The Riemann-Roch Theorem.

Autor(en): **Royden, H.L.**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **34 (1960)**

PDF erstellt am: **18.09.2024**

Persistenter Link: https://doi.org/10.5169/seals-26621

The RIEMANN-ROCH Theorem [1])

by H. L. ROYDEN, Stanford (Calif., U. S. A.)

By a *divisor* $\mathfrak{d}$ on a RIEMANN surface W we shall mean a formal expression $\mathfrak{d} = p_1^{\nu_1} \ldots p_n^{\nu_n}$, where the p_i are distinct points of W and the ν_i are integers. We multiply two divisors by adding the exponents at corresponding points, and agree that a divisor is unchanged by the addition or deletion of a factor p_k^0. The divisors on W then form an ABELian group whose unit is the unit divisor 1 all of whose exponents are zero. A divisor is called *integral* if all of its exponents are non-negative. The integral divisors form a semi-group. The set of points which occur with a non-zero exponent in a divisor is called the *carrier* of the divisor. Two divisors are said to be *disjoint* if their carriers are disjoint. Every divisor is the quotient of two disjoint integral divisors.

A meromorphic function f is called a multiple of a divisor $\mathfrak{d} = p_1^{\nu_1} \ldots p_n^{\nu_n}$ if f is analytic except on the carrier of $\mathfrak{d}$ and the order of f at p_k is at least ν_k, where the order of a meromorphic function at a point is defined as the order of the zero of f if the point is a zero of f, minus the order of the pole if the point is a pole of f, and zero if the point is neither a zero nor a pole of f. Similarly, a meromorphic differential α on W is called a multiple of $\mathfrak{d}$ if α is analytic except on the carrier of $\mathfrak{d}$ and the order of α at p_k is at least ν_k.

The classical RIEMANN-ROCH theorem [6] gives the dimension of the space of meromorphic functions on a compact surface which are multiples of $\mathfrak{d}$ in terms of the number of differentials having a certain relationship to $\mathfrak{d}$. The purpose of the present note is to give a reformulation of the classical version and proof of this theorem which has the advantage that it remains valid for certain types of open RIEMANN surfaces. Since the results of FLORACK [2] imply that for any open RIEMANN surface there are always infinitely many meromorphic functions which are multiples of $\mathfrak{d}$, it is clear that we must restrict the class of meromorphic functions under consideration if we are to obtain non-trivial results. A natural restriction seems to be to consider the class $\mathfrak{M}$ of meromorphic functions on W which are analytic except for a finite number of poles and which have a finite DIRICHLET integral over the exterior of any neighborhood of the poles. For parabolic surfaces this restriction turns out to be sufficient to give a theory similar to that for the compact case, and in the third section we extend this to surfaces of class O_{FD}.

For hyperbolic surfaces in general we must impose a further restriction on

[1]) The research for this paper was carried out under the sponsorship of the Office of Ordnance Research, U. S. Army.

our meromorphic functions which may be thought of as requiring our functions to be "real on the ideal boundary" or to have "constant real part on each boundary component".

In the last two sections we indicate some generalizations.

AHLFORS [1] takes a stand against the imposition of "null-boundary" hypotheses and recommends instead that restrictions be imposed on the class of functions under consideration. The results of the fourth section in the present paper are in line with this program, but it is to be noted that both the results and proofs are more awkward than those of the null-bounded cases considered in sections 2 and 3. This is in part due to the fact that the restricted classes of meromorphic functions do not admit multiplication by complex constants in the general case, while in the null-bounded cases this multiplication is always possible.

1. Relations with respect to and integral divisor. Let $\mathfrak{d} = q_1^{\nu_1} \ldots q_n^{\nu_n}$ be an integral divisor, and consider the space G of all differentials which are analytic in some neighborhood (which may depend on the differential) of the carrier of $\mathfrak{d}$. Let $\mathsf{L}(\mathfrak{d})$ be the set of those linear functionals L on this space G which have the property that they anihilate multiples of $\mathfrak{d}$, i. e. those linear functionals for which $L[\alpha] = 0$ whenever α is a multiple of $\mathfrak{d}$. We shall often refer to elements of $\mathsf{L}(\mathfrak{d})$ as *relations* with respect to $\mathfrak{d}$, and say that α satisfies the relation $L \in \mathsf{L}(\mathfrak{d})$ if $L[\alpha] = 0$. By the carrier of L we shall mean the carrier of $\mathfrak{d}$.

Let $\zeta_1, \ldots, \zeta_n$ be fixed uniformizers at the points $q_1, \ldots, q_n$. Then at q_k each differential in G has the representation $\alpha = \varphi_k(\zeta_k) d\zeta_k$ where φ_k is analytic near q_k. If $L \in \mathsf{L}(\mathfrak{d})$, then $L[\alpha]$ depends only on the value of φ_k and its first $\nu_k - 1$ derivatives at q_k, and so we may express L in the form

$$L[\alpha] = \sum_{k=1}^{n} \sum_{j=1}^{\nu k} \frac{a_{kj}}{(j-1)!} \varphi_k^{(j-1)}, \tag{1}$$

where $\varphi_k^{(j)}$ denotes the *j-th* derivative of φ_k with respect to ζ_k evaluated at q_k. The coefficients a_{kj} depend in a rather complicated fashion on the choice of the uniformizers ζ_k, but for a fixed choice of the ζ_k, the coefficients a_{kj} determine and are uniquely determined by L. Since the a_{kj} can be arbitrarily prescribed, we see that $\mathsf{L}(\mathfrak{d})$ is a vector space of dimension $n(\mathfrak{d})$, where $n(\mathfrak{d})$ denotes the *order* $\Sigma\nu_k$ of $\mathfrak{d}$.

