Factorization of 3-manifolds.

Factorization of 3-Manifolds

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If $M = S \times T$, where M, S and T are topological spaces, then, for each $n \geqslant 1$

$$\pi_n(M) \approx \pi_n(S) + \pi_n(T) .$$

The following conjecture, made by the author, would be a converse to the above theorem:

Conjecture. If M is a compact 3-manifold with no 2-sphere boundary components, and if $\pi_1(M) \approx C \times D$, where C is infinite and $D \neq 1$, then $M = S^1 \times T$, where T is a 2-manifold.

(We note that the 2-sphere boundary components are really irrelevant, since they may be removed or introduced by filling in or cutting out 3-balls. This operation does not change the fundamental group.)

In this paper, the conjecture is investigated and the following is proved:

Theorem. *If M is a compact 3-manifold and $\pi_1(M) = C \times D$, where C is infinite and $D \neq 1$, then either C or D is infinite cyclic.*

Using unpublished work of J. R. Stallings, and this theorem, it is possible to prove the conjecture, modulo composition with a homotopy 3-sphere, except for the case where M is closed and non-orientable. In the case where M is closed and non-orientable, it is possible to prove the conjecture up to homotopy type.

J. R. Stallings has made the following contributions to this paper. His proof (presented here) of Lemma (4.4) is a considerable improvement on the author's proof; and he gave the first proof of Lemma (6b.3) for the case where M is closed. The author would like to express his gratitude for this assistance.

The paper rests heavily on the results of Specker [7] and Hopf [3] and on the theory of the homology of groups.

§ 1. Ends

If K is a locally finite simplicial complex, let $C^*(K)$ and $C_f^*(K)$ be the groups of ordinary and finite cochains (simple integral coefficients will be used throughout this paper). We have an exact sequence

$$0 \to C_f^*(K) \to C^*(K) \to C_e^*(K) \to 0$$

which defines the term on the right. We deduce the exact sequence

$$0 \to H^0_f(K) \to H^0(K) \to H^0_e(K) \to H^1_f(K) \to H^1(K)\,. \qquad (1.1).$$

From results of Specker ([7], Satz III), we easily see that $H^0_e(K)$ is free abelian. *Its rank is equal to the number of ends of* K ([7], Satz IV).

Let G be any finitely generated group and let L be a finite connected simplicial complex, with the property that G is a quotient group of $\pi_1(L)$. Let K be the regular covering space of L which has G as group of covering translations. Then *the number of ends of* K *is independent of the choice of* K *and* L *and depends only on* G. (For proofs see [3]). This number is defined to be *the number of ends of* G.

According to [3], we have the following result:

(1.2). *A necessary and sufficient condition for* G *to have two ends is that it should have an infinite cyclic subgroup of finite index.*

§ 2. Well-Known Remarks

We make the following conventions throughout the paper:

All manifolds are connected and may have boundary, unless otherwise stated.

In any direct or free product of two groups, the factors are identified with he appropriate subgroups, in the natural way.

Z is the cyclic infinite group.

Z_r is the cyclic group of order r.

Lemma (2.1). *Let* M *be a 3-manifold and let* $\pi_1(M)$ *be infinite. A necessary and sufficient condition for* M *to be aspherical is that* $\pi_2(M) = 0$.

This is proved by applying the Hurewicz Isomorphism Theorem to the universal cover of M.

Lemma (2.2). *If* P *is a compact 3-manifold and* $\pi_1(P) = 1$, *then* P *is a homotopy 3-sphere with a number of 3-balls removed. If* Bd P *consists of only one component* S, *then* S *is contractible in* M.

P is orientable, since it has no double coverings. So all the components of Bd P are 2-spheres ([6] p. 223, Satz IV). Filling in all these 2-spheres with 3-balls, we obtain a closed 3-manifold with trivial fundamental group. Such a manifold is a homotopy 3-sphere. If Bd P contains only one component S, then S bounds P. So S is homologically trivial. By the Hurewicz Isomorphism, S is contractible. This proves Lemma (2.2).

