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On the singularities of DIRICHLET series

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1. Introduction. Let us put

$$F(s) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n s) \quad (s = \sigma + it, \ 0 \leqq \lambda_1 < \lambda_2 < \cdots < \lambda_n \to +\infty) \ . \qquad (1.1)$$

The object of this note is to establish the following theorems:

Theorem 1. *Let (1.1) have the finite simple convergence-abscissa σ_s. Then there exists a sequence $\{\varepsilon_n\}$ ($\varepsilon_n = \pm 1$) such that*

$$\sum_{n=1}^{\infty} \varepsilon_n a_n \exp(-\lambda_n s)$$

has $\sigma = \sigma_s$ as the natural boundary.

Theorem 2. *Let (1.1) have the finite simple convergence-abscissa σ_s. Then there exists a new DIRICHLET series $\sum_{n=1}^{\infty} b_n \exp(-\lambda_n s)$ having $\sigma = \sigma_s$ as the natural boundary such that*

$$|b_n| = |a_n| \, (n = 1, 2, \ldots), \qquad \lim_{n\to\infty} |\arg(b_n) - \arg(a_n)| = 0 \qquad \text{(i)}$$

or

$$\arg(b_n) = \arg(a_n) \quad (n = 1, 2, \ldots), \qquad \lim_{n\to\infty} |b_n/a_n| = 1 \ . \qquad \text{(ii)}$$

Under the additional conditions $\lim_{n\to\infty} \log n/\lambda_n = 0$, these two theorems have been proved by O. SZASZ ([1], p. 107) and the author ([2], p. 308) respectively. The method of its proofs is based upon A. OSTROWSKI's criterion of singularities.

2. Lemmas. To establish these theorems, we need some lemmas.

Lemma 1. (A. OSTROWSKI [3, 4], pp. 12–16.) *Let (1.1) have the simple convergence-abscissa $\sigma_s = 0$. For $s = 0$ to be the singular point of (1.1), it is necessary and sufficient that we have*

$$\overline{\lim_{m\to\infty}} |O_m(\sigma, \omega)|^{1/m} \geqq 1 \qquad (m: \textit{positive integer}),$$

where

$$\sigma(>0), \qquad \omega(0<\omega<1): \textit{fixed constants}, \qquad \text{(i)}$$

$$O_m(\sigma, \omega) = \sum_{m/\sigma\cdot(1-\omega) \leqq \lambda_n \leqq m/\sigma\cdot(1+\omega)} a_n (\lambda_n e \sigma/m)^m \exp(-\lambda_n \sigma) \ . \qquad \text{(ii)}$$

Lemma 2. *The simple convergence-abscissa* σ_s *of* (1.1) *is determined by*

$$\sigma_s = \overline{\lim_{x\to+\infty}}\, 1/x\cdot\log\Big|\sum_{[x]+f(x)\leqq\lambda_n<x} a_n\Big| ,$$

where

$$[x]:\ \textit{GAUSS's symbol}, \tag{i}$$

$$f(x) = 1/m\cdot[m(x-m)] , \qquad m = [x] . \tag{ii}$$

Proof. By T. KOJIMA's theorem [5], σ_s is given by

$$\sigma_s = \overline{\lim_{x\to+\infty}}\, 1/x\cdot\log\Big|\sum_{[x]\leqq\lambda_n<x} a_n\Big|$$
$$= \overline{\lim_{x\to+\infty}}\, 1/[x]\cdot\log\Big|\sum_{[x]\leqq\lambda_n<x} a_n\Big| .$$

Hence, for any given ε (>0),

$$\Big|\sum_{[x]\leqq\lambda_n<x} a_n\Big| < \exp\{[x](\sigma_s+\varepsilon)\} \quad \text{for} \quad [x]>K(\varepsilon) .$$

Therefore

$$\Big|\sum_{[x]+f(x)\leqq\lambda_n<x} a_n\Big| = \Big|\sum_{[x]\leqq\lambda_n<x} - \sum_{[x]\leqq\lambda_n<x+f(x)}\Big|$$
$$< 2\exp\{[x](\sigma_s+\varepsilon)\} \quad \text{for} \quad [x]>K(\varepsilon) ,$$

so that

$$\sigma_s^* = \overline{\lim_{x\to+\infty}}\, 1/x\cdot\log\Big|\sum_{[x]+f(x)\leqq\lambda_n<x} a_n\Big| \leqq \sigma_s + \varepsilon .$$

