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Cobordism of Pairs

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This paper extends the results of ordinary cobordism theory to cobordism of pairs of manifolds (a *pair* is a pair (V, M) of closed differentiable manifolds, with V a submanifold of M). We first reduce the cobordism problem for the pair to separate problems for V, M; for V, however, a new structural group must be considered (e.g. $O_k \times O_N$). We then evaluate cobordism theory for the new structural groups. Our more precise results include the following:

In the general case (groups unrestricted); or if V and M are supposed oriented; or if M is weakly almost complex and the normal bundle of V in M is reduced to the unitary group; the characteristic numbers of V and M determine the cobordism class. The characteristic numbers of M are as usual; those of V are mixed products of characteristic classes of the tangent bundle of V, and of the normal bundle of V in M, evaluated on V. These have coefficient groups Z_2 (in the first case), Z (in the third), or both (in the second). Correspondingly, the cobordism groups are direct sums of copies of these groups. Their additive structure is completely determined.

Corresponding extensions of these results also hold for n-tuples (defined as chains $V_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset M$ of submanifolds) but appear rather less interesting. Products of various kinds can be defined; the most natural one appears to be

$$W \times (M, V) \to (W \times M, W \times V)$$

which establishes the cobordism group above as a free module over the usual cobordism group (in cases 1 and 3 only).

1. Cobordism over a sequence of groups

A cobordism theory is defined using a sequence of groups G_n such that (i) G_n is a subgroup of the orthogonal group O_n, (ii) G_n is a subgroup of G_{n+1} (using the usual imbedding of O_n in O_{n+1}). (A similar formulation has been suggested by MILNOR). It is possible to consider the more general situation in which we have maps $G_n \to O_n$, $G_n \to G_{n+1}$ not required to be inclusions, but we shall adhere to the simpler version.

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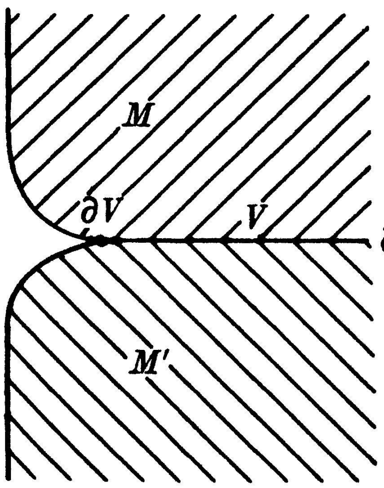
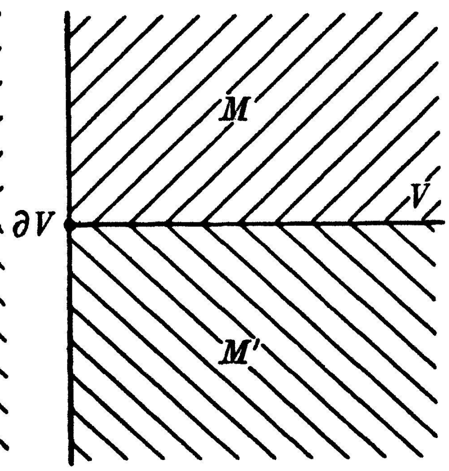
[1]) The results of this paper were announced in a lecture to the International Colloquium on Differential Geometry and Topology, Zürich, June 1960.

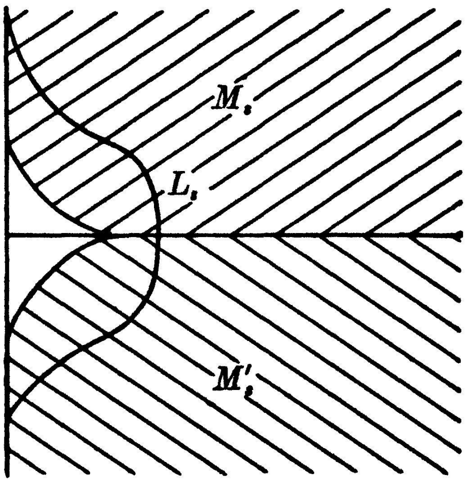
A manifold M_n is said to be endowed with G-structure, or called a G-manifold, if for some imbedding $M_n \to S^{n+K}$, a reduction of the normal bundle to G_K is given. We shall identify a G-structure with those induced from it by suspension (adding a trivial bundle preserves a natural G-structure in virtue of the inclusions above). It is known (STEENROD [12]) that a G-structure may be specified by a homotopy class of cross-sections of the bundle associated to the normal bundle of M and with fibre O_K/G_K.