Lemma (2.3). *Let M be an orientable compact 3-manifold with no 2-sphere boundary components. Let $\pi_2(M) \neq 0$. Then $\pi_1(M) \approx Z$ or $A*B \# (A \neq 1, B \neq 1)$.*

Since $\pi_2(M) \neq 0$, the Sphere Theorem [9] gives us a non-contractible 2-sphere S in M. If S separates M into two components P and Q, with fundamental groups A and B respectively, then $\pi_1(M) \approx A*B$ by VAN KAMPEN's Theorem. If $A = 1$, then, by Lemma (2.2), P has only 2-sphere boundaries. Since M has no 2-sphere boundaries, we must have $\operatorname{Bd} P = S$. By Lemma (2.2), S is contractible in P. But S is not contractible in M, which is a contradiction. So $A \neq 1$ and similarly $B \neq 1$. If S does not separate M, then $\pi_1(M) \approx Z^* \pi_1(M - S)$. This proves Lemma (2.3).

Combining Lemmas (2.1) and (2.3), we obtain:

Lemma (2.4). *If M is not aspherical and is an orientable compact 3-manifold with no 2-sphere boundary components, and if $\pi_1(M)$ is infinite, then*

$$\pi_1(M) \approx Z \quad \text{or} \quad A^*B \# \; (A \neq 1, B \neq 1) .$$

Lemma (2.5). *If T is a component of $\operatorname{Bd} M$, where M is a compact 3-manifold, and if $\pi_1(T) \to \pi_1(M)$ is not a monomorphism, then*

$$\pi_1(M) \approx Z \quad \text{or} \quad A^*B \# \; (A \neq 1, B \neq 1) .$$

By the Loop Theorem [8], there is a disk E in M such that $\operatorname{Bd} E \subset T$ and $\operatorname{Bd} M \cap E = \operatorname{Bd} E$, and $\operatorname{Bd} E$ is essential in T. If E separates M into two components P and Q with fundamental groups A and B respectively, then by VAN KAMPEN's Theorem, $\pi_1(M) \approx A*B$. If $A = 1$, then, by Lemma (2.2), $\operatorname{Bd} P$ consists of 2-spheres. One of these 2-spheres consists of the disk E and a disk which is part of T. Therefore $\operatorname{Bd} E$ is contractible in T, which is a contradiction. We conclude that $A \neq 1$, and similarly $B \neq 1$. If E does not separate M, then $\pi_1(M) \approx Z*\pi_1(M - E)$. This proves Lemma (2.5).

Lemma (2.6). *If M is a closed orientable 3-manifold, and $\pi_1(M)$ is torsion free, but not free, then $H_3(\pi_1(M)) \neq 0$.* (We recall that simple integral coefficients are used throughout).

For any closed orientable 3-manifold, $M = M_1 \# \ldots\ldots \# M_k$, where each M_i is a closed orientable 3-manifold satisfying one of the following three conditions for each i, $1 \leqslant i \leqslant k$:

i) $\pi_1(M_1)$ is finite.

ii) M_i is aspherical.

iii) $M_i = S^1 \times S^2$.

(This theorem is due to Milnor. See [4] and [5] p. 439, (21) and (22)). Therefore $\pi_1(M) \approx \pi_1(M_1) * \ldots\ldots * \pi_1(M_k)$, by van Kampen's Theorem.

In the case under consideration here, we see that if $\pi_1(M_i)$ is finite then it is trivial, since $\pi_1(M)$ is torsion free. So we can assume that $\pi_1(M_i)$ is not finite for any i. Since $\pi_1(M)$ is not free, M_i must be aspherical for at least one i. Then $H_3(\pi_1(M_i)) \approx H_3(M_i) \approx Z$. Now

$$H_3(\pi_1(M)) \approx H_3(\pi_1(M_1)) + \ldots\ldots + H_3(\pi_1(M_k)) .$$

So Lemma (2.6) follows.