Let f be a function which is meromorphic in a neighborhood of the carrier of $\mathfrak{d}$ and is a multiple of $\mathfrak{d}^{-1}$. We can associate to f an element L_f of $\mathsf{L}(\mathfrak{d})$ by defining

$$L_f[\alpha] = \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{\Gamma_k} f\alpha , \tag{2}$$

where Γ_k is the boundary of a disc which contains q_k, but not q_j, $j \neq k$, and which lies in the common domain of meromorphy of f and α. In virtue of CAUCHY's theorem this definition is independent of the choice of Γ_k, and if f is a multiple of $\mathfrak{d}^{-1}$, then for α a multiple of $\mathfrak{d}$ the differential $f\alpha$ is analytic in the disc bounded by Γ_k, and so $L_f[\alpha] = 0$. Thus $L_f \in L(\mathfrak{d})$.

In terms of a fixed uniformizer ζ_k at q_k such that $\zeta_k(q_k) = 0$, the function f has an expansion of the form

$$f = \sum_{j=1}^{\nu k} \frac{a_{kj}}{\zeta_k^j} + \text{regular terms.} \tag{3}$$

If we use the CAUCHY formula to evaluate $L_f[\alpha]$, we see that $L_f[\alpha]$ is again given by the formula (1), whence the space $L(\mathfrak{d})$ is in a natural one-to-one correspondence with the space of "principal parts" of functions which are meromorphic in some neighborhood of the carrier of $\mathfrak{d}$ and which are multiples of $\mathfrak{d}^{-1}$. In fact the coefficients a_{kj} in the expansion (1) of L_f relative to the uniformizers $\zeta_1, \ldots, \zeta_n$ are precisely the same coefficients which occur in the expansions (3) of the principal parts of f. Thus the relations in $L(\mathfrak{d})$ give us a convenient specification of the principal parts of a multiple of $\mathfrak{d}^{-1}$, and this specification has the advantage over the specification (3) in that it is independent of the uniformizers chosen at the points q_k. In line with this we shall often say that f has the principal part L when $L_f = L$, and refer to L as a principal part.

2. The analogue of the RIEMANN-ROCH theorem for parabolic surfaces. We shall say that a meromorphic function f on a RIEMANN surface is of class $\mathfrak{M}$ if f has only a finite number of poles and the DIRICHLET integral of f is finite over the complement of any neighborhood of the poles of f. Similarly a meromorphic differential α is said to be of class $\mathfrak{D}$ if it has only a finite number of poles and is square integrable over the exterior of any neighborhood of its poles. Clearly $f \in \mathfrak{M}$ if and only if $df \in \mathfrak{D}$. On a compact surface every meromorphic function belongs to $\mathfrak{M}$ and every meromorphic differential to $\mathfrak{D}$.

In the remainder of this section, we shall suppose that W is parabolic, where we use the term parabolic to include the compact surfaces as a special case. There are various ways of defining parabolic surfaces, but we shall make use only of the following properties:

1. Every harmonic function on W with a finite DIRICHLET integral is constant.

2. Every bounded harmonic function on W is constant.

3. Let O be an open set on W whose closure is compact and whose boundary Γ consists of a finite number of smooth JORDAN curves. Let f be harmonic

in the complement $\tilde{O}$ of O and have a finite Dirichlet integral over $\tilde{O}$. Let α be a harmonic differential in $\tilde{O}$ which is square integrable over $\tilde{O}$. Then, assuming f and α sufficiently regular on Γ, we have

$$\int_\Gamma f\alpha = \iint_{\tilde{O}} df \wedge \alpha . \tag{4}$$

As an application of this last property, we note that if f and α are analytic in O, then $df \wedge \alpha = 0$. Consequently, for $f \in \mathfrak{M}$ and $\alpha \in \mathfrak{D}$, we have $\int_\Gamma f\alpha = 0$, provided that the poles of α and f are contained in O. Let $\mathfrak{d}_1$ and $\mathfrak{d}_2$ be two disjoint integral divisors, and suppose that f is a multiple of $\mathfrak{d}_1/\mathfrak{d}_2$ and that α is a multiple of $\mathfrak{d}_1^{-1}$. Then $f\alpha$ is analytic except on the carrier of $\mathfrak{d}_2$, and so $\int_\Gamma f\alpha$ is equal by the Cauchy theorem to minus the sum of the integrals of $f\alpha$ over small circles Γ_k about the points of $\mathfrak{d}_2$. Thus

$$0 = \int_\Gamma f\alpha = -\sum_{k=1}^{n} \int_{\Gamma_k} f\alpha = -2\pi i L_f[\alpha]$$

We have thus proved the following proposition:

Proposition 1. Let $\mathfrak{d}_1$ and $\mathfrak{d}_2$ be disjoint integral divisors on a parabolic Riemann surface, and let $f \in \mathfrak{M}$ be a multiple of $\mathfrak{d}_1/\mathfrak{d}_2$ and $\alpha \in \mathfrak{D}$ a multiple of $\mathfrak{d}_1^{-1}$. Then we have $L_f[\alpha] = 0$.