Two closed G-manifolds are G-cobordant if together they form the boundary of another G-manifold, and its G-structure induces theirs. A little care is needed here: we regard the normal bundle of the boundary as the direct sum of the normal bundle of the manifold and the normal bundle of the boundary in the manifold, where the latter must be counted as pointing respectively inwards and outwards for the two manifolds on the boundary. This convention will allow us to show that G-cobordism is an equivalence relation, in view of Lemma 1.

We must now explain 'pasting and straightening', as introduced by MILNOR [7]. We shall adopt different, but equivalent definitions to his. Let M_n, M_n' be two differentiable manifolds, and let V_{n-1} be a submanifold of the boundary of each. Write $L = M + M'$, identified along V; we shall show that L has a natural differentiable structure. Near its boundary M is locally a product, also M'. Fixing a differentiable product structure allows us to define a differentiable structure on the union, except on ∂V. Now a neighbourhood of ∂V in ∂M or $\partial M'$ is also a product with an interval, so a neighbourhood of ∂V in L is a product of ∂V with Fig. 1. L is made differentiable by giving a homeomorphism of this with Fig. 2, diffeomorphic except on ∂V. (This is easy.) We can then form Fig. 3, showing how copies, M_s, M_s' of M, M' can be imbedded in L, and a copy L_s of L differentiably in $M_s \cup M_s'$ without change of differentiable structure.







Similar diagrams can also be drawn in the case when one or both of M, M' has already a corner along ∂V; for our application, M will be straight and M' have a right angle there.

Lemma 1. *Suppose M, M' are G-manifolds, and induce the same G-structure on V. Then L admits a G-structure inducing the given G-structures on M, M'.*

Proof. Suppose N so large that the normal bundle of an n-manifold in S^{N+n} is independent of the choice of imbedding (e.g., $n < N$, WHITNEY [15]). Take an imbedding of L in S^{N+n}; this induces imbeddings of the submanifolds M_s, M'_s, whose normal bundles are reduced to G_N. Hence we have cross-sections of the associated O_N/G_N-bundles. The two structures induce the same G-structure on V_s, so the restrictions of the suspended bundles with fibre O_{N+1}/G_{N+1} to V_s are homotopic. Extending a homotopy on V_s to one on M_s, we can find a cross section over $M_s + M'_s$, and thus reduce the bundle to G_{N+1}. But this is the suspended bundle, which is the same as the normal bundle for the suspended imbedding in S^{N+1}. Hence L_s has a G-structure, and it is clear from its construction that the corresponding G-structure on L induces the given ones on M, M'.

Note. The structure is not unique; we had to make a choice of a homotopy on V_s, and the difference between two such structures can be described by a bundle on SV.

Corollary. *G-cobordism is an equivalence relation.*

For we may paste together manifolds giving G-cobordisms of U to V and of V to W to find one giving a cobordism of U to W.

Examples of sequences (G_n) are O_n, SO_n (THOM [13], MILNOR [8], AVERBUCH [4] and WALL [14]), $U_{\frac{1}{2}n}$ (MILNOR [8] and NOVIKOV [10]), 1 (PONTRJAGIN [11] and KERVAIRE) $Sp_{\frac{1}{4}n}$ (NOVIKOV [10]), $SU_{\frac{1}{2}n}$, and $O_k \times O_{n-k}$, which suggests a general type which we will study below.

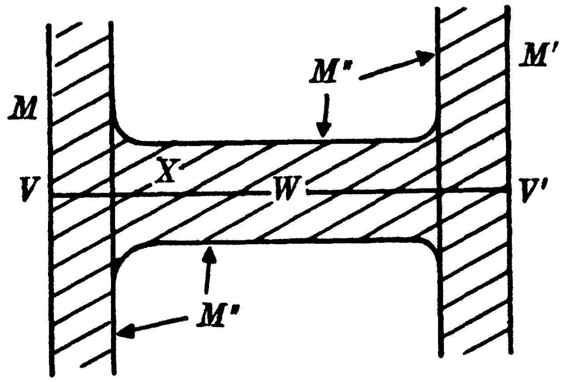
2. The reduction lemma

In accordance with the spirit of the first paragraph, we now make the following definitions. (M, V) is a *(G, L_k)-pair* if M is a G-manifold, and V a submanifold with normal bundle reduced to L_k. Two closed (G, L_k)-pairs (M, V), (M', V') are *(G, L_k)-cobordant* if there is a (G, L_k)-pair (N, W) such that $\partial N = M \cup M'$, $\partial W = V \cup V'$ and the (G, L_k)-structure of (N, W) induces the given (G, L_k)-structures of (M, V) and (M', V').