Lemma (2.7.) *Let M be a compact 3-manifold with all elements of $\pi_1(M)$ of finite order. Then $\pi_1(M)$ is finite.*

Suppose $\pi_1(M)$ is infinite. We may assume M is orientable by taking a double cover, if M is non-orientable. We may also assume M has no boundary 2-spheres by filling in such boundaries with 3-balls. If M is not aspherical, then by Lemma (2.4), $\pi_1(M) \approx Z$ or $A*B$, and so $\pi_1(M)$ has an element of infinite order. It M is aspherical, then $\pi_1(M)$ must be torsion free since if is the fundamental group of a finite dimensional aspherical space. Lemma (2.7) follows.

Lemma (2.8). *If T is an orientable 2-manifold and $\pi_1(T)$ contains a free abelian subgroup of rank two, then T is a torus.* (By a torus, we mean a closed orientable surface of genus one).

Let $S \to T$ be the covering of T such that $\pi_1(S) \approx Z \times Z$. S is obviously not a 2-sphere, and so S is aspherical. So $H_2(S) \approx H_2(Z \times Z) \approx Z$. Therefore S is a closed 2-manifold. In fact S is a torus. Therefore $\chi(S) = 0$. Since the covering map $S \to T$ is onto and a local homeomorphism, T is closed. The covering must be finite sheeted since S is compact. Let there be r sheets. Then $0 = \chi(S) = r\chi(T)$. So $\chi(T) = 0$. Therefore T is a torus and Lemma (2.8) follows.

Lemma (2.9). *If T is a torus, and $\pi_1(T)$ has generators u and v, then u can be represented by a simple closed curve.*

Let a and b be the usual generators of $\pi_1(T)$, whose representatives are two simple closed curves intersecting at one point only. Then $u = a^p b^q$. If $p = 0$ then $q = \pm 1$, and Lemma (2.8) is true. Similarly if $q = 0$. So we assume $p \neq 0$ and $q \neq 0$. Then $(p, q) = 1$ since u, v and a, b are connected by a unimodular transformation.

Let $P \to T$ be the universal covering of T. P is the Euclidean plane. The covering translations corresponding to a and b are translations of one unit parallel to the x-axis and the y-axis respectively. Let us construct a representative of u in T by drawing a straight line l from (0,0) to (p, q) in P and projecting down to T. The image of l under any covering translation only meets l at an end-point. So the representative of u in T is a simple closed curve. This proves Lemma (2.9).

§ 3. The Orientable Case

In this section, we assume that M is a compact, orientable 3-manifold and $\pi_1(M) = C \times D$, where C is infinite and $D \neq 1$. We also assume M has no boundary 2-spheres.

Lemma (3.1). *M is aspherical.*

If M is not aspherical, then, by Lemma (2.4), $\pi_1(M) \approx Z$ or $A * B$ # $(A \neq 1, B \neq 1)$. But neither of these groups can be a non-trivial direct product (Folgerung 4, [1]). Lemma (3.1) follows.

Lemma (3.2). *C and D are torsion free and so D is infinite.*

Since M is a finite dimensional aspherical space, its fundamental group $C \times D$ must be torsion free. Lemma (3.2) follows.

Lemma (3.3). *If $G = C \times D$ is finitely generated, then so are C and D.*

This is proved by projecting the generators of G into the factors C and D in turn.

§ 4. The Closed Orientable Case

In this section, we assume M is a closed orientable 3-manifold and $\pi_1(M) = C \times D$, where C is infinite and $D \neq 1$. From Lemma (3.2), we see that D is also infinite. So we are free to interchange the names C and D of the two direct factors. The aim of this section is to prove:

Theorem (4.1). *C or D is infinite cyclic.* (By interchanging C and D, if necessary, we shall make sure that C is infinite cyclic).