In order to investigate further the structure of the class $\mathfrak{M}$ on a parabolic Riemann surface, we make use of the fundamental potential on the surface. Let $G(p; q) = G(p, p_0; q, q_0)$ be a function of p which is harmonic except at q and q_0 and has a finite Dirichlet integral over the complement of any neighborhood of q and q_0. Then G will be called a fundamental potential if it vanishes at p_0 and has the behavior

$$G = -\log | \zeta(p) - \zeta(q) | + \text{regular terms}$$

at q and the behavior

$$G = \log | \zeta(p) - \zeta(q_0) | + \text{regular terms}$$

at q_0. A fundamental solution always exists (cf. [3] p. 129). On a parabolic surface it is unique, since the difference of two such fundamental potentials is a harmonic function with a finite Dirichlet integral, and so must vanish identically since it vanishes at p_0. The function G has the following symmetry properties:

$$G(p, p_0; q, q_0') = G(p, p_0; q, q_0) + G(p, p_0; q_0, q_0') \tag{5}$$

and

$$G(p, p_0; q, q_0) = G(q, q_0; p, p_0) . \tag{6}$$

Let ζ be a uniformizer at q. Then for q in the domain of ζ we may express G as $G(p, p_0; \zeta, q_0)$, and we may form the derivatives $\frac{\partial^j G}{\partial \zeta^j}$ where $\frac{\partial}{\partial \zeta}$ denotes $\frac{1}{2}\left(\frac{\partial}{\partial \xi} - i\frac{\partial}{\partial \eta}\right)$. It follows from (5) that these derivatives are independent of the choice of the point q_0. As functions of p they are harmonic except at q and have a finite DIRICHLET integral over the exterior of any neighborhood of q (cf. [3] Satz IV.8). At q we have the behavior

$$\frac{\partial^j G}{\partial \zeta^j} = \frac{(j-1)!}{2} \frac{1}{[\zeta(p) - \zeta(q)]^j} + \text{regular terms.} \tag{7}$$

Let $\mathfrak{d} = q_1^{\nu_1} \ldots q_n^{\nu_n}$ be an integral divisor, and let ζ_k be a uniformizer at q_k. Let L be an element of $\boldsymbol{L}(\mathfrak{d})$ and consider L as a principal part in the form (3). Then the function

$$f(p) = 2 \sum_{k=1}^{n} \sum_{j=1}^{\nu k} \frac{a_{kj}}{(j-1)!} \frac{\partial^j G}{\partial \zeta_k^j} , \tag{8}$$

where $\frac{\partial^j G}{\partial \zeta_k^j}$ denotes the derivative with respect to ζ_k of $G(p, p_0; q_k, q_0)$, is a harmonic function except at the carrier of $\mathfrak{d}$, where it has the expansion (3). Moreover, f has a finite DIRICHLET integral over the complement of any neighborhood of the carrier of $\mathfrak{d}$. Since W is parabolic, a harmonic function with these properties is unique to within an additive constant.

Let $\partial_q G = \frac{\partial G}{\partial \zeta} d\zeta$, where ζ is a uniformizer at q. Then it follows from the symmetry relation (7) that $\partial_q G$ is, in its dependence on q, an analytic differential except at p and p_0 where it has simple poles, and that it is square integrable over the complement of any neighborhood of p and p_0. Thus $\partial_q G$ is in $\mathfrak{D}$ and a multiple of $p^{-1} p_0^{-1}$. In terms of $\partial_q G$ we may write the relation (8) as $f(p) = 2L[\partial_q G(p, p_0; q)]$. Since $f(p_0) = 0$, we have the following proposition:

Proposition 2. Let $L \in \boldsymbol{L}(\mathfrak{d})$. Then the unique function f which is harmonic and has a finite DIRICHLET integral in the complement of every neighborhood of the carrier of $\mathfrak{d}$, which vanishes at p_0, and which has the principal part L at $\mathfrak{d}$, is given by $f(p) = 2L[\partial_q G(p, p_0; q)]$.

Corollary. Let $f \in \mathfrak{M}$. Then $f(p) = 2L_f[\partial_q G(p, p_0; q)] + f(p_0)$.

From the uniqueness of f in Proposition 2 we see that there is an f in $\mathfrak{M}$

with the principal part L if and only if the harmonic function $f(p) = 2L[\partial_q G]$ is analytic (apart from its poles). The condition that f be analytic is that at each p we have $\frac{\partial f}{\partial \bar{z}} = 0$, where z is a uniformizer at p. Now $\frac{\partial f}{\partial \bar{z}} = 2L\left[\partial_q \frac{\partial G}{\partial \bar{z}}\right]$. In its dependence on q, the differential $\partial_q\left(\frac{\partial G}{\partial \bar{z}}\right)$ is an everywhere analytic square integrable differential, the singularity at p being eliminated since $\frac{\partial^2}{\partial \zeta\, \partial \bar{z}} \log |z - \zeta| = 0$. This gives us the "if" part of the following proposition. The "only if" part is an immediate consequence of Proposition 1.