Letting $\varepsilon\to 0$,

$$\sigma_s^* \leqq \sigma_s . \tag{2.1}$$

Putting

$$\begin{cases} x = m + y/m \quad (m=[x],\ 0<y<m) , \\ [x] + f(x) = m + [y]/m , \end{cases}$$

we have easily

$$\sigma_s^* = \overline{\lim_{\substack{m\to+\infty\\0<y<m}}}\, 1/m\cdot\log\Big|\sum_{m+[y]/m\leqq\lambda_n<m+y/m} a_n\Big| \qquad (m:\ \text{positive integer}).$$

Hence, for any given ε (>0),

$$\Big|\sum_{\substack{m+[y]/m\leqq\lambda_n<m+y/m\\0<y<m}} a_n\Big| < \exp\{m(\sigma_s^*+\varepsilon)\} \quad \text{for} \quad m>K^*(\varepsilon) .$$

Accordingly

$$\Big|\sum_{[x]\leqq\lambda_n<x} a_n\Big|$$
$$= \Big|\sum_{m\leqq\lambda_n<m+1/m} + \sum_{m+1/m\leqq\lambda_n<m+2/m} + \cdots + \sum_{m+[y]/m\leqq\lambda_n<m+y/m}\Big|$$
$$\leqq ([y]+1)\exp\{m(\sigma_s^*+\varepsilon)\} \leqq m\exp\{m(\sigma_s^*+\varepsilon)\} \quad \text{for} \quad m>K^*(\varepsilon) ,$$

so that

$$\sigma_s = \overline{\lim_{x \to +\infty}} \, 1/x \cdot \log \Big| \sum_{[x] \leqq \lambda_n < x} a_n \Big| \leqq \sigma_s^* + \varepsilon \ .$$

Letting $\varepsilon \to 0$,

$$\sigma_s \leqq \sigma_s^* \ . \tag{2.2}$$

Combining (2.1) with (2.2), lemma 2 is completely established.

Lemma 3. *Put*

$$O(x,t) = \sum_{[x]+f(x) \leqq \lambda_n < x} a_n (\lambda_n e/[x])^{[x]} \exp(-\lambda_n(1+it)) \ .$$

If

$$\Big| \sum_{[x]+f(x) \leqq \lambda_n < x} a_n \Big| \geqq \Big| \sum_{[x]+f(x) \leqq \lambda_n < X} a_n \Big|$$

for every X *such that* $[x] + f(x) < X \leqq x$, *then*

$$|O(x,t)| \geqq \Big| \sum_{[x]+f(x) \leqq \lambda_n < x} a_n \Big| \cdot \{1 - (|t| + K(x))/[x]\} \ ,$$

where $K(x) = 0(1)$.

Proof. Put

$$\begin{cases} x = m + y/m \qquad (m = [x], \ 0 < y < m) \ , \\ [x] + f(x) = m + [y]/m \ . \end{cases}$$

Let us denote by $\{\lambda_{m,i}\}$ $(i = 1,2,\ldots,r)$ the exponents λ_n's contained in $m + [y]/m \leqq \lambda_n < m + y/m$ and by $\{a_{m,i}\}$ the coefficient corresponding to $\lambda_{m,i}$. Setting

$$S_k = \sum_{i=1}^{k} a_{m,i} \quad (1 \leqq k \leqq r), \qquad f(\lambda) = \lambda^m \exp(-\lambda(1+it)) \ ,$$

then by ABEL's transformation

$$\begin{aligned} O(x,t) &= (e/m)^m \cdot \sum_{i=1}^{r} a_{m,i} f(\lambda_{m,i}) \\ &= (e/m)^m \cdot \{S_r f(\lambda_{m,r}) + \sum_{i=1}^{r-1} S_i (f(\lambda_{m,i}) - f(\lambda_{m,i+1}))\} \ . \end{aligned}$$

Since

$$|S_r| \geqq |S_i| \qquad (i = 1,2,\ldots,r-1) \ ,$$

we have

$$|O(x,t)| \geqq (e/m)^m \cdot |S_r| \cdot \{|f(\lambda_{m,r})| - \sum_{i=1}^{r-1} |f(\lambda_{m,i}) - f(\lambda_{m,i+1})|\} \ . \tag{2.3}$$

