Cohomology Operations
Derived from the Symmetric Group [1]

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Dedicated to H. Hopf

1. Introduction

A *cohomology operation*, relative to dimensions q, r and coefficient groups A, B, is a function h which, for each space X, maps the cohomology group $H^q(X; A)$ into $H^r(X; B)$ so as to commute with the homomorphisms induced by mappings of spaces. Examples are the squaring operations

$$\mathrm{Sq}^i : H^q(X^r;\ Z_2) \rightarrow H^{q+i}(X; Z_2) \ ,$$

and the cyclic reduced p^{th} powers

$$\mathcal{P}^i_p : H^q(X; Z_p) \rightarrow H^{q+2i(p-1)}(X^r;\ Z_p)$$

which I have defined elsewhere [13, 14]. Both are defined for all $q, i \geqq 0$; p denotes a prime, and Z_p denotes the integers modulo p. Cohomology operations not only yield new topological invariants, but are of vital importance in solving extension problems, and in homotopy classification. The known operations have been used successfully in diverse situations; but there remain many problems for which they are inadequate.

Several years ago I found a connection between the reduced powers and another development of algebraic topology, namely, the homology theory of groups (initiated by HOPF [11] and, independently, by EILENBERG and MACLANE [8]). Stated roughly, the operations $\mathcal{P}^i_p$ are homology classes of the cyclic group of permutations of degree p. More generally, it was found that each homology class of each permutation group determines a cohomology operation. This gave a potential wealth of new operations. It soon became clear that there were many relations of dependence among them. For example a homology class of a permutation group of degree n gives the same operation as does its image in the homology of the symmetric group $\mathcal{S}(n)$.

This paper contains two main results. The first is an improvement in the construction of reduced power operations based on $\mathcal{S}(n)$. This gives more coho-

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mology operations than the previous construction. In particular it gives the PONTRJAGIN squaring operation (see [18])

$$\mathfrak{p}_2 : \; H^{2q}(X;Z_{2m}) \to H^{4q}(X;Z_{4m}) \; .$$

The second main result asserts, roughly, that all cohomology operations based on $\mathcal{S}(n)$ are generated by those based on the cyclic permutation groups of prime orders and degrees. A precise statement is given in § 4.

In a subsequent paper by P. E. THOMAS and myself, the above result will be refined by an analysis of cohomology operations based on permutation groups of prime orders and degrees. These are relatively few, namely, Sq^i, $\mathfrak{p}_2$, $\mathcal{P}^i_p$ as described above, and the generalizations of the PONTRJAGIN square to primes $p > 2$ which were found by THOMAS [16]. These are functions

$$\mathfrak{p}_p : \; H^{2q}(X;Z_{pm}) \to H^{2pq}(X;Z_{p^2m}) \; .$$

J. ADEM [1, 2, 3] and H. CARTAN [5] have shown, independently, that Sq^i and $\mathcal{P}^i_p$ satisfy certain relations when iterated. These relations have yielded useful results: This raises the problem of determining a basis for *all* relations satisfied by the four types of basic operations listed above.

Serre has shown [12] that the set of all cohomology operations relative to (q, r, A, B) is in $1-1$ correspondence with the cohomology group

$$H^r(K(A,q);B)$$

where $K(A,q)$ is the EILENBERG-MACLANE complex of the abelian group A in the dimension q (see [10]). Recent efforts of EILENBERG-MACLANE and of H. CARTAN [6] appear to be leading to successful calculations of these cohomology groups, and therefore to the determination of all cohomology operations. The preliminary results give some hope that the four types of reduced power operations listed above generate all cohomology operations.

2. The construction of reduced powers

Let K denote a regular cell complex. Regularity means that the closure of any cell is a subcomplex and an acyclic one. K may be infinite, if so it has the CW topology. Let K^* denote the associated cochain complex with integer coefficients, i. e. $K^* = \mathrm{Hom}\,(K, Z)$ where $Z =$ integers. If u is a q-cochain of K^* and c is a (finite) q-chain of K, then $u \cdot c \,\varepsilon\, Z$ denotes the value of u on c. The coboundary operator δ in K^*, and the boundary ∂ in K are related by $\delta u \cdot c = u \cdot \partial c$ where $\dim c = 1 + \dim u$.

Let $\theta \geqq 0$ be an integer; and let Z_θ denote the integers Z reduced mod θ (as usual $Z_0 = Z$). Let $\bar{u} \,\varepsilon\, H^q(K;Z_\theta)$ be a q-dimension cohomology class mod θ of K. We wish to define a *cochain representation* of $\bar{u}$. To this end define the *elementary* cochain complex $M = M(\theta, q)$ as follows. Its cochain groups $C^r(M) = 0$ if $r \neq q$ or $q+1$; $C^q(M)$ is an infinite cyclic group with generator u; $C^{q+1}(M)$ is zero if $\theta = 0$, and otherwise is infinite cyclic with generator v. The coboundary in M is defined by $\delta u = \theta v$. If $f\colon\ M \to K^*$ is a cochain mapping (i. e. $f\delta = \delta f$), then fu is a cocycle mod θ and determines a cohomology class $\bar{u}$. Conversely, starting with $\bar{u}$, there is an integral cochain u_1 which is a cocycle mod θ (i. e. $\delta u_1 = \theta v_1$) and whose cohomology class is $\bar{u}$; hence, setting $fu = u_1$, $fv = v_1$, defines a cochain map $M \to K^*$. Such an f we call a cochain representation of $\bar{u}$.

If f_0, f_1 are two cochain representations of $\bar{u}$, then $f_0 u \sim f_1 u$ mod θ. Stated otherwise there are cochains $a, b \,\varepsilon\, K^*$ such that

$$\delta a = f_1 u - f_0 u - \theta b \ , \qquad \text{and then} \qquad \delta b = f_1 v - f_0 v \ .$$

If we set $Du = a$ and $Dv = b$, it follows that D defines a *cochain homotopy* of f_0 into f_1:

$$\delta D + D\delta = f_1 - f_0 \ .$$

Conversely, it is obvious that homotopic cochain maps $M \to K^*$ represent the same $\bar{u}$. Thus the cohomology class $\bar{u}$ may be regarded as a homotopy class of cochain mappings $M \to K^*$ any one of which is a *representation* of $\bar{u}$.

Let π denote a permutation group of degree n. We shall regard π as a group of permutations of the factors of any n-fold tensor product such as $K^{*n} = K^* \otimes \ldots \otimes K^*$ (n factors). Let W be an acyclic complex on which π operates freely, i. e. each $x \,\varepsilon\, \pi$ acts as an automorphism of W, and, if $x \neq 1$, no cell of W is mapped on itself by x. The existence of such a W is proved in the theory of the homology groups of a group [8]; and, for any π-module A, $H_q(W \otimes_\pi A)$ is the q^{th} homology group of π with coefficients in A. We also denote by W the chain complex it determines; its chain groups are free abelian.

The construction of the π-reduced powers of a cohomology class

$$\bar{u} \,\varepsilon\, H^q(K; Z_\theta) \qquad \text{or} \qquad \bar{u} \,\varepsilon\, H^q(K; Z)$$

is based on the following diagram:

$$W \otimes_\pi M^n \xrightarrow{\psi} W \otimes_\pi K^{*n} \xrightarrow{\zeta} W \otimes_\pi K^{n*} \xrightarrow{\varphi} K^* \ . \tag{2.1}$$

We proceed to explain its undefined terms.

If W is a chain complex, and A is a cochain complex, their tensor product $W \otimes A$ is the cochain complex whose cochain groups are

$$C^r(W \otimes A) = \sum_{i=0}^{\infty} C_i(W) \otimes C^{r+i}(A) \ , \tag{2.2}$$

and whose coboundary operator is defined by

$$\delta(w\otimes a) = \partial w\otimes a + (-1)^i w\otimes \delta a \tag{2.3}$$

where $i = \dim w$. In case π operates on both W and A, we define operations in $W\otimes A$ by

$$x(w\otimes a) = xw\otimes xa\ , \qquad x\,\varepsilon\,\pi\ . \tag{2.4}$$

Then $W\otimes_\pi A$ is the factor complex by the subcomplex generated by cochains of the form $x(w\otimes a) - w\otimes a$. Although the chain groups of W and the cochain groups of A are zero in all negative dimensions, this is not the case for $W\otimes A$. For example, $W\otimes_\pi M^n$ has zero cochain groups in dimensions $> n(q+1)$, and usually non-zero cochain groups in all dimensions from $-\infty$ to $n(q+1)$. Thus, it is not the cochain complex of any geometric complex.

