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On the Euler characteristic of Compact Locally Affine Spaces

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Introduction

Let M be an n dimensional compact manifold endowed with an affine connection whose curvature and torsion tensors vanish identically. Over the years there has been some effort made to determine the Euler characteristic of such spaces. Indeed, J. Milnor [4], has shown that for dimension two the Euler characteristic must always be zero. In our study of this problem, we have been forced to place two additional conditions on our spaces. We have first required that our spaces shall be affinely complete. This is equivalent to assuming (see [3]) that the universal covering space of M, endowed with the lifted affine connection, shall be the affine plane. We will call a manifold with such a structure a locally affine space. The second restriction is topological and will be given in the statement of our main theorem.

Main Theorem. *Let M be a compact, complete locally affine space whose fundamental group contains a non-trivial normal abelian subgroup. Then the Euler characteristic of M is zero.*

Remark. All known examples of compact complete locally affine spaces have the property that their fundamental groups contain non-trivial normal abelian subgroups.

1. Background material

Let A^n denote the n dimensional affine plane and let $A(n)$ denote the group of all affine transformations of A^n. Further, let Γ denote the fundamental group of M, M a locally affine space. Then the affine connection on M induces an isomorphism of Γ into $A(n)$. We will identify Γ with its image in $A(n)$ under this isomorphism and we will identify M with the orbit space A^n/Γ.

Now $A(n)$ has a normal subgroup T, consisting of all pure translations, and $A(n)/T$ is isomorphic to $GL(n, R)$. Now $h(\Gamma) = \Gamma/\Gamma \cap T$ is the holonomy group of the connection.

Before proceeding with the proof of our main theorem we would like to remark that most of the machinery and results we will use have been available

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for some time in [1], [3], [6]. The major new material we will use is the following theorem of A. SELBERG [5].

SELBERG's Theorem. Let G be a finitely generated subgroup of a group of matrices over a field of characteristic zero. Then there exists a subgroup G^* of G, such that G^* is torsion free and of finite index in G.

2. Proof of Main Theorem

By means of homogeneous coordinates $A(n)$ can be given a faithful matrix representation in $GL(n+1, R)$. For the rest of this discussion, we will assume this imbedding has been made. Now let Z be an abelian normal subgroup of Γ. Then by Theorem 12 in [6], Z has all eigen values one. Let $H(Z)$ denote the algebraic hull of Z in $GL(n+1, R)$. Then $H(Z)$ is invariant under inner automorphisms of $GL(n+1, R)$ by elements of Γ. Further, $H(Z)$ is a connected abelian LIE group, let us say of dimension $s, s > 0$. Hence we have a representation Ψ of Γ in $GL(s, R)$. We may now apply the SELBERG Theorem to $\Psi(\Gamma)$ and get a subgroup Γ^* of Γ of finite index such that $\Psi(\Gamma^*)$ is torsion free. Let M^* be the covering space of M corresponding to A^n/Γ^*.

Theorem 1. *M^* is a fiber bundle over a manifold X^* with the s dimensional torus as fiber.*

Remark 1. Theorem 1 will prove that the EULER characteristic of M^* is zero and hence, since M^* is a finite covering of M, it will show that the EULER characteristic of M is also zero.

Remark 2. The proof of Theorem 1 will be essentially a modification of the proof of Theorem 1 in [1]. Because of this, we will just give an outline of the proof.

Proof of Theorem 1. Let us begin by considering $H(Z)$ acting in A^n. Then the orbit space $X = A^n/H(Z)$ is homeomorphic to A^{n-s}. We will now see that Γ^* acts properly discontinuously on X. Let $\gamma \in \Gamma^*$. Then since $\gamma H(Z) = H(Z)\gamma$, γ preserves the orbits of $H(Z)$ acting on A^n. Hence Γ^* may be considered as acting on X. We will denote this action of Γ^* on X by $I(\Gamma^*)$.

Assume now that $I(\gamma)x_0 = x_0$ for some $x_0 \in X$. Then γ^k maps the orbit of $H(Z)$ determining x_0 onto itself for all k. Hence γ^k must be in Z for some k or else Γ^* would not act on A^n properly discontinuously (for greater details see Lemma 1, [1]). This is impossible by our definition of Γ^*. Hence Γ^* acting on X has no fixed points.

It remains to show that $I(\Gamma^*)$ operates without accumulation points. Assume this is false; i.e. there exist $\gamma_1, \ldots, \gamma_n, \ldots$ in Γ^* such that the set $I(\gamma_i)x_0$, $i = 1, \ldots, n, \ldots$, is a CAUCHY sequence. Then there exists $\gamma_i \equiv \gamma_i$ mod. Z such that $\gamma_i' a_0$, $a_0 \in A^n$ would also be CAUCHY. This is impossible. (For greater details see Lemma 2, [1].) Hence we see that $X/I(\Gamma^*) = X^*$ is a manifold. It is trivial to verify that M^* is a fiber bundle over X^* with fiber an s dimensional torus. Hence X^* is compact. This completes the proof of Theorem 1 and hence by Remark 1 the proof of the Main Theorem.

The determination of whether or not X^* is also locally affine seems to be hard due to the difficulties exhibited by the examples in [2].

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