On the other hand, $|f(\lambda)| = \lambda^m \exp(-\lambda)$ is monotone-decreasing for $m \leqq \lambda$. Hence, for $m \leqq \alpha < \beta < m + 1$,

$$|f(\alpha) - f(\beta)| \leqq |f(\alpha)| - |f(\beta)| + |f(\beta)| (\beta - \alpha) |t| \ ,$$

so that

$$|f(\alpha)-f(\beta)|<\int_{\alpha}^{\beta} z^m \exp(-z)\cdot(|t|+1-m/z)dz \ . \tag{2.4}$$

Therefore, by (2.4)

$$\sum_{i=1}^{r-1}|f(\lambda_{m,i})-f(\lambda_{m,i+1})|<\int_{\lambda_{m,1}}^{\lambda_{m,r}} z^m \exp(-z)\cdot(|t|+1-m/z)dz$$
$$<(|t|+1)\cdot\int_{m+[y]/m}^{m+([y]+1)/m} z^m \exp(-z)dz \ .$$

Hence, by (2.3) and $[y]+1\leqq m$

$$|O(x,t)|$$
$$\geqq|S_r|(e/m)^m\{(m+([y]+1)/m)^m\cdot\exp(-(m+([y]+1)/m))$$
$$-(|t|+1)\cdot\int_{m+[y]/m}^{m+([y]+1)/m} z^m\exp(-z)dz\}$$
$$>|S_r|\{(1+([y]+1)/m^2)^m\cdot\exp(-([y]+1)/m)-(|t|+1)\cdot 1/m\}$$
$$>|S_r|\{\exp(-0(1/m))-(|t|+1)\cdot 1/m\}=|S_r|\cdot\{1-(|t|+0(1))/m\} \ ,$$

which proves lemma 3.

Lemma 4. *Let* (1.1) *have the simple convergence-abscissa* $\sigma_s=0$. *Then there exists a sequence* $\{x_\nu\}$ $([x_\nu]\uparrow+\infty)$ *independent of* t $(-\infty<t<+\infty)$ *such that*

$$\left.\begin{array}{ll}
\text{(a)} & [x_\nu](1+\omega)<[x_{\nu+1}](1-\omega) \qquad (\nu=1,2,\dots,0<\omega<1)\ ,\\
\text{(b)} & \lim\limits_{\nu\to+\infty} 1/x_\nu\cdot\log|\sum\limits_{[x_\nu]+f(x_\nu)\leqq\lambda_n<x_\nu} a_n|=0\ ,\\
\text{(c)} & \varlimsup\limits_{\nu\to\infty}|O(x_\nu,t)|^{1/[x_\nu]}\leqq 1 \quad \textit{for arbitrary} \quad t(-\infty<t<+\infty)\ .
\end{array}\right\}$$

Proof. By lemma 2, we can find a sequence $\{y_\nu\}$ $([y_\nu]\uparrow+\infty)$ such that

$$\left.\begin{array}{l}
[y_\nu](1+\omega)<[y_{\nu+1}](1-\omega) \qquad (\nu=1,2,\dots,0<\omega<1)\ ,\\
\lim\limits_{\nu\to\infty} 1/y_\nu\cdot\log|\sum\limits_{[y_\nu]+f(y_\nu)\leqq\lambda_n<y_\nu} a_n|=0\ .
\end{array}\right\} \tag{2.5}$$

Put

$$\operatorname*{Max}_{[y_\nu]+f(y_\nu)\leqq x\leqq y_\nu}|\sum_{[x]+f(x)\leqq\lambda_n<x} a_n|=|\sum_{[x_\nu]+f(x_\nu)\leqq\lambda_n<x_\nu} a_n|\ ,$$

where $[x_\nu]=[y_\nu]$, $f(x_\nu)=f(y_\nu)$. Since

$$0=\lim_{\nu\to\infty} 1/y\cdot\log|\sum_{[y_\nu]+f(y_\nu)\leqq\lambda_n<y_\nu} a_n|\leqq\varlimsup_{\nu\to\infty} 1/x\cdot\log|\sum_{[x_\nu]+f(x_\nu)\leqq\lambda_n<x_\nu} a_n|$$
$$\leqq\varlimsup_{x\to+\infty} 1/x\cdot\log|\sum_{[x]+f(x)\leqq\lambda_n<x} a_n|=0\ ,$$

by (2.5) and (2.6), selecting a suitable subsequence, if necessary, we have easily