The map ψ of 2.1 is induced by a map $f\colon\ M \to K^*$ representing $\bar{u}$, i. e. $\psi = 1\otimes_\pi f^n$.

The map ζ of 2.1 is induced by a natural map $\zeta'\colon K^{*n} \to K^{n*}$. If $u_1, \ldots, u_n$ are cochains of K^*, and $\sigma_1, \ldots, \sigma_n$ are oriented cells of K, then ζ' is defined by

$$\zeta'(u_1\otimes\ldots\otimes u_n)\cdot\sigma_1\times\ldots\times\sigma_n = (u_1\cdot\sigma_1)\ldots(u_n\cdot\sigma_n)\ \varepsilon\ Z\ . \tag{2.5}$$

In case K is a finite complex, ζ' is an isomorphism. In any case, $\delta\zeta' = \zeta'\delta$. Furthermore the action of π in K^n yields a dual action in K^{n*} with respect to which ζ' is equivariant. Thus, finally, $\zeta = 1\otimes_\pi\zeta'$.

In contrast to ψ and ζ which are natural maps, the map φ requires a preliminary construction. It will be defined as the *dual* of an equivariant chain map called a *diagonal approximation*:

$$\varphi'\colon\ W\otimes K \to K^n\ . \tag{2.6}$$

The construction of φ' is based on a general existence lemma which we now state in detail. A proof can be found in [13; §§ 3.5; 5.5].

2.7. Lemma. *Let A, B be complexes on which π operates, and suppose it operates freely in A. Let C be a carrier from A to B which is π-equivariant and acyclic, i. e. for each cell σ of A, $C(\sigma)$ is an acyclic subcomplex of B such that $C(x\sigma) = xC(\sigma)$ for $x\varepsilon\pi$, and σ a face of τ implies $C(\sigma)\subset C(\tau)$. Then there exists a chain map $\varphi'\colon A \to B$ carried by C* (i. e. *$\varphi'(\sigma)$ is a chain of $C(\sigma)$*) *and φ' is equivariant* (i. e. $\varphi'(xc) = x\varphi'(c)$). *Furthermore, if φ_0', φ_1' are two such chain maps, then there is a chain homotopy D of φ_0' into φ_1' which is carried by C and is π-equivariant.*

To apply the lemma, we take A to be the product complex $W\otimes K$ and $B = K^n$. π permutes the factors of K^n in the usual way. It operates in $W\otimes K$

by $x(w\otimes\sigma) = (xw)\otimes\sigma$. Since the action is free in W, it is free in $W\otimes K$. The carrier C is the so-called *diagonal* carrier

$$C(w\otimes\sigma) = |\sigma|^n$$

where $|\sigma|$ is the subcomplex of K consisting of σ and its faces. Since K is regular, C is an acyclic carrier. It is obviously equivariant. Then the lemma gives the map φ' of 2.6.

The map φ of 2.1 dual to φ' is defined by

$$\varphi(w\otimes y)\cdot\sigma = (-1)^{i(i-1)/2}y\cdot\varphi'(w\otimes\sigma) \tag{2.8}$$

where $i = \dim w$, y is a cochain of K^{n*}, and σ is an oriented cell of K with $\dim\sigma = \dim y - i$. From the equivariance of φ', we deduce that $\varphi x = \varphi$ for every $x\varepsilon\pi$; hence φ is defined on $W\otimes_\pi K^{n*}$. It is readily checked that $\varphi\delta = \delta\varphi$; indeed the awkward sign in 2.8 is needed for this.

Having defined completely the terms of 2.1 we are prepared for the final steps in the construction. Let G be an abelian group of coefficients. There is a natural transformation

$$\omega : K^*\otimes G \to \mathrm{Hom}\,(K, G) \tag{2.9}$$

given by $(y\otimes g)\cdot\sigma = (y\cdot\sigma)g$ where $y\varepsilon K^*$, $g\varepsilon G$, and σ is an oriented cell of K. In case K is finitely generated in each dimension, ω is an isomorphism. In any case, $\delta\omega = \omega\delta$ so that ω induces a homomorphism

$$\omega : H^r(K^*\otimes G) \to H^r(K;G) \tag{2.10}$$

where the right side is the ordinary cohomology group of K with coefficients in G.

Now tensor the diagram 2.1 with G and pass to the derived diagram of cohomology groups and induced homomorphisms. The composition of the three induced homomorphisms and the homomorphism ω of 2.10 is a homomorphism denoted by

$$\Phi : H^r(W\otimes_\pi M^n\otimes G) \to H^r(K;G)\ . \tag{2.11}$$

The image of Φ for all dimensions r is called *the set of π-reduced powers* of the cohomology class $\bar{u}$ of K.

The definition of Φ is somewhat unwieldy since it is induced by the composition of four homomorphism (the three of 2.1 and ω of 2.9). Two of these are natural, namely ζ and ω. They commute with all conceivable operations, and are isomorphisms when K is finitely generated in each dimension. We shall suppress them by letting φ denote henceforth the composition $\varphi\zeta$ of 2.1, and by ignoring ω. Thus, on the level of cochains, we have mappings

$$W \otimes_\pi M^n \otimes G \xrightarrow{\psi} W \otimes_\pi K^{*n} \otimes G \xrightarrow{\varphi} K^* \otimes G$$

which induce cohomology homomorphism

$$H^r(W \otimes_\pi M^n \otimes G) \xrightarrow{\psi^*} H^r(W \otimes_\pi K^{*n} \otimes G) \xrightarrow{\varphi^*} H^r(K;G) \tag{2.12}$$

whose composition is Φ.

The role of the ψ^*, φ^* should be emphasized. The first, ψ^*, is induced by the representation $M \to K^*$ of $\bar{u}$. The second, φ^*, is induced by the diagonal approximation φ' of 2.6, and is independent of $\bar{u}$. The image of ψ^* is called the set of *external* π-reduced powers of $\bar{u}$. This is in analogy with the external cohomology cross-product in $H^{p+q}(K^* \otimes K^*)$. The *internal* operations are derived from the external by the use of the diagonal map.

As will be seen, $H(W \otimes_\pi M^n \otimes G)$ depends only on π, θ and G and can be regarded as a kind of homology group of π. In fact, in the special case $\delta u = 0$, M^n is zero in all dimensions save nq, and $C^{nq}(M^n) \approx Z$; hence

$$H(W \otimes_\pi M^n \otimes G)$$

is just the system of homology groups of π with coefficients G suitable reindexed. Thus an element $\xi \varepsilon H(W \otimes_\pi M^n \otimes G)$ can be interpreted as a universal cohomology operation which can be applied to a cohomology class $\bar{u}$ mod θ to give the external reduced power $\psi^*(\xi)$ and the internal reduced power $\varphi^* \psi^*(\xi) = \Phi(\xi)$. To emphasize its dependence on $\bar{u}$ and ξ, we shall write $\xi(\bar{u})$ for $\Phi(\xi)$:

$$\xi(\bar{u}) = \Phi(\xi) \; . \tag{2.12}$$

As an example, let $\theta = 2$, and let π be the symmetric group of degree 2, having x as its generator. The simplest W has, as π-base, a single element e_i in each dimension i, and boundary relations

$$\partial e_{2i} = (1 + x) e_{2i-1} \; , \qquad \partial e_{2i+1} = (x - 1) e_{2i} \; .$$

Then $e_{2i} \otimes_\pi u^2$ is a cocycle mod 2, and its Φ-image is $\mathrm{Sq}_{2i} \bar{u}$. Likewise $e_0 \otimes_\pi u^2 + 2 e_1 \otimes_\pi uv$ is a cocycle mod 4, and its Φ-image is the PONTRJAGIN square of $\bar{u}$ (see [18, p. 83]).

3. General properties

The construction of reduced powers involves a number of choices at various stages. We state now a series of "invariance theorems" which establish the degree of independence of the resulting operations. The proofs are found in subsequent sections.