From Lemma (3.1), we see that any covering space of M is aspherical. Let M_C and M_D be the regular covering spaces of M with fundamental groups C and D respectively. Then M is homotopy equivalent to $M_C \times M_D$, since they are aspherical spaces with the same fundamental groups.

Lemma (4.2). *On interchanging C and D, if necessary, $H_1(M_C) \approx Z$ and $H_2(M_D) \approx Z$.*

By Lemma (3.2), M_C and M_D are infinite sheeted coverings of M, and are therefore non-compact 3-manifolds. So $H_3(M_C) = 0$ and $H_3(M_D) = 0$.

By the KÜNNETH formula

$$Z \approx H_3(M) \approx H_1(M_C) \otimes H_2(M_D) + H_2(M_C) \otimes H_1(M_D) + \\ + \operatorname{Tor}(H_1(M_C), H_1(M_D)) .$$

By Lemma (3.3), C and D are finitely generated. Therefore

$$H_1(M_C) \approx C/[C, C] \quad \text{and} \quad H_1(M_D) \approx D/[D, D]$$

are finitely generated abelian groups. Therefore $\operatorname{Tor}(H_1(M_C), H_1(M_D))$ is finite, and, as a subgroup of Z, is zero. Therefore

$$Z \approx H_1(M_C) \otimes H_2(M_D) + H_2(M_C) \otimes H_1(M_D) .$$

We can assume, by interchanging C and D if necessary that

$$Z \approx H_1(M_C) \otimes H_2(M_D) .$$

Then $H_2(M_D) \neq 0$ and $H_1(M_C)$ cannot be finite. So, the finitely generated abelian group $H_1(M_C) \approx Z + H$, for some abelian group H. Therefore

$$Z \approx H_1(M_C) \otimes H_2(M_D) \approx (Z + H) \otimes H_2(M_D) \approx H_2(M_D) + H \otimes H_2(M_D) .$$

Since $H_2(M_D) \neq 0, H_2(M_D) \approx Z$. Therefore

$$Z \approx H_1(M_C) \otimes H_2(M_D) \approx H_1(M_C) \otimes Z \approx H_1(M_C) .$$

This proves Lemma (4.2).

Lemma (4.3). *C has two ends.*

M_D is a regular covering space of M with C as a group of covering translations. So the number of ends of C is equal to the number of ends of M_D (see § 1). Since M_D is infinite, $H_f^0(M_D) = 0$. Also $H_f^1(M_D) \approx H_2(M_D) \approx Z$, by POINCARÉ Duality and Lemma (4.2). The sequence (1.1) therefore becomes

$$0 \to Z \to H_e^0(M_D) \to Z .$$

By [7], last line of Satz V,

$$\operatorname{Coker}(H_e^0(M_D) \to H_f^1(M_D))$$

is either zero or free abelian of infinite rank. In this case, we must have zero, and so the map is an epimorphism. From the exact sequence, we deduce that $H_e^0(M_D) \approx Z + Z$. Therefore M_D has two ends, and Lemma (4.3) is proved.

Lemma (4.4). $C \approx Z$.

By (1.2) and Lemma (4.3), C has an infinite cyclic subgroup G of finite index. We have an epimorphism

$$C \to C/[C, C] \approx Z$$

by Lemma (4.2). C is torsion free by Lemma (3.2).

Any map $Z \to Z$ is either a monomorphism or trivial. The composition

$$Z \approx G \to C \to C/[C, C] \approx Z$$

cannot be trivial. For if it were, then, since G is of finite index in C, the image of C in $C/[C, C]$ would be finite. So the composition must be a monomorphism.

Consider the map $C \to C/[C, C]$. Each coset of G in C is mapped monomorphically. Since G has finite index in C, the kernel of this map must be finite. Since C is torsion free, the kernel must be trivial. Therefore the map $C \to C/[C, C]$ is an isomorphism. This proves Lemma (4.4) and completes the proof of Theorem (4.1).