$$\left.\begin{array}{l} [x_\nu](1+\omega)<[x_{\nu+1}](1-\omega) \quad (\nu=1,2,\ldots,0<\omega<1)\ , \\ \lim\limits_{\nu\to\infty} 1/x_\nu\cdot\log\Big|\sum\limits_{[x_\nu]+f(x_\nu)\leqq\lambda_n<x_\nu} a_n\Big|=0\ , \\ \Big|\sum\limits_{[x_\nu]+f(x_\nu)\leqq\lambda_n<x_\nu} a_n\Big| \geqq \Big|\sum\limits_{[x_\nu]+f(x_\nu)\leqq\lambda_n<X} a_n\Big| \\ \text{for every } X \text{ such that } [x_\nu]+f(x_\nu)<X\leqq x_\nu\ . \end{array}\right\} \tag{2.7}$$

On account of (2.7) and lemma 3

$$\left.\begin{array}{l} [x_\nu](1+\omega)<[x_{\nu+1}](1-\omega) \quad (\nu=1,2,\ldots,0<\omega<1)\ , \\ \lim\limits_{\nu\to\infty} 1/x_\nu\cdot\log\Big|\sum\limits_{[x_\nu]+f(x_\nu)\leqq\lambda_n<x_\nu} a_n\Big|=0\ , \\ \varliminf\limits_{\nu\to\infty} |O(x_\nu,t)|^{1/[x_\nu]} \geqq \lim\limits_{\nu\to\infty}\Big|\sum\limits_{[x_\nu]+f(x_\nu)\leqq\lambda_n<x_\nu} a_n\Big|^{1/[x_\nu]}=1 \\ \text{for arbitrary } t\ (-\infty<t<+\infty)\ , \end{array}\right\}$$

which is to be proved.

Lemma 5. *Let* (1.1) *have the simple convergence-abscisse* $\sigma_s=0$. *Then, for any sequence* $\{x_\nu\}$ $([x_\nu]\uparrow+\infty)$, $\sum\limits_{\nu=1}^{+\infty} u_\nu(s)$ *converges absolutely for* $\sigma>0$, *where*

$$u_\nu(s)=\sum_{[x_\nu]+f(x_\nu)\leqq\lambda_n<x_\nu} a_n\exp(-\lambda_n s)\ .$$

Proof. By lemma 2

$$0=\varlimsup_{x\to+\infty} 1/x\cdot\log\Big|\sum_{[x]+f(x)\leqq\lambda_n<x} a_n\Big| = \varlimsup_{x\to+\infty} 1/[x]\cdot\log\Big|\sum_{[x]+f(x)\leqq\lambda_n<x} a_n\Big|\ .$$

Hence, for any given ε (>0)

$$\Big|\sum_{[x]+f(x)\leqq\lambda_n<x} a_n\Big|<\exp(\varepsilon[x]) \quad \text{for} \quad [x]>X(\varepsilon)\ . \tag{2.8}$$

Using the same notations as in lemma 3, by ABEL's transformation

$$\begin{aligned} u_\nu(s) &= \sum_{i=1}^{r_\nu} a_{m_\nu,i}\exp(-\lambda_{m_\nu,i}s) \\ &= S_{r_\nu}\exp(-\lambda_{m_\nu,r_\nu}s) \\ &\quad+\sum_{i=1}^{r_\nu-1} S_i\{\exp(-\lambda_{m_\nu,i}s)-\exp(-\lambda_{m_\nu,i+1}s)\}\ , \end{aligned}$$

where
$$x_\nu = m_\nu + y_\nu/m_\nu \quad (m_\nu = [x_\nu]\ ,\quad 0<y_\nu<m_\nu)\ ,$$
$$[x_\nu] + f(x_\nu) = m_\nu + [y_\nu]/m_\nu\ .$$
Since
$$|\exp(-\alpha s) - \exp(-\beta s)| \leqq |s|/\sigma\cdot\{\exp(-\alpha\sigma) - \exp(-\beta\sigma)\}$$
for $\alpha<\beta$, $\sigma>0$, by (2.8)
$$|u_\nu(s)| < |s|/\sigma\cdot\exp(\varepsilon[x_\nu]\cdot\{\exp(-\lambda_{m_\nu, r_\nu}\sigma)$$
$$+\sum_{i=1}^{r_\nu-1}\exp(-\lambda_{m_\nu, i}\sigma) - \exp(-\lambda_{m_\nu, i+1}\sigma)\}$$
$$\leqq |s|/\sigma\cdot\exp([x_\nu](\varepsilon-\sigma)) \quad \text{for} \quad \sigma>0,\quad [x_\nu]>X(\varepsilon)\ .$$
Therefore, putting $\varepsilon = \sigma/2$
$$|u_\nu(s)| \leqq |s|/\sigma\cdot\exp(-\sigma/2\cdot[x_\nu])\ ,$$
so that $\sum_{\nu=1}^{+\infty} u_\nu(s)$ is absolutely convergent for $\sigma>0$. q. e. d.