3.1. Theorem. *Any two representations $f_0, f_1 : M \to K^*$ of the cohomology class $\bar{u}$ induce the same homomorphism*

$$\psi^* : H^r(W \otimes_\pi M^n \otimes G) \to H^r(W \otimes_\pi K^{*n} \otimes G) \ .$$

3.2. Theorem. *The homomorphism*

$$\varphi^* : H^r(W \otimes_\pi K^{*n} \otimes G) \to H^r(K; G)$$

is independent of the choice of the diagonal approximation 2.6.

Now let π, ϱ be permutation groups of degree n with $\pi \subset \varrho$. Let V, W be π-free and ϱ-free acyclic complexes respectively. Let $C(v) = W$ for each cell $v \varepsilon V$. By 2.7, there is a π-equivariant chain mapping $V \to W$ which induces a cohomology homomorphism

$$H^r(V \otimes_\pi M^n \otimes G) \to H^r(W \otimes_\varrho M^n \otimes G) \ . \qquad (3.3)$$

If ξ is in the left group, let ξ' be its image on the right. Then we have

3.4. Theorem. *For any $\bar{u} \varepsilon H^q(K; Z_\theta)$, we have $\xi(\bar{u}) = \xi'(\bar{u})$.*

Taking $\pi = \varrho$, we have as a corollary that the π-reduced powers do not depend on the choice of the acyclic π-free complex W used in their construction. In this case, 3.3 is an isomorphism since there is a reverse mapping $W \to V$, and, by 2.7, the two compositions $W \to W$ and $V \to V$ are equivariantly homotopic to the respective identity maps. As a consequence, $H(W \otimes_\pi M^n \otimes G)$ depends only on π, G and θ, and its dimensional indexing depends on the dimension q of $\bar{u}$.

3.5. Corollary. *Letting $\mathcal{S}(n)$ denote the symmetric group of degree n, we have that each π-reduced power of $\bar{u}$ is an $\mathcal{S}(n)$-reduced power of $\bar{u}$.*

Therefore the collection of reduced powers of $\bar{u}$ associated with all groups of degree n form a single subgroup of $H(K; G)$ which we may call the set of reduced n^{th} powers of $\bar{u}$.

3.6. Theorem. *If $f : K \to L$ is a continuous map, f^* is the induced homomorphism of cohomology, $\bar{u} \varepsilon H^q(L; Z_\theta)$ and $\xi \varepsilon H^r(W \otimes_\pi M^n \otimes G)$, then*

$$f^* \xi(\bar{u}) = \xi(f^* \bar{u}) \quad \text{in} \quad H^r(K; G) \ .$$

This, of course, asserts the topological invariance of the cohomology operation ξ.

3.7. Theorem. *If $\bar{u} \varepsilon H^q(K; Z_\theta)$, then the non-zero reduced n^{th} powers of $\bar{u}$ have dimensions in the range from q to $n(q+1)$ inclusive. If $\bar{u} \varepsilon H^q(K; Z)$, the corresponding range is q to nq.*

4. Primitive operations and the main theorem

The following five types of operations on cohomology classes are called *primitive*:

1. Addition in $H^r(K;G)$.

2. The homomorphism $H^r(K;G) \to H^r(K;G')$ induced by a coefficient homomorphism $G \to G'$.

3. The BOCKSTEIN-WHITNEY coboundary operator of an exact coefficient sequence $0 \to A \to B \to C \to 0$ which is a homomorphism

$$\delta^*: \ H^r(K;C) \to H^{r+1}(K;A) \ .$$

4. The cup product which gives a pairing

$$H^p(K;A) \otimes H^q(K;B) \to H^{p+q}(K;\ A \otimes B) \ .$$

5. A π-reduced power of an element of $H^q(K;Z_\theta)$ or $H^q(K;Z)$ where π is a *cyclic* group of *prime* order p and degree p.

If A is a set of cohomology classes, and λ is a set of operations on classes, let λA denote the set of cohomology classes each of which can be obtained by a single application of some operation of λ on one or more classes of A. Define, inductively, $\lambda^k A = \lambda\lambda^{k-1}A$. The union $\bigcup_{k=0}^{\infty}\lambda^k A$ is called the set of classes *generated by* A *and the operations of* λ. Any subset of this union is also said to be generated by A and λ.

Main theorem. *If* $\bar{u}\varepsilon H^q(K;Z_\theta)$ *or if* $\bar{u}\varepsilon H^q(K;Z)$, *then the set of all reduced powers of* $\bar{u}$ *is generated by* $\bar{u}$ *and the primitive operations* 1 *through* 5.

It is easily seen that this theorem is a consequence of the following four propositions whose proofs are given in subsequent sections. We denote the symmetric group of degree n by $\mathcal{S}(n)$; and for a prime p, $\mathcal{S}(n,p)$ denotes a p-Sylow subgroup of $\mathcal{S}(n)$.

4.1. If $\pi \subset \mathcal{S}(n)$, then each π-reduced power of $\bar{u}$ is an $\mathcal{S}(n)$-reduced power of $\bar{u}$.

This is a restatement of 3.5.

4.2. Each $\mathcal{S}(n)$-reduced power of $\bar{u}$ is a sum of $\mathcal{S}(n,p)$-reduced powers of $\bar{u}$ for the various primes $p \leqq n$.

4.3. If p is a prime $\leqq n$, and k is such that $p^k \leqq n < p^{k+1}$, then the set of $\mathcal{S}(n,p)$-reduced powers of $\bar{u}$ is generated by the set of $\mathcal{S}(p^i,p)$-reduced powers of $\bar{u}$ $(i = 1, \ldots, k)$ and the primitive operations 1 through 4.

4.4. If p is a prime and $i \geqq 2$, then the set of $\mathcal{S}(p^i,p)$-reduced powers of $\bar{u}$

is generated by the set of $\mathcal{S}(p^{i-1},p)$-reduced powers of $\bar{u}$ and the primitive operations 1 through 5.

To see that these imply the theorem, we apply the steps in reverse order. Since any $\mathcal{S}(p,p)$-power of $\bar{u}$ is an operation of type 5, an induction based on 4.4 shows that the main theorem holds for $\mathcal{S}(p^i,p)$-powers of $\bar{u}$. Then 4.3 implies the same for $\mathcal{S}(n,p)$-powers, then 4.2 for $\mathcal{S}(n)$-powers, and finally 4.1 for arbitrary powers.

5. Proof of 3.1 [2])

Let I be the chain complex of the 1-simplex $[0,1]$. $C_0(I)$ is generated by two vertices e_0, e_1, C_1 has a single generator e, with $\partial e = e_1 - e_0$; and all other chain groups are zero. Let Z denote the chain complex of a single point, and let $\epsilon: I \to Z$ be the unique chain map (it is sometimes called the *augmentation* map). Passing to n-fold tensor products, we have $\epsilon^n: I^n \to Z^n$ and $Z^n \approx Z$. Thus ϵ^n is the augmentation map of I^n. Let Q be the kernel of ϵ^n. Since I^n is acyclic, all the homology groups of Q are zero including the 0^{th}. Since ϵ^n is π-equivariant, Q is a π-subcomplex of I^n.

5.1. Lemma. *If W is a π-free complex, then there exists a π-equivariant chain mapping*

$$g: \quad W \otimes I \to W \otimes I^n$$

such that $g(w \otimes e_0) = w \otimes e_0^n$ and $g(w \otimes e_1) = w \otimes e_1^n$ for all chains w in W.

Define a map $f_1: W \to W \otimes Q$ by

$$f_1(w) = g(w \otimes e_1) - g(w \otimes e_0) = w \otimes (e_1^n - e_0^n) .$$

Define $f_0: W \to W \otimes Q$ by $f_0(w) = 0$ for all w. Then f_0, f_1 are π-equivariant chain maps. Since $H_i(Q) = 0$ for all $i \geqq 0$, the Künneth relations for a product, show that $H_i(W \otimes Q) = 0$ for all $i \geqq 0$. Since W is π-free, we may apply 2.7 to obtain an equivariant chain homotopy D of f_0 into f_1. Explicitly, $D: C_i(W) \to C_{i+1}(W \otimes Q)$ for $i \geqq 0$ is such that $Dx = xD$ for $x \varepsilon \pi$, and

$$\partial D w + D \partial w = f_1 w - f_0 w = f_1 w .$$

Extend g over $W \otimes I$ by defining

$$g(w \otimes e) = (-1)^i D w , \quad i = \dim w$$

and the requirements on g are readily verified.