§ 5. The Orientable Non-Closed Case

In this section we assume M is an orientable compact 3-manifold with boundary and $\pi_1(M) = C \times D$ where C is infinite and $D \neq 1$. The aim of this section is to prove:

Theorem (5.1). *C or D is infinite cyclic.*

We fill in each boundary 2-sphere of M with a 3-ball. This does not change the fundamental group. If the resulting manifold is closed, we can apply Theorem (4.1). So we assume throughout this section that M *has no 2-sphere boundaries.*

Lemma (5.2). *If G is any subgroup of C or of D, then $H_i(G) = 0$ for $i \geqslant 2$.*

For suppose $G \subset C$ and $H_i(G) \neq 0$, for some $i \geqslant 2$. Let $d \in D$ be a non-trivial element. By Lemma (3.2), G and d generate a subgroup H of $\pi_1(M)$ isomorphic to $G \times Z$. By the KÜNNETH formula, $H_{i+1}(H) \neq 0$. Let N be the covering space of M, with fundamental group H. M is aspherical by Lemma (3.1). Therefore N is aspherical. Therefore $H_{i+1}(N) \approx H_{i+1}(H) \neq 0$. But this is a contradiction since N is a non-closed 3-manifold. Lemma (5.2) follows.

Lemma (5.3). *If T is a component of Bd M, then $\pi_1(T) \to \pi_1(M)$ is a monomorphism.*

For if not, $\pi_1(M) \approx Z$ or $A*B \# (A \neq 1, B \neq 1)$. But neither of these groups can be a non-trivial direct product (Folgerung 4, [1]). Lemma (5.3) follows.

We now take a fixed maximal tree in a triangulation of M and a fixed vertex x_0 as base-point in M. Let γ be a loop in M based on a vertex y. Then, drawing a path in the tree from x_0 to y, we see that γ represents a unique element of $\pi_1(M, x_0)$. With this understood, we again drop the x_0 and write simply $\pi_1(M)$. The representatives of elements of $\pi_1(M)$ are loops based on any vertex of M.

By Lemma (5.3), if T is a component of Bd M, then $\pi_1(T) \to \pi_1(M)$ is a monomorphism. For each T, *we identify $\pi_1(T)$ with a subgroup of $\pi_1(M)$.*

Lemma (5.4). *If T is a component of* Bd M, *then* $1 \neq \pi_1(T) \cap C \neq \pi_1(T)$ *and* $1 \neq \pi_1(T) \cap D \neq \pi_1(T)$.

Since T is not a 2-sphere, $H_2(\pi_1(T)) \approx H_2(T) \approx Z$. The map $\pi_1(T) \to C$ induced by the projection $C \times D \to C$ has kernel $\pi_1(T) \cap D$. By Lemma (5.2), $\pi_1(T)$ cannot be isomorphic to a subgroup of C. Therefore the map $\pi_1(T) \to C$ is not a monomorphism and so $1 \neq \pi_1(T) \cap D$. Similarly $1 \neq \pi_1(T) \cap C$. By Lemma (5.2), $\pi_1(T)$ cannot be a subgroup of C. Therefore $\pi_1(T) \cap C \neq \pi_1(T)$. Similarly $\pi_1(T) \cap D \neq \pi_1(T)$. This proves Lemma (5.4).

Lemma (5.5). *If T is a component of* Bd M, *then T is a torus.*

By Lemma (5.4), there are non-trivial elements $c \in C \cap \pi_1(T)$ and $d \in D \cap \pi_1(T)$. The elements c and d generate a free abelian subgroup of $\pi_1(T)$ of rank two, by Lemma (3.2). By Lemma (2.8), T is a torus. This proves Lemma (5.5).