Lemma 6. *Let* (1.1) *have the simple convergence-abscissa* $\sigma_s = 0$.

(A) *Let us put*
$$\sum_{\nu=1}^{+\infty}\delta(m_\nu)u_\nu(s) = \sum_{n=1}^{+\infty}\varepsilon_n a_n \exp(-\lambda_n s)\ , \tag{2.9}$$
where

(a) $\overline{\lim}_{\nu\to+\infty} 1/x_\nu\cdot\log|\sum_{[x_\nu]+f(x_\nu)\leqq\lambda_n<x_\nu} a_n| = 0$ [1] $\quad ([x_\nu] = m_\nu \uparrow +\infty)$

(b) $\lim_{x\to+\infty} 1/x\cdot\log|\delta(x)| = 0$,

(c) $u_\nu(s) = \sum_{[x_\nu]+f(x_\nu)\leqq\lambda_n<x_\nu} a_n\exp(-\lambda_n s)$

(d) $\begin{cases}\varepsilon_n = \delta(m_\nu) & \textit{for} \quad \lambda_n\in\{I_\nu\} \quad (I_\nu\colon [x_\nu]+f(x_\nu)\leqq x<x_\nu) \\ \varepsilon_n = 0 & \textit{for} \quad \lambda_n\bar{\in}\{I_\nu\}\ .\end{cases}$

Then (2.9) *has the simple convergence-abcissa* $\sigma = 0$.

(B) *Set*
$$f_0(s) + \sum_{\nu=1}^{+\infty}\delta(m_\nu)u_\nu(s) = \sum_{n=1}^{+\infty}\varepsilon'_n a_n\exp(-\lambda_n s)\ , \tag{2.10}$$
where

(e) $f_0(s) = \sum_{\lambda_n\bar{\in}\{I_\nu\}} a_n\exp(-\lambda_n s)$

(f) $\begin{cases}\varepsilon'_n = \delta(m_\nu) & \textit{for} \quad \lambda_n\in\{I_\nu\}\ , \\ \varepsilon'_n = 1 & \textit{for} \quad \lambda_n\bar{\in}\{I_\nu\}\ .\end{cases}$

Then (2.10) *has also the simple convergence-abscissa* $\sigma = 0$.

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[1]) By lemma 1, there exists a sequence $\{x_\nu\}$ satisfying (a).

Proof. (A) By lemma 2

$$\varlimsup_{x\to+\infty} 1/x\cdot\log\Big|\sum_{[x]+f(x)\leqq\lambda_n<x} a_n\Big| = 0\ . \tag{2.11}$$

Taking account of (a), (b), (c) and (d), we get

$$\varlimsup_{\substack{x\to+\infty\\ [x]\neq m_\nu}} 1/x\cdot\log\Big|\sum_{[x]+f(x)\leqq\lambda_n<x} \varepsilon_n a_n\Big| = -\infty\ ,$$

$$\varlimsup_{\substack{x\to+\infty\\ [x]=m_\nu\\ x_\nu\leqq x<m_\nu+1}} 1/x\cdot\log\Big|\sum_{[x]+f(x)\leqq\lambda_n<x} \varepsilon_n a_n\Big| = \varlimsup_{\nu\to+\infty} 1/x_\nu\cdot\log\Big\{|\delta(m_\nu)|\cdot\Big|\sum_{m_\nu+f(x_\nu)\leqq\lambda_n<x_\nu} a_n\Big|\Big\},$$
$$= 0\ ,$$

$$\varlimsup_{\substack{x\to+\infty\\ [x]=m_\nu\\ m_\nu+f(x_\nu)<x<x_\nu}} 1/x\cdot\log\Big|\sum_{[x]+f(x)\leqq\lambda_n<x} \varepsilon_n a_n\Big| = \varlimsup_{\nu\to+\infty} 1/x_\nu\cdot\log\Big\{|\delta(m_\nu)|\cdot\Big|\sum_{[x_\nu+f(x_\nu)]\leqq\lambda_n<x} a_n\Big|\Big\},$$
$$\leqq 0 \quad (\text{by } (2.11))\ ,$$