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[2]) I am indebted to *S. Eilenberg* for the much simplified form of this proof.

5.2. Lemma. *Let M, N be cochain complexes, and let $f_0, f_1 : M \to N$ be cochain maps which are cochain homotopic. Let W be a π-free chain complex. Then the cochain mappings*

$$1 \otimes_\pi f_0^n \,,\ 1 \otimes_\pi f_1^n : \ W \otimes_\pi M^n \to W \otimes_\pi N^n$$

are cochain homotopic.

The tensor product $I \otimes M$ of a chain and cochain complex is a cochain complex (see 2.2). The cochain homotopy $f_0 \simeq f_1$ is a cochain map F: $I \otimes M \to N$ such that $F(e_0 \otimes m) = f_0(m)$ and $F(e_1 \otimes m) = f_1(m)$. The required cochain homotopy is the composition of the three mappings

$$(W \otimes I) \otimes_\pi M^n \xrightarrow{g \otimes 1} (W \otimes I^n) \otimes_\pi M^n \xrightarrow{\mu} W \otimes_\pi (I \otimes M)^n \xrightarrow{1 \otimes F^n} W \otimes_\pi N^n \ .$$

The mapping g is given by 5.1, and μ is the isomorphism obtained by shuffling the factors of I^n and M^n.

The theorem 3.1 follows immediately from 5.2; for, as observed in § 2, any two cochain representations of $\bar{u}$ are homotopic.

6. Proof of 3.2

Let φ_0', φ_1': $W \otimes K \to K^n$ be two π-equivariant diagonal approximations. According to 2.7, there is a π-equivariant chain homotopy D of φ_0' into φ_1'. Let I be as in § 5, and define

$$F' : \ I \otimes W \otimes K \to K^n$$

by $F'(e_i \otimes w \otimes \sigma) = \varphi_i'(w \otimes \sigma)$ for $i = 0, 1$, and $F'(e \otimes w \otimes \sigma) = D(w \otimes \sigma)$. Then F' is an equivariant chain map. Its dual, as defined in 2.8, is a cochain mapping

$$F : \ I \otimes W \otimes_\pi K^{n*} \to K^* \ ;$$

and F reinterprets in the obvious way as a cochain homotopy of the dual φ_0 of φ_0' into the dual φ_1 of φ_1'. Therefore φ_0, φ_1 induce the same homomorphism φ^* of cohomology.

7. Proof of 3.4

Let $g: V \to W$ be the π-equivariant map given by 2.7. Let $\varphi' : W \otimes K \to K^n$ be a ϱ-equivariant diagonal approximation. Then $\varphi_1' = \varphi' g \otimes 1$ is a π-equivariant diagonal approximation $V \otimes K \to K^n$. Passing to the duals φ, φ_1 of φ', φ_1' we obtain a commutative diagram

$$\begin{tikzcd}
V\otimes_\pi M^n \arrow[r, "\psi"] \arrow[d, "g\otimes 1"] & V\otimes_\pi K^{*n} \arrow[dr, "\varphi"] \arrow[d, "g\otimes 1"] & \\
W\otimes_\rho M^n \arrow[r, "\psi_1"] & W\otimes_\rho K^{*n} \arrow[r, "\varphi_1"'] & K^*
\end{tikzcd}$$

Tensoring the diagram with G and passing to cohomology, we obtain a diagram whose commutativity yields the assertion 3.4.

8. Proof of 3.6

A map $f: K \to L$ is called *proper* if, for each cell σ of K, the least closed subcomplex $C(\sigma)$ of L containing $f(\sigma)$ is acyclic. It is shown in [15; p. 162], that, if K is a finite complex, then any map can be factored into the composition of three proper maps: $K \to K' \to L' \to L$ where K', L' are subdivisions of K, L and the first and last maps are identities. In [17; p. 317], it is shown that the required subdivisions can be found also when K, L are infinite CW-complexes. It suffices therefore to prove 3.6 for a proper map f.

Since the minimal carrier C is acyclic, there is a chain mapping $f_\# : K \to L$ carried by C (this is given by 2.7 with $\pi =$ identity). Let $f^\# : L^* \to K^*$ be the dual homomorphism of cochains. Then $f^\#$ induces the homomorphism f^* of cohomology.

Let φ', φ_1' be diagonal approximations $W\otimes K \to K^n$ and $W\otimes L \to L^n$, and let φ, φ_1 be their duals. Let $g: M \to L^*$ be a representation of $\bar{u}$; then $f^\# g: M \to K^*$ is a representation of $f^*\bar{u}$. Let $\psi_1 = 1\otimes_\pi g^n$ and $\psi = 1\otimes_\pi (f^\# g)^n$. We obtain then the diagram

$$\begin{tikzcd}
& W\otimes_\pi K^{*n} \arrow[dl, "\psi"'] \arrow[r, "\varphi"] & K^* \\
W\otimes_\pi M^n & & \\
& W\otimes_\pi L^{*n} \arrow[ul, "\psi_1"] \arrow[uu, "1\otimes f^{\#n}"] \arrow[r, "\varphi_1"] & L^* \arrow[uu, "f^\#"']
\end{tikzcd}$$

If this diagram were commutative we could tensor it with G, pass to cohomology, and the desired result would be immediate. By its construction, the triangle on the left is commutative. The square on the right need not be; however to obtain commutativity of the cohomology diagram, it suffices to prove that $\varphi(1\otimes f^{\#n})$ and $f^\# \varphi_1$ are cochain homotopic.

Consider then the dual diagram

$$\begin{array}{ccc} W\otimes K & \xrightarrow{\varphi'} & K^n \\ \downarrow 1\otimes f_{\#} & & \downarrow f^n_{\#} \\ W\otimes L & \xrightarrow{\varphi'_1} & L^n \end{array}$$

For each cell $w\times\sigma$ of $W\times K$, define $C'(w\times\sigma)$ to be the product

$$C(\sigma)\times\ldots\times C(\sigma) \qquad (n \text{ factors}).$$

This is an acyclic subcomplex of L^n. This carrier C' from $W\times K$ to L^n is clearly equivariant. It is obvious that it carries the two equivariant chain mappings $f^n_{\#}\varphi'$ and $\varphi'_1(1\otimes f_{\#})$. Hence, by 2.7, it carries an equivariant chain homotopy of $f^n_{\#}\varphi'$ into $\varphi'_1(1\otimes f_{\#})$. As in § 5, the homotopy can be regarded as an equivariant map

$$F'\colon\ I\otimes W\otimes K \to L^n\ .$$

Its dual, as defined in 2.8, is a map

$$F\colon\ I\otimes W\otimes_\pi L^{*n} \to K^*$$

which reinterprets as a cochain homotopy of $\varphi(1\otimes f^{\#n})$ into $f^{\#}\varphi_1$. This completes the proof.

9. Proof of 3.7

Now M has zero cochain groups in dimensions $>q+1$, and $>q$ if $v=0$. Thus M^n is zero in dimension $>n(q+1)$ (nq if $v=0$). If the indexing of the groups of $W\otimes_\pi M^n$, as defined in 2.2, is examined, it is seen that its cochain groups are zero in dimensions $>n(q+1)$ (nq if $v=0$). The same conclusion holds for the cohomology groups; and this establishes half of the assertion 3.7.

The same reasoning shows that any non-zero element of $W\otimes_\pi M^n$ is a sum of terms of the form $w\otimes z$ where $nq\leqq \dim z\leqq n(q+1)$. If

$$r=\dim(w\otimes z)=\dim z-\dim w\ ,$$

we obtain $\dim w\geqq nq-r$. Let σ be any r-cell of K. Then $\varphi'(w\otimes\sigma)$ is a chain of dimension $\geqq nq$ on the carrier $|\sigma|^n$ whose dimension is nr. Now $q>r$ implies $nq>nr$ and this implies $\varphi'(w\otimes\sigma)=0$. This implies

$$\varphi\psi(w\otimes z)\cdot\sigma=0\ ,$$

and therefore $\varphi^*\psi^*$ maps $H^r(W\otimes_\pi M^n\otimes G)$ into zero.

10. The primitive operations 1 through 4

In preparation for the more elaborate constructions later, we dispose of certain simple special cases in this section.