Proposition 3. There is an $f \in \mathfrak{M}$ with the principal part L if and only if $L[\alpha] = 0$ for all square-integrable analytic differentials α.

From the fact that $L[\partial_q G]$ is a bounded harmonic function in the complement of any neighborhood of the carrier of L, we see that the functions of $\mathfrak{M}$ have the property that they are bounded in the complement of any neighborhood of their poles If on the other hand f is a meromorphic function on W which has only a finite number of poles and is bounded in the complement of each neighborhood of its poles, then $L_f[\partial_q G]$ is a harmonic function with the same principal part as f and bounded except near the carrier of L. Hence it differs from f by a bounded harmonic function. Since W is parabolic, this difference is a constant, and so f must be in $\mathfrak{M}$, since $L_f[\partial_q G]$ has a finite DIRICHLET integral over the complement of any neighborhood of the poles of f. Thus we have the following proposition, which, however, we shall not use in the remainder of the paper.

Proposition 4. On a parabolic surface the class $\mathfrak{M}$ coincides with the class of meromorphic functions which have only a finite number of poles and are bounded in the complement of each neighborhood of these poles.

Corollary. On a parabolic surface the product of two functions in $\mathfrak{M}$ is again in $\mathfrak{M}$, and the product of a function in $\mathfrak{M}$ and a differential in $\mathfrak{D}$ is again a differential in $\mathfrak{D}$.

Thus the class $\mathfrak{M}$ is a ring of functions which serves as a ring of operators on $\mathfrak{D}$. Unfortunately, $\mathfrak{M}$ is not a field, since the reciprocal of a function in $\mathfrak{M}$ may well have an infinite number of poles, and even if it has only a finite number of poles, there is no guarantee that it will have a finite DIRICHLET integral in the exterior of a neighborhood of its poles. The fact that $\mathfrak{M}$ is not a field should prepare us for the different roles played by the numerator and denominator of the divisor $\mathfrak{d}$ in Theorem 1.

It is perhaps worth noting that if f is a meromorphic function on a parabolic surface such that $d \log f$ belongs to $\mathfrak{D}$, then f belongs to $\mathfrak{M}$. Such meromorphic functions are called quasi-rational by AHLFORS, and he has established [1] a generalization of ABEL's theorem for them. Unfortunately, the sum of two quasi-rational functions need not be quasi-rational.

The following theorem, whose corollaries may be thought of as an analogue of the RIEMANN-ROCH theorem for general parabolic surfaces, gives us a criterion for determining which principal parts associated with a divisor $\mathfrak{d}$ belong to functions in $\mathfrak{M}$ which are multiples of $\mathfrak{d}$.

Theorem 1. *Let W be a parabolic* RIEMANN *surface and $\mathfrak{d}_1$ and $\mathfrak{d}_2$ disjoint integral divisors on W. In order that $L \in \mathsf{L}(\mathfrak{d}_2)$ be the principal part of a function in $\mathfrak{M}$ which is a multiple of $\mathfrak{d} = \mathfrak{d}_1/\mathfrak{d}_2$, it is necessary and sufficient that $L[\alpha] = 0$ for all differentials α in $\mathfrak{D}$ which are multiples of $\mathfrak{d}_1^{-1}$.*

Proof. The necessity of this condition is given by Proposition 1. To prove sufficiency, we suppose that $L[\alpha] = 0$ for all differentials in $\mathfrak{D}$ which are multiples of $\mathfrak{d}_1^{-1}$. Since this includes all square integral analytic differentials, Proposition 3 asserts the existence of a function f in $\mathfrak{M}$ with the principal part L. If $\mathfrak{d}_1 = 1$, this completes the proof.

If $\mathfrak{d}_1 = p_1^{\mu_1} \dots p_m^{\mu_m}$, and $\mu_1 > 0$, we can represent f as $f = L[\partial_q G(p, p_1; q)]$, and this is an analytic function which vanishes at p_1. Let $z_1, \dots, z_m$ be uniformizers at $p_1, \dots p_m$. Then $\frac{\partial^j f}{\partial z_k^j}$ is given by $L\left[\partial_q \frac{\partial^j G}{\partial z_k^j}\right]$, where $\frac{\partial^j G}{\partial z_k^j}$ denotes the *j–th* derivative with respect to z_k of $G(z_k, p_1; q)$ evaluated at the point p_k. But for $0 \leqslant j \leqslant \mu_k - 1$, the expression $\partial_q \frac{\partial^j G}{\partial z_k^j}$ is, in its dependence on q, a differential in $\mathfrak{D}$ and a multiple of $\mathfrak{d}_1^{-1}$. Hence by hypothesis $L\left[\partial_q \frac{\partial^j G}{\partial z_k^j}\right] = 0$, and so f has a zero of order μ_k at p_k. Thus f is a multiple of $\mathfrak{d}_1/\mathfrak{d}_2$, proving the theorem.

Since functions in $\mathfrak{M}$ which are multiples of $\mathfrak{d}_1/\mathfrak{d}_2$ are completely determined by their principal parts if $\mathfrak{d}_2 \neq 1$, and otherwise determined to within an additive constant, we have the following corollaries:

Corollary 1. Let $\mathfrak{d}$ be an integral divisor on the parabolic surface W. Then the number of linearly independent functions $f \in \mathfrak{M}$ which are multiples of $\mathfrak{d}^{-1}$ is one more than the number of linearly independent relations $L \in \mathsf{L}(\mathfrak{d})$ such that $L[\alpha] = 0$ for all square integrable differentials α.