Lemma (5.6). *For each component T of* Bd M, *we can choose generators c_T and g of $\pi_1(T)$, such that $c_T \in C$.*

The map $\pi_1(T) \to D$, induced by the projection $C \times D \to D$, has kernel $\pi_1(T) \cap C$. So, by Lemma (5.4), the map $\pi_1(T) \to D$ is neither trivial nor a monomorphism. D is torsion free, by Lemma (3.2) and $\pi_1(T)$ is free abelian of rank two by Lemma (5.5). So the map has an infinite cyclic group as image. Let $d_1 \in D$ be a generator of this image. Then any element of $\pi_1(T)$ can be written in the form cd_1^n where $c \in C$.

Let generators of $\pi_1(T)$ be $u = c_1 d_1^p$ and $v = c_2 d_1^q$. If p or q is zero, Lemma (5.6) is true. So we assume both p and q are non-zero. Then $(p,q)=1$, since d_1 is in the image of $\pi_1(T) \to D$. Therefore $u^q v^{-p} = c_1^q c_2^{-p} \in C$. Since $(p, q) = 1$ this element can be taken as one of two generators of $\pi_1(T)$. (Different pairs of generators of $\pi_1(T)$ are related to each other by unimodular transformations). This proves Lemma (5.6).

Lemma (5.7). *There is a simple closed curve* γ_T *on each component* T *of* Bd M, *which represents a non-trivial element in* C.

This follows immediately from Lemmas (2.9) and (5.6).

Each γ_T has a neighbourhood on T, homeomorphic to $\gamma_T \times I$, where $I = [-1, 1]$ and γ_T is identified with $\gamma_T \times 0$. Let E be the closed unit disk in the Euclidean plane, and $\mathrm{Bd}\, E = S^1$. Now $S^1 \times I \subset \mathrm{Bd}\,(E \times I)$. Selecting one particular T, we glue $E + I$ onto M in the obvious way, by identifying $\gamma_T \times I$ with $S^1 \times I$. An easy calculation with the Euler characteristic shows that we have converted the torus boundary component T into a 2-sphere. We fill in the 2-sphere with a 3-ball. This last step does not change the fundamental group. The effect of the operation we have performed is therefore to reduce the number of components of Bd M by one and to add an extra relation to C.

We repeat this operation until all the components of Bd M have been eliminated. If there are n components of Bd M to begin with, the total effect of these operations is to give us a closed orientable 3-manifold M^1 and to add n extra relations to C, giving us a group C^1. Then $\pi_1(M^1) \approx C^1 \times D$.

Lemma (5.8). *If Theorem* (5.1) *is false, then* $C^1 = 1$.

If $C^1 \neq 1$, Theorem (4.1) tells us that C^1 or D is cyclic infinite. Since we are assuming Theorem (5.1) is false, D is not cyclic infinite. Therefore C^1 is cyclic infinite. By Lemma (3.1), M^1 is aspherical. Also $H_i(D) = 0$ for $i \geqslant 2$, by Lemma (5.2). Therefore, by the Künneth formula

$$Z \approx H_3(M^1) \approx H_3(C^1 \times D) = 0 .$$

This is a contradiction, which proves Lemma (5.8).

Lemma (5.9). *If Theorem* (5.1) *is false, then* C *and* D *are free.*

M^1 is a closed orientable 3-manifold and, by Lemma (5.8), $\pi_1(M^1) \approx D$. By Lemma (3.2), D is torsion free. By Lemma (5.2), $H_3(D) = 0$. Therefore, by Lemma (2.6), D is free. Interchanging the names C and D, we see that C is free also. Lemma (5.9) follows.

Lemma (5.10). $\chi(M) = 0$.

If we fill in each boundary torus of M with a solid torus, we do not change the Euler characteristic. But the Euler characteristic of a closed 3-manifold is zero, by Poincaré Duality. Lemma (5.10) follows.