10.1. Lemma. *If $f: M \to K^*$ represents $\bar{u} \varepsilon H^q(K; Z_\theta)$, then the image of $f^*: H^r(M \otimes G) \to H^r(K; G)$ is generated by $\bar{u}$ and the primitive operations* 2 *and* 3 *of* § 4.

Suppose first that $\theta = 0$. Then $H^r(M \otimes G) = 0$ if $r \neq q$; and

$$H^q(M \otimes G) = C^q(M \otimes G) \approx C^q(M) \otimes G .$$

So each q-cocycle may be given the form $u \otimes g$. For a fixed g, define $\eta: Z \to G$ by $\eta(1) = g$. If η^* is the cohomology homomorphism induced by η, it is clear that $f^*(u \otimes g) = \eta^*(\bar{u})$.

Suppose now that $\theta > 0$. Then $H^r(M \otimes G) = 0$ if $r \neq q$ or $q + 1$. Again each q-cochain can be given the form $u \otimes g$, and it is a cocycle if and only if $\theta g = 0$. For a fixed such g, we set $\eta(1) = g$ and obtain a homomorphism $\eta: Z_\theta \to G$. And then $f^*(u \otimes g) = \eta^*(\bar{u})$.

In the dimension $q + 1$, v is an integral cocycle. Let $v' \varepsilon H^{q+1}(M)$ be its class, and set $\bar{v} = f^* v' \varepsilon H^{q+1}(K; Z)$. Let δ^* be the BOCKSTEIN-WHITNEY coboundary operator associated with the exact sequence $0 \to Z \to Z \to Z_\theta \to 0$. Since $\delta u = \theta v$, it follows that $\delta^* \bar{u} = \bar{v}$. Now any $(q + 1)$-cocycle of $M \otimes G$ has the form $v \otimes g$. For a fixed g, define $\eta;\ Z \to G$ by $\eta(1) = g$. Then $f^*(v \otimes g) = \eta^* \delta^*(\bar{u})$.

10.2. Lemma. *Let $f: M \to K^*$ and $f': M' \to K^*$ represent cohomology class $\bar{u}$ mod θ and $\bar{u}'$ mod θ' respectively.* Let $\psi = f \otimes f'$. *Let $d: K \to K \otimes K$ be a chain mapping having the diagonal carrier. Then the set of images for all r of the composition*

$$H^r(M \otimes M' \otimes G) \xrightarrow{\psi^*} H^r(K^* \otimes K^* \otimes G) \xrightarrow{d^*} H^r(K; G)$$

is generated by $\bar{u}, \bar{u}'$ and the primitive operations 1 *to* 4.

We shall express $M \otimes M'$ as a sum $P + Q$ of elementary subcomplexes as follows. Let τ be the G. C. D. of θ, θ' and set $a = \theta/\tau$, $a' = \theta'/\tau$. Then there are integers m, n such that $ma + na' = 1$. Define cochains w, w' by

$$w = av \otimes u' + (-1)^q a' u \otimes v' , \qquad q = \dim u ,$$

$$w' = (-1)^{q+1} nv \otimes u' + mu \otimes v' . \tag{10.3}$$

Then w, w' form a base for the cochains of the intermediate dimension. Let P be generated by $u \otimes u'$ and w with $\delta(u \otimes u') = \tau w$; and let Q be generated by w' and $v \otimes v'$ with $\delta w' = \tau v \otimes v'$. This is the required form.

Let ψ_P, ψ_Q be the restrictions of ψ to P and Q respectively. Now

$$\psi(u \otimes u') = fu \otimes f'u' \ ;$$

and therefore $\psi_P : P \to K^* \otimes K^*$ represents the external cohomology product $\bar{u} \otimes \bar{u}'$ computed using the coefficient pairing Z_θ and $Z_{\theta'}$ to $Z_\theta \otimes Z_{\theta'} \approx Z_\tau$. Let d^* be the cochain map dual to d. Since $d^*(\bar{u} \otimes \bar{u}') = \bar{u} \smile \bar{u}'$ (the internal cup-product), we have that $d^* \psi_P : P \to K^*$ represents $\bar{u} \smile \bar{u}'$. Then, by 10.1, $d^* \psi^*(H^r(P \otimes G))$ is generated by $\bar{u} \smile \bar{u}'$ and the primitive operations 2 and 3.

Let $\bar{v}$ be the cohomology class of fv. As shown in 10.1, $\bar{v} = \delta^* \bar{u}$ using the BOCKSTEIN operator associated with θ. Then $\bar{v} \smile \bar{u}'$ has coefficient group $Z_{\theta'}$. Since τ divides θ', we have a natural coefficient homomorphism $\eta' : Z_{\theta'} \to Z_\tau$. Similarly, form $\bar{u} \smile \bar{v}'$ with coefficient group Z_θ, and the natural map η: $Z_\theta \to Z$. Then

$$\bar{w}' = (-1)^{q+1} n \eta'(\bar{v} \smile \bar{u}') + m \eta(\bar{u} \smile \bar{v}')$$

is generated by $\bar{u}, \bar{u}'$ and the operations 1 to 4. Comparing $\bar{w}'$ with 10.3, it is evident that $d^* \psi_Q : Q \to K^*$ represents $\bar{w}'$. By 10.1, $\bar{w}'$ generates

$$d^* \psi^* H^r(Q \otimes G) \ ;$$

hence also $\bar{u}, \bar{u}'$. Since $H^r(M \otimes M' \otimes G) = H^r(P \otimes G) + H^r(Q \otimes G)$, the proof is complete.

10.4. Lemma. *If the permutation group π of degree n consists of the identity alone, then the set of π-reduced powers of $\bar{u}$ is generated by $\bar{u}$ and the primitive operations* 1 *to* 4.

The complex W consisting of a single vertex is π-free and acyclic. Then we have natural isomorphisms $W \otimes K \approx K$, etc. Hence the diagram 2.1 reduces to

$$M^n \xrightarrow{\psi} K^{*n} \xrightarrow{\varphi} K^* \ . \tag{10.5}$$

The cases $n = 1, 2$ are seen now to be covered by 10.1 and 10.2. Assume, inductively, that the lemma is true for the integer $n - 1$. Let

$$d : \ K \to K \otimes K \ , \quad \varphi_1' : \ K \to K^{n-1}$$

be chain maps with the diagonal carrier. Define

$$\varphi' : \ K \to K^n \quad \text{by} \quad \varphi' = (1 \otimes \varphi_1')d \ .$$

Then φ' has the diagonal carrier. Taking duals, 10.5 decomposes into

$$M \otimes M^{n-1} \xrightarrow{f \otimes \psi_1} K^* \otimes K^{*n-1} \xrightarrow{1 \otimes \varphi_1} K^* \otimes K^* \xrightarrow{d^*} K^*$$

where $\psi_1 = f^{n-1}$. Setting $\Phi_1 = \varphi_1 \psi_1$, the diagram reduces to

$$M \otimes M^{n-1} \xrightarrow{f \otimes \Phi_1} K^* \otimes K^* \xrightarrow{d^\#} K^* \ .$$

Since M^{n-1} is finitely generated, it is a direct sum of elementary subcomplexes $M^{n-1} = \sum N_j$. Hence

$$H^r(M^n \otimes G) \approx \sum_j H^r(M \otimes N_j \otimes G) \ .$$

By the inductive hypothesis, $\Phi_1 \mid N_j$ represents a cohomology class $\bar{u}'$ obtained from $\bar{u}$ by the primitive operations 1 to 4. And, by 10.2, $\bar{u}$ and $\bar{u}'$ generate the image of $H^r(M \otimes N_j \otimes G)$ under $d^*(f \otimes \Phi_1)^* = \varphi^* \psi^*$. This completes the proof.

10.5. Lemma. *Let ϱ be a permutation group of degree n, and*

$$\bar{u} \ \varepsilon \ H^q(K; Z_\theta) \ .$$

Then the ϱ-reduced powers of $\bar{u}$ of dimension $n(q+1)$ is generated by $\bar{u}$ and the primitive operations 1 to 4. In case $\theta = 0$, the same is true of the ϱ-reduced powers of dimension nq.