Corollary 2. Let $\mathfrak{d}_1$ and $\mathfrak{d}_2$ be disjoint integral divisors on the parabolic surface W, and suppose that $\mathfrak{d}_1 \neq 1$. Then the number of linearly independent functions $f \in \mathfrak{M}$ which are multiples of $\mathfrak{d} = \mathfrak{d}_1/\mathfrak{d}_2$ is equal to the number of linearly independent relations $L \in \mathcal{L}(\mathfrak{d}_2)$ such that $L[\alpha] = 0$ for all $\alpha \in \mathfrak{D}$ which are multiples of $\mathfrak{d}_1^{-1}$.

On a compact surface W of genus g we can formulate our theorem in a slightly different fashion. The number of linearly independent analytic differentials is g. Every meromorphic differential is determined by its principal part to within the addition of an everywhere analytic differential. On the other hand, if a principal part of a meromorphic differential satisfies the condition that the sum of the residues is zero, we can construct a meromorphic differential having this principal part by taking linear combinations of $\partial_q \dfrac{\partial^j G}{\partial z_k^j}(p_k, p_1; q)$. Thus we have the following lemma:

Lemma. On a compact surface W of genus g, the dimension of the space $V(\mathfrak{d})$ of meromorphic differentials which are multiples of the reciprocal $\mathfrak{d}^{-1}$ of an integral divisor $\mathfrak{d}$ is $g + n(\mathfrak{d}) - 1$, if $n(\mathfrak{d}) > 0$, and g if $n(\mathfrak{d}) = 0$.

Let $\mathfrak{d}_1$ and $\mathfrak{d}_2$ be disjoint integral divisors. Each element in $L(\mathfrak{d}_2)$ can be considered as a linear functional on the space $V(\mathfrak{d}_1)$ of differentials which are multiples of $\mathfrak{d}_1^{-1}$. Thus we have a natural linear mapping T of $\mathcal{L}(\mathfrak{d}_2)$ into the adjoint space V^* of $V(\mathfrak{d}_1)$. By Theorem 1, the null space of T consists of those L which are principal parts of meromorphic functions which are multiples of $\mathfrak{d} = \mathfrak{d}_1/\mathfrak{d}_2$. Thus if we let A denote the number of linearly independent meromorphic functions which are multiples of $\mathfrak{d}$, we see that the dimension of the null space of T is A for $\mathfrak{d}_1 \neq 1$ and $A - 1$ for $\mathfrak{d}_1 = 1$. Since the dimension of the null space of T plus the rank R of T is equal to the dimension of $\mathcal{L}(\mathfrak{d}_2)$, we have

$$A + R = \begin{cases} n(\mathfrak{d}_2), & \mathfrak{d}_1 \neq 1, \\ n(\mathfrak{d}_2) + 1, & \mathfrak{d}_1 = 1. \end{cases}$$

Let B be the number of linearly independent differentials in $V(\mathfrak{d}_1)$ which annihilate the range of T. Then $B + R$ is the dimension of $V(\mathfrak{d}_1)$, and we have

$$B + R = \begin{cases} g + n(\mathfrak{d}_1) - 1, & \mathfrak{d}_1 \neq 1, \\ g, & \mathfrak{d}_1 = 1. \end{cases}$$

Subtracting, we have

$$A - B = -n(\mathfrak{d}) - g + 1,$$

where $n(\mathfrak{d}) = n(\mathfrak{d}_1) - n(\mathfrak{d}_2)$ is the order of $\mathfrak{d}$.

Now the differentials in $V(\mathfrak{d}_1)$ which annihilate the range of T are just those for which $L[\alpha] = 0$ for all $L \in \mathcal{L}(\mathfrak{d}_2)$. But this is clearly the set of all meromorphic differentials which are multiples of $\mathfrak{d}_2/\mathfrak{d}_1$. Thus we have established the classical RIEMANN-ROCH Theorem:

Theorem (RIEMANN-ROCH). *Let $\mathfrak{d}$ be a divisor of order n on a compact* RIEMANN *surface of genus g, and let A denote the number of linearly independent meromorphic functions which are multiples of $\mathfrak{d}$ and B the number of linearly independent meromorphic differentials which are multiples of $\mathfrak{d}^{-1}$. Then $A = B - n - g + 1$.*

3. An extension to surfaces of class O_{FD}. We denote by F the space of harmonic functions u on a RIEMANN surface W with the property that $*du$ is semi-exact, i. e. has period zero over each dividing cycle of W. We shall denote by O_{FD} the class of those RIEMANN surfaces on which every function of class F which has a finite DIRICHLET integral is constant. Throughout this section we shall assume that W is of class O_{FD}. Surfaces of this class have been considered by SARIO [5], who uses the letter K where we use F. Some of the properties of surfaces of the class O_{FD} are investigated in [4]. In addition to the definition we shall use here only the following property ([4], Proposition 2):

Let O be an open set on $W \in O_{FD}$ whose closure is compact and whose boundary Γ is composed of a finite number of smooth JORDAN curves. Let f be a function of class FD in the complement $\tilde{O}$ of O, and α a semi-exact square integrable differential in $\tilde{O}$. Then, assuming sufficient regularity of f and α on Γ, we have

$$\int_{\Gamma} f\alpha = -\iint_{\tilde{O}} df \wedge \alpha . \tag{9}$$

By a semi-exact differential, we mean a closed differential whose periods over each dividing cycle are zero. We shall extend the notion of semi-exactness to differentials with a finite number of singularities by saying that such a differential is semi-exact if it is closed and its periods vanish over each dividing cycle which does not separate its singularities. Let $\mathfrak{D}_{SE}$ denote the subspace of $\mathfrak{D}$ consisting of differentials which are semi-exact in this sense.