Now if (M, V) is a (G, L_k)-pair, we may choose an imbedding of M in a sphere with normal bundle reduced to G_N. But then the normal bundle of V is reduced to $L_k \times G_N$, so V has a natural H-structure, where H is defined by $H_{k+N} = L_k \times G_N$ (the definition of H_i for $i < k$ does not matter, but could be made by $H_i = 1$ or by $H_i = L_k \cap O_i$). We then see that if (N, W) provides a (G, L_k)-cobordism of (M, V) with (M', V'), W provides an H-cobordism of V with V'. We can now state the reduction lemma.

Lemma 2. *Two (G, L_k)-pairs (M, V), (M', V') are (G, L_k)-cobordant if and only if M, M' are G-cobordant and V, V' are H-cobordant.*

Proof. We have just seen the necessity of the condition. Now suppose it satisfied. Let W provide an H-cobordism of V, V'. Choose an imbedding of W in a sphere with normal bundle reduced to H_{k+N}. A neighbourhood of W is diffeomorphic to the associated bundle with fibre $E^{k+N} = E^k \times E^N$ (a small cell) and the subset E^k of this is stable under the operations of the group $H_{k+N} = L_k \times G_N$, so we can select a corresponding submanifold with boundary X, whose normal bundle, we note, is reduced to G_N, and the normal bundle of W in which is reduced to L_k. Now form the manifold $P = M \times I + X + M' \times I$.



Since by definition X induces the correct G-structure of a neighbourhood of V in M, we may apply Lemma 1 to deduce that P (with corners rounded as in the figure) is a G-manifold. If M'' denotes the middle components of the boundary of P, P induces a G-structure on it, and provides a G-cobordism of M'' (endowed with this G-structure) to $M \cup M'$, hence, since M is G-cobordant to M' and G-cobordism is transitive, M'' is G-cobordant to zero. If Q provides this G-cobordism, $P + Q = R$, which by Lemma 1 may be endowed with G-structure inducing the given structures on P, Q, provides a G-cobordism of M to M', and contains the submanifold $Y = V \times I + W + V' \times I$ whose normal bundle is reduced to L_k by construction, so that (R, Y) provides the required (G, L_k)-cobordism.

This lemma reduces the general problem of cobordism of pairs to the consideration of a single cobordism theory. It is now easy to see that by the same method we can now provide similar reductions for the problem of cobordism of n-tuples $M_1 \subset M_2 \subset \ldots \subset M_n$, with (if we so desire) assigned structural groups at each stage. For suppose inductively a cobordism given for M_{n-1}

(and the smaller manifolds); then this gives also (as above) a cobordism of the neighbourhood of M_{n-1} in M_n, and a glueing argument as above extends this to a cobordism of M_n as well. It is with this application in mind that we keep both G and L_k quite general.

In subsequent paragraphs we shall compute many of these groups of cobordism classes. We close this section by noting that the lemma has direct consequences such as the following:

The only obstruction to extending a cobordism W of V to one of M is the obstruction to extending the normal bundle of V in M to a bundle on W.

For once this extension is made, the above method will give an extension of W.

3. Algebraic Preliminaries

We now wish to compute cobordism theory for certain groups, which results of Thom [13] reduce to computing certain stable homotopy groups. These we shall evaluate using the homology of the appropriate spaces and the Adams spectral sequence. For this we must study certain modules over the Steenrod algebra. Our main tool will be the following lemma.

Lemma 3. *Let A be a connected graded Hopf algebra over a field k, F a free graded A-module, and M any graded A-module. Then $F \otimes_k M$ is a free A-module. If (f_n) is an A-base for F, and (m_r) a k-base for M, then $(f_n \otimes m_r)$ is an A-base for $F \otimes M$.*

Proof. Let (a_i) be a k-base for A. Then by hypothesis $(a_i f_n)$ is a k-base for F. Hence $(a_i f_n \otimes m_r)$ is a k-base for $F \otimes M$. The lemma states that $(a_i(f_n \otimes m_r))$ is a k-base for $F \otimes M$. To prove this we filter $F \otimes M$ by $C_p = \Sigma_0^p F_i \otimes M$. Then prove by induction on p that the $a_i(f_n \otimes m_r)$ with $\dim (a_i f_n) \leq p$ form a base for C_p. For $p = 0$ this is trivial, A being connected.