Let g: $W \otimes M^n \to W \otimes_\varrho M^n$ be the natural factorization. Let π consist solely of the identity permutation of degree n. Then $W \otimes M^n = W \otimes_\pi M^n$, and we may apply 3.4 to infer that $\xi(\bar{u}) = \xi'(\bar{u})$ if $\xi \ \varepsilon \ H^r(W \otimes M^n \otimes G)$ and $\xi' = g^* \xi \ \varepsilon \ H^r(W \otimes_\varrho M^n \otimes G)$. Let c be an r-cocycle of $W \otimes_\varrho M^n \otimes G$. Since g is a factorization, there is an r-cochain b of $W \otimes M^n \otimes G$ such that $gb = c$. Let $r = n(q+1)$ if $\theta \neq 0$, and $r = nq$ if $\theta = 0$. Then r is the highest non-zero dimension for the cochain groups of $W \otimes M^n$. Therefore $\delta b = 0$. Then the class ξ of b maps onto the class ξ' of c under g^*. Since $\xi(\bar{u})$ is a π-reduced power and $\pi = 1$, the conclusion follows from 10.4.

11. The transfer

Let π be a subgroup of a group ϱ subject to the restriction that the index m of π in ϱ is finite. Let W and L be ϱ-complexes where W is a chain and L a cochain complex. The inclusion map $\pi \subset \varrho$ induces a factorization

$$g: \ W \otimes_\pi L \to W \otimes_\varrho L \ , \qquad g(w \otimes_\pi u) = w \otimes_\varrho u \ .$$

The *transfer* T: $W \otimes_\varrho L \to W \otimes_\pi L$ is defined by

$$T(w \otimes_\varrho u) = \sum_x xw \otimes_\pi xu \ , \tag{11.1}$$

where the summation is taken as x ranges over a set of representatives of right cosets of π in ϱ. A second set of representatives would have the form $\{y_x x\}$ where $y_x \varepsilon \pi$. But

$$y_x xw \otimes_\pi y_x xu = xw \otimes_\pi xu \ ,$$

so T does not depend on the choice of representatives. Clearly

$$\begin{aligned} gT(w \otimes_\varrho u) &= \Sigma_x g(xw \otimes_\pi xu) = \Sigma_x xw \otimes_\varrho xu \\ &= \Sigma_x w \otimes_\varrho u = mw \otimes_\varrho u \ . \end{aligned} \tag{11.2}$$

It is easily verified that $T\delta = \delta T$ so that both T and g induce homomorphism of cohomology groups. By 11.2, we have

11.3. Lemma. *For each* $\xi \,\varepsilon\, H^r(W \otimes_\varrho L \otimes G)$, $g^* T \xi = m\xi$.

An obvious corollary is

11.4. Lemma. $mH^r(W \otimes_\varrho L \otimes G) \subset g^* H^r(W \otimes_\pi L \otimes G)$.

11.5. Theorem. *Let* $\pi \subset \varrho$ *be permutation groups of degree* n, *and let* m $=$ *index of* π *in* ϱ. *Then each* ϱ*-reduced power* $\xi(\bar{u})$ *is such that* $m\xi(u)$ *is a* π*-reduced power.*

By 11.3, $m\xi = g^*\xi'$ where $\xi' = T\xi$. The conclusion follows from 3.4.

Taking $\pi = 1$, 11.5 and 10.4 yield

11.6. Corollary. *If* m = *order of* ϱ, *then each* ϱ*-reduced power* $\xi(\bar{u})$ *is such that* $m\xi(\bar{u})$ *is generated by* $\bar{u}$ *and the primitive operations* 1 *to* 4.

11.7. Theorem. *Let* ϱ *be a permutation group of degree* n, *let* m = *order* ϱ, *and let* $M = M(\theta, q)$ *be as in* § 2. *If* $\theta \neq 0$, *then*

$$m\theta H^r(W \otimes_\varrho M^n \otimes G) = 0 \ ;$$

and therefore each ϱ*-reduced power of* $\bar{u}$ *has an order dividing* $m\theta$. *If* $\theta = 0$ *and* $r \neq nq$, *then*

$$mH^r(W \otimes_\varrho M^n \otimes G) = 0$$

and each ϱ*-reduced power of dimension* $\neq nq$ *has an order dividing* m.

If $\theta \neq 0$, it is obvious that $\theta H^r(M) = 0$ for all r. By the KÜNNETH relations for products, this implies $\theta H^r(M^n) = 0$. By the universal coefficient theorem, this implies $\theta H^r(M^n \otimes G) = 0$. Since W is acyclic,

$$H^r(W \otimes M^n \otimes G) \approx H^r(M^n \otimes G) \ .$$

Therefore $\theta H^r(W \otimes M^n \otimes G) = 0$. Apply 11.4 with $L = M^n$ and $\pi = 1$, and obtain $m\theta H^r(W \otimes_\varrho M^n \otimes G) = 0$.

If $\theta = 0$, and therefore $v = 0$, we have $H^r(M^n) = C^r(M^n) = 0$ if $r \neq nq$. Reasoning as above gives $H^r(W \otimes M^n \otimes G) = 0$ if $r \neq nq$. Then 11.4 yields the desired result.

Remark. The notion of transfer used above is a slight generalization of the concept first defined by ECKMANN [7].

12. Proof of 4.2

In case $\theta = 0$, we have by 10.5 that each $\mathcal{S}(n)$-reduced power of dimension nq is a 1-reduced power, and is therefore an $\mathcal{S}(n,p)$-reduced power for each prime p.

Suppose therefore that $\theta \neq 0$ or that $r \neq nq$. Let $\varrho = \mathcal{S}(n)$, then $n! =$ order of ϱ. By 11.7, each element of $H^r(W \otimes_\varrho M^n \otimes G)$ has a finite order dividing $n!\theta$. Therefore this group is the sum of its p-primary components for all primes p dividing $n!\theta$. An element ξ of the group is therefore a sum $\xi = \sum_p \xi_p$ where ξ_p lies in the p-primary part.

For a fixed prime p, let $\pi = \mathcal{S}(n,p)$ be a p-Sylow subgroup. Write $n! = mp^i$ where m is prime to p. Then, in the p-primary component, division by m is possible so that $\xi_p = m\xi'_p$. By 11.3, we have $\xi_p = g^* T \xi'_p$ where

$$T\xi'_p \in H^r(W \otimes_\pi M^n \otimes G) \ .$$

Therefore, by 3.4, $\xi_p(\bar{u})$ is an $\mathcal{S}(n,p)$-reduced power.

13. Proof of 4.3

The main idea behind the proofs of 4.3 and 4.4 is the known fact that the SYLOW subgroups $\mathcal{S}(n,p)$ of the symmetric groups have a relatively simple structure which can be described in detail. Briefly, $\mathcal{S}(n,p)$ can be built from cyclic groups of order p by the iterated use of direct products and split extensions. This is not generally the case for the SYLOW subgroups of a subgroup π of $\mathcal{S}(n)$; hence the necessity for the step 4.1. In this section we give the „direct product" part of the decomposition.

If $n = \sum_{i=0}^{k} a_i p^i$ is the p-adic expansion of n, then the integer part of n/p^j is $[n/p^j] = \sum_{i=j}^{k} a_i p^{i-j}$. It is easily seen that the exponent of p in the prime factorization of $n!$ is $\sum_{j=1}^{k} [n/p^j]$. It follows that the order of a SYLOW group $\mathcal{S}(n,p)$ is p raised to the power

$$a_1 + a_2(p+1) + a_3(p^2+p+1) + \cdots + a_k(p^{k-1} + \cdots + p + 1) \ . \qquad (13.1)$$

Now divide the integers $0, 1, \ldots, n-1$ into disjoint sets so that there are precisely a_i sets having p^i elements $(i = 0, 1, \ldots, k)$. For each such set choose a p-SYLOW subgroup of the permutation group of that set, and let it act as the identity in the remaining sets. Then the groups for distinct sets commute. Hence the direct product τ of all these groups is a subgroup of $\mathcal{S}(n)$. The order of $\mathcal{S}(p^i,p)$ is p raised to the power $\sum_{j=0}^{i-1} p^j$ (this follows from 13.1 by taking $n = p^i$). The order of the direct product τ is p raised to the power which is the sum of the exponents of the factors. Since this exponent coincides with 13.1, it follows that τ is an $\mathcal{S}(n,p)$. Thus $\mathcal{S}(n,p)$ decomposes into a direct product of the special SYLOW groups $\mathcal{S}(p^i,p)$:

$$\mathcal{S}(n,p) \approx \prod_{i=1}^{k} [\mathcal{S}(p^i,p)]^{a_i} . \tag{13.2}$$

In view of this decomposition, the assertion of 4.3 is a consequence of an induction whose general step is provided by the following lemma concerning the direct product of two groups.