Since each analytic function belongs to $\mathfrak{M}$, while $df \wedge \alpha = 0$ if f and α are analytic, (9) gives us the following proposition:

Proposition 5. On a RIEMANN surface W of class O_{FD}, let $f \in \mathfrak{M}$ and $\alpha \in \mathfrak{D}_{SE}$. Then $L_f[\alpha] = 0$.

In order to investigate the structure of $\mathfrak{M}$ on surface of class O_{FD}, we make use of the NEUMANN's function $N(p, p_0; q, q_0)$ on W. As a function of p, N is harmonic except at q and q_0 where it has the behavior

$$N(p, p_0; q, q_0) = -\log|\zeta(p) - \zeta(q)| + \text{regular terms},$$

and

$$N(p, p_0; q, q_0) = \log|\zeta(p) - \zeta(q_0)| + \text{regular terms},$$

respectively. Moreover, $N(p_0, p_0; q, q_0) = 0$, and N has a finite DIRICHLET integral over the complement of any neighborhood of q and q_0, and

$$f(q) - f(q_0) = \frac{1}{2\pi}\iint df \wedge * d_p N \tag{10}$$

for any f on W with a finite DIRICHLET integral. The NEUMANN's function is completely determined by the above properties, and is easily constructed by "projecting" the differential of a fundamental solution away from the space of all exact square integrable differentials. That is to say $N = G + K$, where G is the GREEN's function (or fundamental potential) and K is the BERGMAN kernel of the space HD. On a parabolic surface N coincides with the fundamental potential, and on a finite surface (10) is equivalent to the requirement that the normal derivatives of N vanish on the boundary.

The NEUMANN's function has the symmetry properties

$$N(p, p_0; q, q_0) = N(q, q_0; p, p_0) \tag{11}$$

and

$$N(p, p_0; q, q_0') = N(p, p_0; q, q_0) + N(p, p_0; q_0, q_0') . \tag{12}$$

From (10) we can deduce that $* d_p N$ is semi-exact, for let C be a dividing cycle which does not separate q from q_0. Then C may be taken as one boundary of a ring domain (i. e. a union of a finite number of annuli) which does not contain q or q_0. Let f be a C^2 function which is identically one on one side of R and identically zero on the other. Then

$$\begin{aligned}\int_C * d_p N &= \iint df \wedge * d_p N \\ &= 2\pi[f(q) - f(q_0)] = 0 .\end{aligned}$$

From these properties of N it follows that if L is any principal part, the function $f = 2L[\partial_q N]$ is a harmonic function with the principal part L, has a finite DIRICHLET integral over the complement of any neighborhood of the carrier of L, and has the property that $* df$ is semi-exact. Since the harmonic function with these properties is uniquely determined apart from an additive constant, we see that for any $f \in \mathfrak{M}$ we have

$$f(p) = 2 L_f[\partial_q N(p, p_0; q)] + f(p_0) .$$

Thus there is a meromorphic function in $\mathfrak{M}$ with the principal part L if and only if $L[\partial_q N]$ is analytic except on the carrier of L. Letting z be a uniformizer at p, we see that $L[\partial_q N]$ will be analytic if and only if $L\left[\partial_q \frac{\partial N}{\partial \bar{z}}\right] \equiv 0$. In its dependence on q, $\partial_q \frac{\partial N}{\partial \bar{z}}$ is an everywhere regular semi-exact analytic square integrable differential. Combining this with Proposition 5, we have the following proposition:

Proposition 6. On a RIEMANN surface W of class O_{FD} there is an $f \in \mathfrak{M}$ with the principal part L if and only if $L[\alpha] = 0$ for all semi-exact square integrable analytic differentials α.

Theorem 2. *Let W be a* RIEMANN *surface of class O_{FD}, and let $\mathfrak{d}_1$ and $\mathfrak{d}_2$ be disjoint integral divisors on W. In order that $L \in \mathsf{L}(\mathfrak{d}_2)$ be the principal part of a function $f \in \mathfrak{M}$ which is a multiple of $\mathfrak{d} = \mathfrak{d}_1/\mathfrak{d}_2$, it is necessary and sufficient that $L[\alpha] = 0$ for all α in $\mathfrak{D}_{SE}$ which are multiples of $\mathfrak{d}_1^{-1}$.*

Proof. The necessity is given by Proposition 5. Suppose that $L[\alpha] = 0$ for all α in $\mathfrak{D}_{SE}$ which are multiples of $\mathfrak{d}_1^{-1}$. Then Proposition 6 states that there is an $f \in \mathfrak{M}$ with the principal part L. If $\mathfrak{d}_1 = 1$, this completes the proof. If $\mathfrak{d}_1 \neq 1$, the remainder of the proof is the same as in Theorem 1 with G replaced by N and making use of the fact that as a differential in q, $\partial_q \frac{\partial^i N}{\partial \bar{z}_k^i}$ is in $\mathfrak{D}_{SE}$.