Suppose it true for $p-1$. Now if $\dim (a_i f_n) = p$,

$$\begin{aligned} a_i(f_n \otimes m_r) &= \Sigma a' f_n \otimes a'' m_r \\ &\equiv a_i f_n \otimes m_r \pmod{C_{p-1}} \end{aligned}$$

(using what we know about the diagonal homomorphism for A), and as it is clear that the $(a_i f_n \otimes m_r)$ with $\dim (a_i f_n) = p$ form a base of C_p modulo C_{p-1}, the result follows.

Complement. *Let A_2 be the Steenrod algebra over Z_2, G an A_2-module on one generator x and one relation $Sq^1 x = 0$, and M any A_2-module. Then $G \otimes M$ is a direct sum of a free module, and modules of type G.*

Proof. A_2, with the differential operator induced by right multiplication by Sq^1, is a free chain complex (first shown by Adem [2], but also follows from general results about HOPF algebras and subalgebras [9]; we have a base consisting of elements a_j, $a_j Sq^1$, and the $a_j x$ form a base of G.

M, with the differential operator induced by Sq^1, is also a chain complex. We take it in normal form, i.e. take a homogeneous base (l_s, m_t, n_t) of elements such that

$$Sq^1 l_s = O\,, \quad Sq^1 m_t = n_t\,, \quad Sq^1 n_t = O\,.$$

We now assert that $G \otimes M$ is the direct sum of free modules on the $x \otimes m_t$ and modules of type G on the $x \otimes l_s$. In fact, $a_j x$ is a base of G; $a_j Sq^1(x \otimes l_s) = O$; and the top terms of $a_j(x \otimes l_s)$, $a_j(x \otimes m_t)$ and $a_j Sq^1(x \otimes m_t) = a_j(x \otimes n_t)$ with respect to the same filtration as used in the proof of the lemma are $a_j x \otimes l_s$, $a_j x \otimes m_t$, and $a_j x \otimes n_t$ respectively, so the same inductive argument as before shows that we have a base of $G \otimes M$.

Let us call an A_2-module *simple* if it is the direct sum of a free module and modules of type G.

Corollary. *If F is a simple graded A_2-module, and M any graded A_2-module, then $F \otimes M$ is simple.*

This follows at once from the lemma and complement, on taking direct sums.

4. Application of THOM theory

It follows from the work of THOM [13], (which we shall suppose known), that cobordism groups for the structural group G_n are given by the homotopy groups of $M(G_n)$. Now in all the cases with which we shall be concerned, $\{G_n\}$ satisfies a certain stability condition. We shall denote the classifying space of a group L by $B(L)$; over it there is a canonical L-bundle [5]. Given a linear representation of L, we may form the associated vector bundle over $B(L)$; its one point compactification is the THOM space $M(L)$.

$\{G_n\}$ is said to be *stable* if for each n there exists a q such that for $q < p$ the induced cohomology map $H^n(B(G_p)) \to H^n(B(G_q))$ is an isomorphism. $H^n(B(G_q))$ is called a stable group, and denoted by $H^n(B(G))$.

Now G_n is a subgroup of O_n, so we have a linear representation ready to hand, and can define $M(G_n)$. Suspending the representation has the effect of suspending the THOM space [3], so the inclusion of G_q in G_{q+1} defines a natural map of $SM(G_q)$ to $M(G_{q+1})$ which, since the cohomology groups of a THOM space are isomorphic to those of the classifying space, but with a dimensional shift, induces an isomorphism of cohomology in dimension $n + q + 1$. Or supposing, as we clearly may, that q is chosen to increase with n, we have an isomorphism up to dimension $n + q + 1$. By the Universal Coefficient Theorem, we have an isomorphism of homology up to dimension $n+q$, and by a theorem of J. H. C. Whitehead, of homotopy up to dimension $n+q-1$. Thus we have homotopy groups of the $M(G_q)$ which become stable (for as $M(G_q)$ is q-connected, its homotopy groups up to dimension $2q-2$ are stable under suspension [13]). We shall refer to these as stable homotopy groups of $M(G)$. We may now state the following corollary of Thom theory.