Let ϱ, σ be permutation groups of sets R,S having r and s elements respectively. Let $\tau = \varrho \times \sigma$, and imbed τ in the permutation group of $T = R \cup S$ by the rule: if $x \varepsilon \varrho$, $y \varepsilon \sigma$, $a \varepsilon R$ and $b \varepsilon S$, then

$$(x,y)(a) = x(a) , \qquad (x,y)(b) = y(b) .$$

Thus τ is a permutation group of degree $r + s$.

13.3. Theorem. *If ϱ, σ and τ are as described above, then any τ-reduced power of $\bar{u}$ is generated by the set of ϱ and σ-reduced powers of $\bar{u}$ and the primitive operations* 1 *to* 4 *of* § 4.

Let U,V be ϱ and σ-free acyclic complexes respectively. Set $W = U \otimes V$, and define operations of τ in W by

$$(x,y)(d \otimes e) = xd \otimes ye$$

where $x \varepsilon \varrho$, $y \varepsilon \sigma$ and $d \otimes e$ is a generator of W. Since U and V are acyclic, so is W. It is easily verified that W is τ-free.

Now choose ϱ and σ-equivariant chain maps

$$\varphi_1' : \; U \otimes K \to K^R , \qquad \varphi_2' : \; V \otimes K \to K^S$$

having the diagonal carriers. Let

$$d : \; K \to K \otimes K$$

be a chain map with the diagonal carrier. We define the chain map

$$\varphi' : \; W \otimes K \to K^T = K^R \otimes K^S$$

to be the composition of the maps

$$U \otimes V \otimes K \xrightarrow{1 \otimes d} U \otimes V \otimes K \otimes K \xrightarrow{\lambda} U \otimes K \otimes V \otimes K \xrightarrow{\varphi_1' \otimes \varphi_2'} K^T ,$$

where λ is the isomorphism interchanging the second and third factors. Elementary calculations show that φ' is τ-equivariant and has the diagonal carrier.

Referring to the diagram of Fig. 1, $\varphi_1, \varphi_2, \varphi$ are the duals of $\varphi_1', \varphi_2', \varphi'$ as defined in § 2; ψ_1, ψ_2, ψ are induced by a representation $M \to K^*$ of $\bar{u}$. $d^\#$ is the cochain map dual to d, and λ is the isomorphism interchanging V with K^{*R}. The upper rectangle is obviously commutative. For the lower rectangle we have

$$\varphi = d^\#(\varphi_1 \otimes \varphi_2)\lambda .$$

The proof of this is omitted as it is a lengthy but mechanical computation based on the above definition of φ' as a composition.

Now tensor the diagram with a coefficient group G and pass to the related diagram of cohomology.

$$\begin{array}{ccc}
(U \otimes V) \otimes_\tau M^T & \xrightarrow{\lambda} & (U \otimes_\varrho M^R) \otimes (V \otimes_\sigma M^S) \\
\downarrow \psi & & \downarrow \psi_1 \otimes \psi_2 \\
(U \otimes V) \otimes_\tau K^{*T} & \xrightarrow{\lambda} & (U \otimes_\tau K^{*R}) \otimes (V \otimes_\sigma K^{*S}) \\
\downarrow \varphi & & \downarrow \varphi_1 \otimes \varphi_2 \\
K^* & \xleftarrow{d^\#} & K^* \otimes K^*
\end{array}$$

Fig. 1

By definition, a τ-reduced power of $\bar{u}$ has the form

$$\varphi \psi \xi = d^*(\varphi_1 \otimes \varphi_2)^*(\psi_1 \otimes \psi_2)^* \lambda^* \xi \tag{13.4}$$

for some

$$\xi \in H^t(U \otimes V \otimes_\tau M^T \otimes G) .$$

We proceed now to split $\lambda^* \xi$ into a suitable sum. For each dimension, the cochain groups of $U \otimes_\varrho M^R$ and $V \otimes_\sigma M^S$ are free abelian groups on finite bases; hence both complexes may be reduced to normal form, and written as a direct sum of elementary subcomplexes:

$$U \otimes_\varrho M^R \approx \Sigma_i P_i , \qquad V \otimes_\sigma M^S \approx \Sigma_i Q_j .$$

These yield a direct sum splitting of the cohomology

$$H^t(U \otimes_\varrho M^R \otimes V \otimes_\sigma M^S \otimes G) \approx \Sigma_{i,j} H^t(P_i \otimes Q_j \otimes G) .$$

Therefore

$$\lambda^* \xi = \sum_{i,j} \xi_{ij} \ , \qquad \xi_{ij} \varepsilon H^t (P_i \otimes Q_j \otimes G) \ . \tag{13.5}$$

and the number of non-zero terms is finite.

Since P_i is elementary, $\varphi_1 \psi_1$, restricted to P_i, is a map $P_i \to K^*$, and therefore is a representation of a cohomology class $\bar{u}_i$ modulo some θ_i. This element lies in the cohomology image of $\varphi_1 \psi_1$: $U \otimes_\varrho M^R \to K^*$ using coefficients Z_{θ_i}. Hence $\bar{u}_i$ is a ϱ-reduced r^{th} power of $\bar{u}$. Similarly, the map $Q_j \to K^*$ obtained by restricting $\varphi_2 \psi_2$ represents a σ-reduced s^{th} power $\bar{u}'_j$ of $\bar{u}$. We can now apply 10.2 to show that

$$d^* (\varphi_1 \otimes \varphi_2)^* (\psi_1 \otimes \psi_2)^* \xi_{ij} = d^* (\varphi_1 \psi_1 \otimes \varphi_2 \psi_2)^* \xi_{ij}$$

is generated by $\bar{u}_i, \bar{u}'_j$ and the primitive operations 1 to 4. Combining this result with 13.5 and 13.4 yields the conclusion of the theorem.

14. Proof of 4.4

Let π be the group of cyclic permutations of the integers $0, 1, \ldots, p-1$ with a generator x defined by $x(k) = k+1 \mod p$. Let ϱ be any permutation group of the integers $0, 1, \ldots, r-1$. Let ϱ^p be the direct product of p copies of ϱ. Imbed π and ϱ^p in the permutation group of the integers $0, 1, \ldots, pr-1$ as follows. If $0 \leqq h < pr$, write $h = kr + l$ where $0 \leqq l < r$; then $x(kr + l) = x(k) r + l$, and if $(y_0, \ldots, y_{p-1}) \varepsilon \varrho^p$ we have

$$(y_0, \ldots, y_{p-1})(kr + l) = kr + y_k(l) \ .$$

These are faithful representations. It is easy to verify the commutation rule

$$x(y_0, \ldots, y_{p-1}) = (y_{p-1}, y_0, y_1, \ldots, y_{p-2}) x \ . \tag{14.1}$$

Let σ be the permutation group of degree pr generated by these representations of π and ϱ^p. In view of 14.1, σ has the order pm^p where m = order of ϱ. Furthermore, ϱ^p is a normal subgroup of σ, $\sigma / \varrho^p \approx \pi$, and the inner automorphisms of σ given by elements of π permute the factors of ϱ^p cyclicly. Thus σ is the „split extension" of ϱ^p by π with respect to these operations of π on ϱ^p.

In the application to the proof of 4.4, p is a prime,

$$r = p^{i-1} \quad \text{and} \quad \varrho = \mathcal{S}(p^{i-1}, p)$$

is a p-SYLOW subgroup of the symmetric group of degree p^{i-1}. As is well known, the order of ϱ is p raised to the power $\sum_{j=0}^{i-2} p^j$ (the number of factors p in $p^{i-1}!$). Then σ is a permutation group of degree p^i and its order is p raised to the power $\sum_{j=0}^{i-1} p^j$. Hence $\sigma = \mathcal{S}(p^i, p)$.