It is left to the reader to formulate corollaries similar to those of Theorem 1. We note that Theorem 2 implies Theorem 1 if we make use of the fact that on a parabolic surface every harmonic differential is semi-exact if it is square integrable.

5. An extension to general hyperbolic surfaces. If W is an arbitrary hyperbolic surface, we define the space $\mathfrak{M}_0$ to consist of those functions f in $\mathfrak{M}$ for which

$$\operatorname{Re} \iint_W df \wedge \alpha = 0 \tag{13}$$

for every real closed square integrable differential α which vanishes in some neighborhood of the poles of f. We define $\mathfrak{D}_0$ to consist of those α in $\mathfrak{D}$ for which

$$\operatorname{Im} \iint_W df \wedge \alpha = 0 \tag{14}$$

for each real function f with a finite DIRICHLET integral which vanishes in a neighborhood of the poles of α. If W is a finite RIEMANN surface, then $\mathfrak{M}_0$ consists of those meromorphic functions which are imaginary on the boundary of W, and $\mathfrak{D}_0$ consists of those meromorphic differentials which are real along the boundary. If W is a parabolic surface, then $\mathfrak{M}_0$ and $\mathfrak{D}_0$ coincide with $\mathfrak{M}$ and $\mathfrak{D}$. We sketch briefly the extension of Theorem 1 to the classes $\mathfrak{M}_0$ and $\mathfrak{D}_0$ on an arbitrary RIEMANN surface. As an easy consequence of the properties (13) and (14) we have the following proposition:

Proposition 7. Let $\mathfrak{d}_1$ and $\mathfrak{d}_2$ be disjoint integral divisors, and let $f \in \mathfrak{M}_0$ be a multiple of $\mathfrak{d} = \mathfrak{d}_1/\mathfrak{d}_2$ and $\alpha \in \mathfrak{D}_0$ a multiple of $\mathfrak{d}_1^{-1}$. Then $\operatorname{Re} L_f[\alpha] = 0$.

Let $G(p, q)$ be the GREEN's function of W. Then $\partial_q G$ is, in its dependence on q, a differential of class $\mathfrak{D}_0$ which is a multiple of p^{-1}. As a function of p it is harmonic, has a finite DIRICHLET integral outside any neighborhood of q, and satisfies (13). We can then establish the following propositions:

Proposition 8. Let $L \in L(\mathfrak{d})$. Then the unique real harmonic function u which has a finite DIRICHLET integral over the complement of any neighborhood of the carrier of d, which satisfies (13), and which has the principal part $\operatorname{Re} L$ is given by $u(p) = 2 \operatorname{Re} L[\partial_q G(p, q)]$.

Proposition 9. There is an $f \in \mathfrak{M}_0$ with principal part L if and only if $\operatorname{Re} L[\alpha] = 0$ for each α which is everywhere regular and belongs to $\mathfrak{D}_0$.

The proof of Proposition 9 differs from that of Proposition 3 only in that we make use of the fact the periods of $* d_p \partial_q G$ as a differential in p are everywhere regular differentials of the class $\mathfrak{D}_0$.

Theorem 3. *Let $\mathfrak{d}_1$ and $\mathfrak{d}_2$ be disjoint integral divisors on the* RIEMANN *surface W. In order that $L \in L(\mathfrak{d}_2)$ be the principal part of a function $f \in \mathfrak{M}_0$ which is a multiple of $\mathfrak{d} = \mathfrak{d}_1/\mathfrak{d}_2$, it is necessary and sufficient that* $\operatorname{Re} L[\alpha] = 0$ *for all differentials $\alpha \in \mathfrak{D}_0$ which are multiples of $\mathfrak{d}_1^{-1}$.*

The proof is similar to that of Theorem 1, but makes use of the fact that an analytic function $u + iv$ which vanishes at p_k has a zero of order μ there if $\frac{\partial^j u}{\partial x^j} = 0$ and $\frac{\partial^j u}{\partial y^j} = 0$ for $j < \mu$. We also make use of the fact that if $\tilde{G}$ is the harmonic conjugate of G then $\partial_q \tilde{G}$ and $\partial_q \frac{\partial^j G}{\partial x^j}$ and $\partial_q \frac{\partial^j G}{\partial y^j}$ are in $\mathfrak{D}_0$ when considered as differentials in q.

