already established. The formula proposed for de is a cycle of $A(K_1 \times S_2^q)$ in the kernel of j and hence a boundary; arguing as in theorem 4.1, we see that it is a legitimate choice for de; similarly we justify the formula for $d\bar{e}$. Then it follows, from corollary 4.1, that we may take $\varphi(b) = mnb$, $\varphi(e) = ne$, $\varphi(\bar{e}) = m\bar{e}$. By theorem 3.2 we may now take $\varphi(t) = t$, getting the given formula for dt.

From theorem 4.4 we may calculate $H_*(Z)$ and $H_*(P)$. In particular we consider the injection $H_{p+q-1}(\Omega Z) \to H_{p+q-1}(\Omega P)$. Let h be the $g \cdot c \cdot d$ of m, n, so that $m = hm'$, $n = hn'$. We will restrict attention to the case $p, q \geqslant 3$, though, by complicating the argument, it would be possible to include the cases $p = 2$ or $q = 2$ (or both). With this restriction we have $H_{p+q-1}(\Omega Z) \equiv Z_h + Z_h$, with generators

$$\{\xi\} = \{m' a_1 c_2 + (-1)^p n' c_1 a_2\}, \qquad \{\eta\} = \{n' a_2 c_1 + (-1)^q m' c_2 a_1\} .$$

On the other hand,

$$H_{p+q-1}(\Omega P) \equiv \operatorname{Tor}\left(H_{p-1}(\Omega X_1), H_{q-1}(\Omega X_2)\right) = Z_h ,$$

generated by $\{\xi\}$ or $\{\eta\}$. In fact, we see that

$$\xi - (-1)^{pq}\eta = d\left((-1)^p n' e - m'\bar{e}\right) .$$

It follows that the injection $H_{p+q-1}(\Omega Z) \to H_{p+q-1}(\Omega P)$ is onto $H_{p+q-1}(\Omega P)$ with kernel[25] $\{\xi - (-1)^{pq}\eta\}$.

Consider the diagram

$$\begin{array}{ccc}
\pi_{p+q+1}(P, Z) & \xrightarrow{d} & \pi_{p+q}(Z) \\
\big\downarrow \omega_1 & & \big\downarrow \omega_2 \\
\pi_{p+q}(\Omega P, \Omega Z) & \xrightarrow{d'} & \pi_{p+q-1}(\Omega Z) \\
\big\downarrow h_1 & & \big\downarrow h_2 \\
H_{p+q}(\Omega P, \Omega Z) & \xrightarrow{d''} & H_{p+q-1}(\Omega Z)
\end{array}$$

where ω_i are the usual isomorphisms and h_i are Hurewicz homomorphisms, $i = 1, 2$. Then each square is commutative or anti-commutative and h_1 is onto $H_{p+q}(\Omega P, \Omega Z)$. Moreover d'' maps $H_{p+q}(\Omega P, \Omega Z)$

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[25]) We permit ourselves here and subsequently to identify $A(Z)$ with $C_*(\Omega Z)$, and thus to omit the maps θ, θ_*.

onto Z_h, generated by $\{\xi - (-1)^{pq}\eta\}$. The group

$$\pi_{p+q}(\Omega P, \Omega Z) \cong \pi_{p+q+1}(P, Z)$$

was computed in [3]; we have

$$\begin{aligned} \pi_{p+q}(\Omega P, \Omega Z) &= Z_h\ , && \text{if} \quad h \text{ is } odd\ , \\ &= Z_{2h}\ , && \text{if} \quad h = 4k\ , \\ &= Z_h + Z_2\ , && \text{if} \quad h = 4k+2\ . \end{aligned}$$

If h is even the Z_2 subgroup (direct factor if $h = 4k+2$) is certainly annihilated by $h_2 d'$. Thus there is an element, ϱ, in $\pi_{p+q-1}(\Omega Z)$ which is mapped by h_2 to $\{\xi - (-1)^{pq}\eta\}$; it follows from arguments in [3] that the image of $(d'\omega_1)^{-1}\varrho$ in the Hurewicz homomorphism

$$\pi_{p+q+1}(P, Z) \to H_{p+q+1}(P, Z)$$

generates the latter group which is isomorphic to Tor $(H_p(X_1), H_q(X_2))$.

For further simplicity we now take $m = n$; we leave the slight modifications in the general case to the reader. Let S be any path-connected space and let $\alpha \in \pi_p(S)$, $\beta \in \pi_q(S)$ be elements whose order divides m. Then we may map Z to S by a map g which, restricted to S^p, represents α and, restricted to S^q, represents β. Let $f : S^{p+q} \to Z$ represent $\omega_2^{-1}\varrho$. Then $gf : S^{p+q} \to S$ represents an element $\{\alpha, \beta\} \in \pi_{p+q}(S)$ which is determined modulo the subgroup generated by elements $[\alpha, \varkappa]$, $[\lambda, \beta]$, $\varkappa \in \pi_{q+1}(S)$, $\lambda \in \pi_{p+1}(S)$. Let $u \in A_{p-1}(S)$, $v \in A_{q-1}(S)$ be cycles such that[26]) $\{u\} = h_2\omega_2\alpha$, $\{v\} = h_2\omega_2\beta$, and let $-mu = du'$, $-mv = dv'$. Then $uv' + (-1)^p u'v - (-1)^{pq}(vu' + (-1)^q v'u)$ is a $(p+q-1)$-cycle of $A(S)$ whose homology class is determined modulo the ideal generated by $\{u\}$ and $\{v\}$. We call the element of

$$H_{p+q-1}(\Omega S)/(\{u\}, \{v\})$$

so determined $|\alpha, \beta|$. Since $h_2\omega_2[\alpha, \pi_{q+1}(S)]$ lies in the ideal generated by $\{u\}$ and $h_2\omega_2\ [\pi_{p+1}(S), \beta]$ lies in the ideal generated by $\{v\}$, we may discuss unambiguously the element $h_2\omega_2\ \{\alpha, \beta\}$ in the quotient ring $H_{p+q-1}(\Omega S)/(\{u\}, \{v\})$. It follows by naturality that

$$h_2\omega_2\{\alpha, \beta\} = |\alpha, \beta|\ ,$$

the space Z being a universal example for the construction $\{\alpha, \beta\}$.

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[26]) We use ω_2, h_2 for the maps $\pi_r(Y) \to \pi_{r-1}(\Omega Y)$, $\pi_{r-1}(\Omega Y) \to H_{r-1}(\Omega Y)$ for any r and any space Y. See also the previous footnote.

A direct computation shows that

$$(-1)^{pr} \| \alpha, \beta |, \gamma | + (-1)^{qp} \| \beta, \gamma |, \alpha | + (-1)^{rq} \| \gamma, \alpha |, \beta | = 0 ,$$

where $\gamma \in \pi_r(S)$, $r \geqslant 3$, $m\gamma = 0$ and the calculation is made in $H_{p+q+r-1}(\Omega S)$ modulo the ideal generated by $\{u\}$, $\{v\}$, and $\{w\} = h_2 \omega_2 \gamma$.

Note added in proof. W. S. Massey [Annals of Mathematics 62 (1955) p. 327] has raised (as problem 18) the question of homotopy operations of higher kinds. It is clear that the product $\{\alpha, \beta\}$ introduced above is an operation of the sort indicated.

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