If L is any chain or cochain complex on which ϱ operates, we shall define the *canonical* operations of σ in L^p = the tensor product of p copies of L. If $e_0 \otimes \ldots \otimes e_{p-1}$ is a generator of L^p, set

$$x(e_0 \otimes \ldots \otimes e_{p-1}) = \pm\, e_{p-1} \otimes e_0 \otimes \ldots \otimes e_{p-2} \,,$$
$$(y_0, \ldots, y_{p-1})(e_0 \otimes \ldots \otimes e_{p-1}) = (y_0 e_0) \otimes \ldots \otimes (y_{p-1} e_{p-1}) \,.$$

The sign in the first line is minus when $\dim e_{p-1}$ and $\dim(e_0 \otimes \ldots \otimes e_{p-2})$ are both odd, otherwise it is plus. The relation 14.1 is readily verified.

Let U be a π-free acyclic complex; and let σ operate in U through the natural factorization $\sigma \to \pi$; hence ϱ^p operates as the identity in U. Let V be a ϱ-free acyclic complex. Form the product V^p of p copies of V, and let σ operate canonically in V^p. Set $W = U \otimes V^p$ and let σ operate in W by operating as prescribed above on each factor. As a product of acyclic complexes, W is acyclic. Finally, W is σ-free; for each element of ϱ^p operates freely in V^p, hence in W, and an element not in ϱ^p operates freely in U, hence also in W.

Let K be a complex. Let ϱ operate as the identity in K, and let σ operate canonically in K^p. Let

$$\varphi_1' : \; U \otimes K \to K^p \,, \qquad \varphi_2' : \; V \otimes K \to K^r$$

be π and ϱ equivariant chain maps, respectively, having the diagonal carriers. With respect to the canonical operations of σ on $(V \otimes K)^p$ and on $(K^r)^p$ the map $(\varphi_2')^p$ is σ-equivariant. We define the chain map $\varphi' : W \otimes K \to K^{rp}$ to be the composition

$$U \otimes V^p \otimes K \xrightarrow{\eta} V^p \otimes U \otimes K \xrightarrow{1 \otimes \varphi_1'} V^p \otimes K^p \xrightarrow{\zeta} (V \otimes K)^p \xrightarrow{\varphi_2'^p} K^{rp}$$

where η is the isomorphism interchanging U with V^p, and ζ is the isomorphism *shuffling* the two sets of p factors. It is to be observed that σ operates on each of the five complexes, and each chain map is σ-equivariant. Hence φ' is σ-equivariant. An obvious calculation shows that φ' has the diagonal carrier.

Let $\varphi_1, \varphi_2, \varphi$ be the duals of $\varphi_1', \varphi_2', \varphi'$ as defined in 2.8. Referring to Fig. 2, we note that the three mappings at the bottom of the diagram have been defined. Let $f : M \to K^*$ represent $\bar{u}$, and set

$$\psi = 1 \otimes_\sigma f^{rp} \,, \qquad \psi' = 1 \otimes_\pi (1 \otimes_\varrho f^r)^p$$

The inclusions $\varrho^p \subset \sigma$ and $1 \subset \pi$ induce the factorizations ν and ν' respectively. The maps λ are isomorphisms obtained by shuffling the p factors of V^p with the p factors of $(M^r)^p$ and $(K^{*r})^p$. Commutativity holds obviously in the two upper squares. The relation $\varphi = \varphi_1(1 \otimes \varphi_2^p)\lambda$ likewise holds. The proof of this is omitted as it is a lengthy but mechanical computation based on the definition of φ' as a composition.

$$\begin{tikzcd}
(U\otimes V^p)\otimes_{\varrho^p} M^{rp} \arrow[r, "\lambda"] \arrow[d, "\nu"] & U\otimes(V\otimes_\varrho M^r)^p \arrow[d, "\nu'"] \\
(U\otimes V^p)\otimes_\sigma M^{rp}) \arrow[r, "\lambda"] \arrow[d, "\psi"] & U\otimes_\pi(V\otimes_\varrho M^r)^p \arrow[d, "\psi'"] \\
(U\otimes V^p)\otimes_\sigma K^{*rp} \arrow[r, "\lambda"] \arrow[d, "\varphi"] & U\otimes_\pi(V\otimes_\varrho K^{*r})^p \arrow[d, "1\otimes\varphi_2^p"] \\
K^* & U\otimes_\pi K^{*p} \arrow[l, "\varphi_1"']
\end{tikzcd}$$

Fig. 2

Now tensor the diagram with a coefficient group G, and pass to the related diagram of cohomology groups and induced homomorphisms. By definition, a σ-reduced power of $\bar{u}$ has the form

$$\varphi\psi\xi = \varphi_1(1\otimes\varphi_2{}^p)\,\psi'\lambda\xi \tag{14.2}$$

for some

$$\xi \in H^s(U\otimes V^p\otimes_\sigma M^{rp}\otimes G) .$$

We proceed now to split $\lambda\xi$ into a suitable sum. For each dimension, the cochain group of $V\otimes_\varrho M^r$ is a free abelian group on a finite base; hence we may reduce the complex to normal form, and express it as a direct sum of elementary subcomplexes:

$$V\otimes_\varrho M^r = \textstyle\sum_{j=1}^\infty N_j .$$

The tensor product of p copies of this complex can be written

$$(V\otimes_\varrho M^r)^p = \textstyle\sum_{j=1}^\infty N_j^p + \sum_\alpha Z(\pi)\otimes P_\alpha . \tag{14.3}$$

The second term accounts for all cross-product terms. Here $\alpha = (j_1, \ldots, j_p)$ is a sequence of positive integers not all equal, and P_α is the tensor product of the corresponding N's. $Z(\pi)$ denotes the group ring of π. The range of α in the sum is a set of representatives of equivalence classes of sequences $(j_1, \ldots, j_p)$ under cyclic permutations. Since p is prime, each equivalence class has p elements. Thus 14.3 is a decomposition into a direct sum of π-invariant subcomplexes.

Tensor each term of 14.3 over π with U. Using the natural identification $U\otimes_\pi Z(\pi) \approx U$, we have

$$U\otimes_\pi(V\otimes_\varrho M^r)^p = \textstyle\sum_{j=1}^\infty U\otimes_\pi N_j^p + \sum_\alpha U\otimes P_\alpha . \tag{14.4}$$

Tensoring with G and passing to cohomology, we obtain a corresponding direct sum of cohomology groups. Referring to 14.2, $\lambda\xi$ decomposes under this direct sum splitting into

$$\lambda\xi = \Sigma_j \xi_j + \Sigma_\alpha \xi_\alpha \tag{14.5}$$

where only a finite number of terms are non-zero.

Since $U \otimes P_\alpha$ appears also as a subcomplex of $U \otimes (V \otimes_\varrho M^r)^p$ and $\nu' \mid U \otimes P_\alpha$ is the identity, it follows that ξ_α is in the image of ν'. As λ is an isomorphism, it is in the image of $\nu'\lambda = \lambda\nu$. So it may be written $\xi_\alpha = \lambda\nu\xi'_\alpha$. Then

$$\varphi_1(1 \otimes \varphi_2^p)\,\psi'\,\xi_\alpha = \varphi\,\psi\nu\,\xi'_\alpha \tag{14.6}$$

is a ϱ^p-reduced power of $\bar{u}$. Since distinct factors of ϱ^p operate on disjoint sets of indices, we can apply 13.3 to infer that this ϱ^p-reduced power of $\bar{u}$ is generated by ϱ-reduced powers of $\bar{u}$ and the operations of type 1 to 4 of § 4.

Let g_j be the map $\varphi_2(1 \otimes_\varrho f^r)$ restricted to N_j. It represents, by definition, a ϱ-reduced power $\bar{v}_j$ modulo some θ_j of $\bar{u}$. Also, by definition,

$$\varphi_1(1 \otimes_\pi g_j^p)\,\xi_j = \varphi_1(1 \otimes_\pi \varphi_2^p)\,\psi'\,\xi_j$$

is a π-reduced power of $\bar{v}_j$. Therefore it is a π-reduced power of a ϱ-reduced power of $\bar{u}$. Thus $\varphi\,\psi\,\xi$ splits into a sum using 14.2 and 14.5, and each term of the sum is generated by ϱ-reduced powers of $\bar{u}$ and the primitive operations 1 to 5. This proves 4.4.

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