$$\varlimsup_{\substack{x\to+\infty\\ [x]=m_\nu\\ m_\nu\leqq x\leqq m_\nu+f(x_\nu)}} 1/x\cdot\log\Big|\sum_{[x]+f(x)\leqq\lambda_n<x} \varepsilon_n a_n\Big| = -\infty\ ,$$

so that

$$\varlimsup_{x\to+\infty} 1/x\cdot\log\Big|\sum_{[x]+f(x)\leqq\lambda_n<x} \varepsilon_n a_n\Big| = 0\ ,$$

which proves that the simple convergence-abscissa of (2.9) is $\sigma=0$.

(B) Since $f_0(s)=f(s)-\sum_{\nu=1}^{+\infty}u_\nu(s)$, by lemma 5 $f_0(s)$ is simply convergent at least for $\sigma>0$. On account of (A), $\sum_{\nu=1}^{+\infty}\delta(m_\nu)\cdot u_\nu(s)=\sum_{n=1}^{+\infty}\varepsilon_n a_n\exp(-\lambda_n s)$ is simply convergent exactly for $\sigma>0$, so that (2.10) is simply convergent at least for $\sigma>0$. In other words, by lemma 2

$$\varlimsup_{x\to+\infty} 1/x\cdot\log\Big|\sum_{[x]+f(x)\leqq\lambda_n<x} \varepsilon'_n a_n\Big| \leqq 0\ .$$

On the other hand, by (a), (b), and (f)

$$\varlimsup_{\nu\to+\infty} 1/x_\nu\cdot\log\Big|\sum_{[x_\nu]+f(x_\nu)\leqq\lambda_n<x_\nu} \varepsilon'_n a_n\Big| = \varlimsup_{\nu\to+\infty} 1/x_\nu\cdot\log\Big\{|\delta(m_\nu)|\cdot\Big|\sum_{[x_\nu]+f(x_\nu)\leqq\lambda_n<x_\nu} a_n\Big|\Big\}\ .$$
$$= 0\ .$$

Therefore

$$\varlimsup_{x\to+\infty} 1/x\cdot\log\Big|\sum_{[x]+f(x)\leqq\lambda_n<x} \varepsilon'_n a_n\Big| = 0\ ,$$

which proves that (2.10) has also the simple convergence-abscissa $\sigma=0$.
q. e. d.

3. Proofs of theorems

Proof of theorem 1. Without any loss of generality, we can assume $\sigma_s = 0$. Let us denote by $\{x_\nu\}$ the sequence of lemma 4. Then, by lemma 5, we can put

$$\sum_{\nu=1}^{+\infty} u_\nu(s) = f_1(s) + f_2(s) + f_3(s)\cdots + f_n(s) + \cdots ,$$

where

(i) $u_\nu(s) = \sum_{[x_\nu]+f(x_\nu)\leqq\lambda_n<x_\nu} a_n \exp(-\lambda_n s)$,

(ii) each $f_n(s)$ $(n = 1, 2, \ldots)$ contains infinite number of $\{u_\nu(s)\}$.

For any given sequence $\{\delta_n\}$ $(\delta_n = \pm 1)$, set

$$\begin{aligned} f(s:\{\delta_n\}) &= f_0(s) + \delta_1 f_1(s) + \cdots + \delta_n f_n(s) + \cdots \\ &= f_0(s) + \sum_{\nu=1}^{+\infty} \alpha_\nu u_\nu(s) , \\ &= \sum_{n=1}^{+\infty} \varepsilon'_n s_n \exp(-\lambda_n s) , \end{aligned} \tag{3.1}$$

where

(i) $f_0(s) = \sum_{\lambda_n \bar{\in} \{I_\nu\}} a_n \exp(-\lambda_n s)$ $\quad (I_\nu:\ [x_\nu] + f(x_\nu) \leqq x < x_\nu)$,

(ii) $\alpha_\nu = \pm 1$,

(iii) $\varepsilon'_n = \alpha_\nu$ for $\lambda_n \in \{I_\nu\}$,
$\varepsilon'_n = 1$ for $\lambda_n \bar{\in} \{I_\nu\}$.

Then, by lemma 6 (B), (3.1) has the simple convergence-abscissa $\sigma = 0$.