Proposition. *If* $\{G_n\}$ *is stable, the cobordism groups for* G *are the stable homotopy groups of* $M(G)$.

We also note that the stability of cohomology of $M(G_q)$ allows us to define stable cohomology groups $H^*(M(G))$, and that since the isomorphisms are defined by induced homomorphisms and the suspension isomorphism, we can define stable cohomology operations acting in $H^*(M(G))$.

We now consider $H_{N+k} = L_k \times G_N$. Since $B(H_{N+k}) = B(L_k) \times B(G_N)$, the Künneth relations show that if G is stable, so is H. By [3], $M(H_{N+k})$ is the collapsed product $M(L_k) \# M(G_N)$. Hence for any coefficient field K, using reduced cohomology, we have

$$H^*(M(H_{N+k}), K) = H^*(M(L_k), K) \otimes H^*(M(G_N), K) \tag{1}$$

and if $K = Z_p$, the two sides of this equation are even isomorphic as modules over the Steenrod algebra A_p, as its action on an algebraic tensor product, using the diagonal homomorphism, was originally defined from the topological product (or cup product, which is the same thing) [6].

Before mentioning special cases, we shall define characteristic numbers in the general case. These we define as invariants of cobordism class directly; in fact given a map of a sphere S^N into a Thom space $M(G_n)$ defining a class, the corresponding characteristic numbers are the inverse images of classes in $H^N(M(G_n))$, evaluated on the fundamental homology class of S^N. To see the connection with the usual definition of characteristic numbers, we recall Thom's procedure; given a manifold V, we take a classifying map for its normal bundle in some S^N, and extend to a map of a tubular neighbourhood of V into the associated vector bundle; the rest of S^N is then mapped to the point at infinity. Hence the inverse image of the class in $H^N(M(G_n))$, which can be regarded as a class on the tubular neighbourhood of V, is the result of lifting the inverse image of the corresponding class in $H^k(B(G_n))$ in V_k. But this is the usual definition of characteristic classes; take a classifying map for the normal bundle of V, and evaluate the inverse image of a class on $B(G_n)$ on the fundamental class of V.

We are now ready to consider the orthogonal group. Note that in (1), there is a module multiplication by $H^*(M(G_N), K)$; this is the algebraic counterpart of the module multiplication mentioned in the introduction.

The Orthogonal Group

We now take $G_n = O_n$. It is well known that $H^*(M(O))$ is zero over any field K of characteristic not 2, and $H^*(M(O), Z_2)$ is a free A_2-module (THOM [13]). By Lemma 3, $H^*(M(H), Z_2)$ is also a free A_2-module, and we know how to choose a base. Hence there is a map of $M(H)$ to a product of EILENBERG-MACLANE spaces $K(Z_2, r)$ inducing an isomorphism of mod 2 cohomology, and hence also of integer homology since (by the analogue of (1) for Z as coefficient group) this consists entirely of torsion of order 2. Since the spaces are both simply connected, by the theorem of WHITEHEAD already mentioned, the map induces also an isomorphism of homotopy.

This determines the cobordism groups for the sequence $H_{N+k} = L_k \times O_N$. The full cobordism group is the graded direct sum of the cobordism groups in the various dimensions. Using the remark at the end of the previous paragraph, and the A_2-base constructed in Lemma 3, we may now enunciate

Theorem 1. *The cobordism groups for the sequence H are all of exponent 2. The cobordism class of a manifold is determined by its characteristic numbers. The full cobordism group is a free $\mathfrak{N}$-module, with a base corresponding to a Z_2-base of $H^*(B(L_k), Z_2)$.*

The Unitary Group

We now take $G_{2n} = G_{2n+1} = U_n$. It is convenient to work only with groups L_k satisfying a certain condition:
(B) *$H^*(B(L_k))$ is torsion free, and zero in odd dimensions.* We note that (B) holds if L_k itself is a unitary group, or a product of such. If (B) holds only up to a certain dimension, or modulo a certain set of primes, then our results will also hold up to nearly that dimension, or modulo those primes; such refinements we leave to the reader.