We have the following corollary, which may be considered a version of the RIEMANN-ROCH theorem for finite surfaces:

Corollary. Let W be a non-compact finite RIEMANN surface of genus g with h boundary contours. Let $\mathfrak{d}$ be a divisor on W, and let A_0 denote the number of linearly independent meromorphic functions on W which are multiples of $\mathfrak{d}$ and which are imaginary on the boundary (linearly independent in the real sense). Let B_0 denote the number of meromorphic differentials on W which are real along the boundary, multiples of $\mathfrak{d}^{-1}$, and linearly independent in the real sense. Then

$$A_0 = B_0 - 2n(\mathfrak{d}) - 2g - h + 2 .$$

Rather than considering the space $\mathfrak{M}_0$, we might equally well consider the space $\mathfrak{M}_m$ consisting of those functions f in $\mathfrak{M}$ for which

$$\operatorname{Re} \iint_W df \wedge \alpha = 0 \tag{15}$$

for all real semi-exact square integrable differentials α which vanish in a neighborhood of the poles of f. We have $\mathfrak{M}_0 \subset \mathfrak{M}_m \subset \mathfrak{M}$. The space $\mathfrak{M}_m$ consists of those meromorphic functions for which $\operatorname{Re} df$ is canonical in the sense of AHLFORS [1]. On a finite RIEMANN surface $\mathfrak{M}_m$ consists of those meromorphic functions whose real parts are constant on each boundary continuum.

Denote by $\mathfrak{D}_{OSE}$ that subspace of $\mathfrak{D}_0$ which consists of semi-exact differentials. If instead of the GREEN's function G, we use a kernel H which differs from G by the BERGMAN kernel for the space of harmonic measures, we can modify the proof of Theorem 3 to obtain the following theorem which we state without proof:

Theorem 4. *Let $\mathfrak{d}_1$ and $\mathfrak{d}_2$ be disjoint integral divisors on the* RIEMANN *surface W. In order that $L \in L(\mathfrak{d}_2)$ be the principal part of a function $f \in M_m$ which is a multiple of $\mathfrak{d} = \mathfrak{d}_1/\mathfrak{d}_2$, it is necessary and sufficient that* $\operatorname{Re} L[\alpha] = 0$ *for all differentials $\alpha \in \mathfrak{D}_{OSE}$ which are multiples of $\mathfrak{d}_1^{-1}$.*

Corollary. Let W be a non-compact finite RIEMANN surface of genus g. Let $\mathfrak{d}$ be a divisor on W, and let A_m denote the number of meromorphic functions on W which are multiples of $\mathfrak{d}$, whose real parts are constant on each boundary contour, and which are linearly independent in the real sense. Let B_{SE} denote the number of semi-exact meromorphic differentials on W which are real along the boundary, multiples of $\mathfrak{d}^{-1}$, and linearly independent in the real sense. Then

$$A_m = B_{SE} - 2g - 2n(\mathfrak{d}) + 2 .$$

6. Infinite divisors. We can generalize the results of the preceeding sections somewhat by considering infinite divisors. By an infinite divisor (or briefly divisor) $\mathfrak{d}$ we mean an integer valued function $\nu(p)$ defined on W. We multiply and divide divisors by adding and subtracting the corresponding functions, and the unit divisor corresponds to the function which is identically zero. A divisor is called integral if $\nu(p) \geqslant 0$ for all p, and two divisors $\mathfrak{d}_1$ and $\mathfrak{d}_2$ are said to be disjoint if $\nu_1(p) \cdot \nu_2(p) = 0$ for all p. A function $f \in \mathfrak{M}$ is said to be a multiple of a divisor $\mathfrak{d}$ if at each p the order of f is at least $\nu(p)$, and similarly for differentials $\alpha \in \mathfrak{D}$.

It is then readily verified that Theorems 1 through 4 remain valid if we allow infinite divisors. Note that our functions and differentials are still required to be of class $\mathfrak{M}$ or $\mathfrak{D}$, i. e. to have only a finite number of poles.

The corollaries to Theorem 1 remain valid for infinite divisors if by a relation with respect to $\mathfrak{d}$ we mean a relation with respect to some finite divisor contained in $\mathfrak{d}$. Here linear independence must be taken in the algebraic sense, i. e. only finite sums permitted.

It would be very desireable to have some corresponding theory for meromorphic functions with an infinite number of poles, but I have been unable to find a suitable replacement for the class $\mathfrak{M}$.

7. Essential Singularities. Rather than considering meromorphic principal parts, we could equally well have considered more general singularities which are given by Laurent expansions at the points $q_1, \ldots, q_n$. We can still define a linear functional L by

$$L[\alpha] = \int_{\Gamma_k} f\alpha$$

and with this definition, analogues of Theorems 1 through 4 remain valid if we allow $\mathfrak{M}$ to contain functions with essential singularities at a finite number of points and omit reference to the divisor $\mathfrak{d}_2$.

BIBLIOGRAPHY

[1] L. V. AHLFORS, *The method of orthogonal decomposition for differentials on open* RIEMANN *surfaces.* Ann. Acad. Sci. Fennicae, ser. A I no. 249/7 (1958).

[2] H. FLORACK, *Reguläre und meromorphe Funktionen auf nicht geschlossenen* RIEMANN*schen Flächen.* Schriftenreihe Math. Inst. Münster, H. 1 (1948).

[3] A. PFLUGER, *Theorie der* RIEMANN*schen Flächen.* Berlin 1957.

[4] H. L. ROYDEN, *A class of null-bounded* RIEMANN *surfaces.* Comm. Math. Helv. 34/1 (1959), p. 52–66.

[5] L. SARIO, *An extremal method on arbitrary* RIEMANN *surfaces.* Trans. Amer. Math. Soc. 73 (1952), pp. 459–470.

[6] H. WEYL, *Die Idee der* RIEMANN*schen Fläche.* Berlin 1923.

(Received July 28, 1959)