For $\{\delta_n\} \not\equiv \{\delta'_n\} (\delta_n, \delta'_n = \pm 1)$, we have

$$\left.\begin{aligned} f(s:\{\delta_n\}) - f(s:\{\delta'_n\}) &= \sum_{n=1}^{+\infty} \delta''_n f_n(s) \\ &= \sum_{\nu=1}^{+\infty} \pm 2\cdot u^*_\nu(s) \\ &= \sum_{n=1}^{+\infty} \varepsilon_n a_n \exp(-\lambda_n s) , \end{aligned}\right\} \tag{3.2}$$

where

(i) $\delta''_n = \delta_n - \delta'_n$ (Since $\{\delta_n\} \not\equiv \{\delta'_n\}$, there exists at least one n such that $\delta''_n = \pm 2$),

(ii) $u^*_\nu(r) = \sum_{[x'_\nu]+f(x'_\nu)\leqq\lambda_n<x'_\nu} a_n \exp(-\lambda_n s)$ $\quad (\{x'_\nu\}$: subsequence of $\{x_\nu\})$,

(iii) $\varepsilon_n = \pm 2$ for $\lambda_n \in \{I'_\nu\}$ $\quad (I'_\nu:\ [x'_\nu] + f(x'_\nu) \leqq x < x'_\nu)$
$\varepsilon_n = 0$ for $\lambda_n \bar{\in} \{I_\nu\}$.

On account of lemma 4 (b) and lemma 6 (A), (3.2) has also the simple convergence-abscissa $\sigma = 0$.

(3.2) has $\sigma = 0$ as the natural boundary. To prove this, by lemma 1, it is sufficient to show that

$$\overline{\lim_{m\to+\infty}} \, |O_m(1,\omega\,;t)|^{1/m} \geqq 1 \quad \text{for arbitrary } t \quad (-\infty<t<+\infty)\ , \tag{3.3}$$

where

$$O_m(1,\omega\,;t) = \sum_{m(1-\omega)\leqq\lambda_n\leqq m(1+\omega)} \varepsilon_n a_n (\lambda_n e/m)^m \cdot \exp(-\lambda_n(1+it))\ .$$

Since $O_{[x'_\nu]}(1,\omega\,;t) = \pm 2O(x'_\nu,t)$ by lemma 4 (a), we have by lemma 4 (c)

$$\begin{aligned}\overline{\lim_{m\to\infty}}\,|O_m(1,\omega\,;t)|^{1/m} &\geqq \overline{\lim_{\nu\to+\infty}}\,|O_{[x'_\nu]}(1,\omega\,;t)|^{1/[x'\nu]} \\ &= \overline{\lim_{\nu\to+\infty}}\,|O(x'_\nu,t)|^{1/[x'\nu]} \\ &\geqq \underline{\lim_{\nu\to+\infty}}\,|O(x_\nu,t)|^{1/[x\nu]} \geqq 1\ ,\end{aligned}$$

which proves (3.3).

Let us denote by $E(\{\delta_n\})$ the set of regular points on $\sigma = 0$ of (3.1), which is evidently an open set. Then we have

$$E(\{\delta_n\})\frown E(\{\delta'_n\}) = 0 \quad \text{for} \quad \{\delta_n\} \not\equiv \{\delta'_n\}\ . \tag{3.4}$$

In fact, if $s_0 \epsilon E(\{\delta_n\})\frown E(\{\delta'_n\}) \neq 0$, (3.2), would be regular at $s = s_0$, which is impossible by (3.3). If $E(\{\delta_n\}) \neq 0$ for all $\{\delta_n\}$, by (3.4) the set of all functions $\{f(s:\{\delta_n\})\}$ is at most enumerable, which contradicts the power of continuum of the set of all $\{\delta_n\}$. Hence $E(\{\delta_n\})=0$ for at least one $\{\delta_n\}$, which shows that $f(s:\{\delta_n\})$ has $\sigma = 0$ as the natural boundary. (3.1) is evidently of the form $\sum_{n=1}^{+\infty} \varepsilon'_n a_n \exp(-\lambda_n s)$ $(\varepsilon'_n = \pm 1)$. q. e. d.