We shall now follow the arguments of MILNOR [8]. Now $H^*(B_U)$ is itself torsion free. Hence $H^*(M_U, Z_p)$ has a BOCKSTEIN operator Q_0 identically zero, and hence can be considered as a module over the algebra $A_p/(Q_0)$, quotient of the STEENROD algebra by the ideal generated by Q_0. We now use the fundamental result of MILNOR [8], Theorem 2 to the effect that it is a free module. By (B), $H^*(M(L_k), Z_p)$ also has zero BOCKSTEINS. We now apply Lemma 3 to the HOPF algebra $A_p/(Q_0)$, the free module $H^*(M(U), Z_p)$, and the module $H^*(M(L_k), Z_p)$ to deduce that their product $H^*(M(H), Z_p)$ is again a free module.

Moreover, since the generators of the Z-module $H^*(B(L_k))$ are even-dimensional, those of the $A_p/(Q_0)$-module $H^*(M(H), Z_p)$ are also even-dimensional.

It now follows from MILNOR, loc. cit., Theorem 1, that the stable homotopy groups are torsion free, and from the fact that the stable HUREWICZ homomorphism is an isomorphism mod finite groups we can deduce what the groups are.

Finally, we consider the module multiplication by the unitary group. We now know that the ADAMS spectral sequence [1] is trivial, so the fact that we have a free module in the E_2-term leads to one also in the E_∞-term, from which we can see that we not only have the cobordism group as a free module over the unitary one but also that all the divisibility conditions for characteristic numbers are implied by this product structure. We can now sum up our results in

Theorem 2. *The cobordism groups for the sequence H are torsion free, and zero in odd dimension. The cobordism class of a manifold is determined by its characteristic numbers. The full cobordism group is a free module over the unitary cobordism group, with generators in $(1-1)$ correspondence with a base for $H^*(B(L_k))$. Characteristic numbers satisfy just those divisibility conditions which are implied by the module structure.*

The Special Orthogonal Group

The case $G_n = SO_n$, as for straight cobordism theory, presents features which are a mixture of the two previous cases. The odd torsion behaves precisely as in the case of the unitary groups: if $H^*(B(L_k))$ is free of odd torsion, and in odd dimensions consists entirely of 2-torsion, the arguments of the preceding section go through without modification (even the references are the same) as fas as odd torsion is concerned. To make further progress, we now require in addition that all torsion in $H^*(B(L_k))$ be of order 2. Using the corollary to Lemma 3, and the fact that $H^*(M(SO), Z_2)$ is simple [14], we deduce that $H^*(M(H), Z_2)$ is also simple. In fact, the G-type generators for $H^*(M(SO))$ correspond to the free part of Ω, hence, since the l_s of the complement to Lemma 3 are the mod 2 restrictions of the free generators of $H^*(B(L_k))$, the G-type generators of $H^*(M(H))$ stand in correspondence with generators of the free part of stable homotopy (as determined above by considering the odd primes). Take then generators of the free part of $H^*(M(H))$ and corresponding maps to EILENBERG-MACLANE spaces $K(Z, n)$; then we already know that the product map induces isomorphisms of homotopy mod finite groups. Now these generators restrict mod 2 to precisely the generators of type G of the simple A_2-module. Choose a set of free generators, and corresponding maps to EILENBERG-MACLANE spaces $K(Z_2, m)$; then the map to the product of all these EILENBERG-MACLANE spaces induces mod 2 cohomology isomorphisms, and hence also mod 2 homology isomorphisms. We already know (using the above cited theorem of WHITEHEAD) that the map induces isomorphisms of homology mod finite groups; it now follows that it induces isomorphisms mod finite groups of odd order. Hence again it does also for homotopy, so all the 2-torsion has order 2, and is all captured by the homology.

Module multiplication follows as before; however we no longer have a free module unless $H^*(B(L_k))$ is completely torsion free; in the contrary case the Z_2-structure is different. (In fact, if we follow the appearances of all the Z_2's, it would appear that we should have the direct sum of a free module over Ω, and one over the algebra $\mathfrak{W}$ of [14].)

Theorem 3. *Cobordism groups for the sequence H are sums of free groups (which all occur in even dimensions) and groups of order 2. The cobordism class of a manifold is determined by its characteristic numbers. The full cobordism group contains a free Ω-submodule, with generators corresponding to a base of the free part of $H^*(B(L_k))$, as direct summand, its complement having exponent 2. Divisibility conditions (all by odd primes) are those implied by the module structure.*

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