Proof of theorem 2. Without any loss of generality, we can assume $\sigma_s = 0$. $\sigma_s = 0$. Let us denote by $\{x_\nu\}$ the sequence of lemma 4. Let us put

$$f(s,\theta,\alpha) = f_0(s) + f_r(s,\theta,\alpha) = \sum_{n=1}^{\infty} \varepsilon'_n a_n \exp(-\lambda_n s)\ , \tag{3.5}$$

where

(i) $\quad f_0(s) = \sum_{\lambda_n \overline{\epsilon}\{I\nu\}} a_n \exp(-\lambda_n s) \quad (I_\nu:\ [x_\nu] + f(x_\nu) \leqq x < x_\nu)\ ,$

(ii) $\quad f_1(s,\theta,\alpha) = \sum_{\nu=1}^{+\infty} \exp(\alpha\theta/[x_\nu]) \cdot u_\nu(s)\ ,$

$\quad u_\nu(s) = \sum_{\lambda_n \epsilon I_\nu} a_n \exp(-\lambda_n s)\ ,$

$\quad \theta$: a real constant, α: a parameter $(=\sqrt{-1}$ or 1),

$$\text{(iii)} \quad \varepsilon'_n = \exp(\alpha\theta/[x_\nu]) \quad \text{for} \quad \lambda_n \in \{I_\nu\} ,$$
$$\varepsilon'_n = 1 \quad \text{for} \quad \lambda_n \bar{\in} \{I_\nu\} .$$

Then, by lemma 6 (B), (3.5) has the simple convergence-abscissa $\sigma = 0$. Put

$$f(s,\theta,\alpha) - f(s,\theta',\alpha) \tag{3.6}$$
$$= \sum_{\nu=1}^{+\infty} \{\exp(\alpha\theta/[x_\nu]) - \exp(\alpha\theta'/[x_\nu])\} \cdot u_\nu(s)$$
$$= \sum_{n=1}^{+\infty} \varepsilon_n a_n \exp(-\lambda_n s) \quad \text{for} \quad \theta \neq \theta' ,$$

where

$$\varepsilon_n = 0 \quad \text{for} \quad \lambda_n \bar{\in} \{I_\nu\} ,$$
$$\varepsilon_n = \exp(\alpha\theta/[x_\nu]) - \exp(\alpha\theta'/[x_\nu])$$
$$= \alpha(\theta - \theta')/[x_\nu] + 0(1/x_\nu^2) \quad \text{for} \quad \lambda_n \in \{I_\nu\} .$$

Then, by lemma 4 (b) and lemma 6 (A), (3.6) has also the simple convergence-abscissa $\sigma = 0$.

(3.6) has $\sigma = 0$ as the natural boundary. To establish this fact, by lemma 1, it is sufficient to show that

$$\overline{\lim_{m\to\infty}} |O_m(1,\omega,t)|^{1/m} \geqq 1 \quad \text{for arbitrary } t \quad (-\infty < t < +\infty) . \tag{3.7}$$

Since

$$O_{[x\nu]}(1,\omega,t) = \{\alpha(\theta - \theta')/[x_\nu] + O(1/x_\nu^2)\} \cdot O(x_\nu, t) ,$$

by lemma 4 (c)

$$\overline{\lim_{m\to\infty}} |O_m(1,\omega,t)|^{1/m} \geqq \overline{\lim_{\nu\to\infty}} |O_{[x\nu]}(1,\omega,t)|^{1/[x\nu]}$$
$$= \overline{\lim_{\nu\to\infty}} |O(x_\nu,t)|^{1/[x\nu]}$$
$$\geqq \underline{\lim_{\nu\to\infty}} |O(x_\nu,t)|^{1/[x\nu]} \geqq 1 ,$$

from which follows (3.7).

Let us denote by $E(\theta,\alpha)$ the set of regular points on $\sigma = 0$ of (3.5), which is an open set. Then we have

$$E(\theta,\alpha) \frown E(\theta',\alpha) = 0 \quad \text{for} \quad \theta \neq \theta' . \tag{3.8}$$

In fact, if $s_0 \in E(\theta,\alpha) \frown E(\theta',\alpha) \neq 0$ (3.6), would be regular at $s = s_0$, which is impossible by (3.7). If $E(\theta,\alpha) \neq 0$ for all θ $(0 < \theta < \gamma,\ \gamma:$ a fixed constant), by (3.8) the set of all functions $\{f(s,\theta,\alpha)\}$ is at most enumerable, which contradicts the power of continuum of the set of θ. Hence $E(\theta,\alpha)=0$ for at least one θ, which shows that $f(s,\theta,\alpha)$ has $\sigma=0$ as the natural boundary. If $\alpha=\sqrt{-1}$ (or $=1$), (i) (or (ii)) of theorem 2 is established. q. e. d.

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