Lemma (5.11). *If Theorem* (5.1) *is false, then we obtain a contradiction.*

By Lemmas (5.9) and (3.3), C and D are free of finite ranks m and n respectively. Since we are assuming Theorem (5.1) is false, $m \geqslant 2$ and $n \geqslant 2$. $H_0(M)$ is free abelian of rank one, and $H_1(M)$ is free abelian of rank $(m+n)$. $H_2(M) \approx H_2(C \times D)$ since M is aspherical by Lemma (3.1). By the Künneth formula, $H_2(M)$ has rank mn. $H_i(M) = 0$ for $i \geqslant 3$, since M is not closed. Therefore

$$\chi(M) = 1 - m - n + mn = (n-1)(m-1) > 0 .$$

But this contradicts Lemma (5.10). Lemma (5.11) follows.

Theorem (5.1) follows from Lemma (5.11).

§ 6. The Non-Orientable Case

In this section we assume M is a non-orientable compact 3-manifold, and that $\pi_1(M) = C \times D$, where C is infinite and $D \neq 1$. The aim of this section is to prove:

Theorem (6.1). *C or D is cyclic infinite.*

Theorem (6.1) will be proved separately for the two cases, M aspherical and M not aspherical.

Let $p: \tilde{M} \to M$ be the orientable double covering of M. Then $p_*(\pi_1(\tilde{M}))$ has index two in $\pi_1(M)$. Let $C_1 = p_*(\pi_1(\tilde{M})) \cap C$ and $D_1 = p_*(\pi_1(\tilde{M})) \cap D$. Then C_1 and D_1 have index at most two in C and D respectively. C_1 is infinite, since C is infinite. Let $\overline{M}$ be the (orientable) covering space of M with fundamental group $C_1 \times D_1$. The covering $M \to \overline{M}$ is at most four sheeted, so $\overline{M}$ is compact.

§ 6a. The Non-Orientable Aspherical Case

Here we assume in addition that M is aspherical.

Lemma (6a.1). *C and D are torsion free, and so D is infinite and $D_1 \neq 1$.*

This follows as in Lemma (3.2).

Lemma (6a.2). *C or D is cyclic infinite.*

We note that $\overline{M}$ is a compact orientable 3-manifold and $\pi_1(\overline{M}) \approx C_1 \times D_1$ where, by Lemma (6a.1), C_1 is infinite and $D_1 \neq 1$. So we can apply Theorems (4.1) and (5.1) to deduce that C_1 or D_1 is cyclic infinite. Without loss of generality, we assume C_1 is cyclic infinite.

Either $C_1 = C$, in which case we are done, or C_1 has index two in C. In the latter case, C has a presentation

$$\{\alpha, \beta / \beta^{-1}\alpha\beta = \alpha^{\varepsilon}, \beta^2 = \alpha^n\}$$

where α generates C_1, $\beta \notin C_1$ and $\varepsilon = \pm 1$. Then $\beta^2 = \beta^{-1}\alpha^n\beta = \alpha^{n\varepsilon} = \beta^{2\varepsilon}$. Since C is torsion free, $\varepsilon = 1$. Therefore C is abelian. If $n = 2m$, then $(\beta^{-1}\alpha^m)^2 = 1$ and so $\beta = \alpha^m$. This is impossible since β / C_1. Therefore $n = 2m + 1$ and C is generated by $\beta^{-1}\alpha^m$. This proves Lemma (6a.2).

§ 6b. The Non-Aspherical Case

Here we assume, in addition to the assumptions at the beginning of § 6, that M is not aspherical. We can also assume that M *has no 2-sphere boundaries*, since we can fill them in with 3-balls. This operation does not change the fundamental group. If M becomes aspherical, we can apply Lemma (6a.2) to obtain Theorem (6.1).

Lemma (6b.1). *If* $D_1 = 1$, *then* $\pi_1(M)$ *has a subgroup isomorphic to* $Z \times Z_2$.

$D \approx Z_2$ since D_1 is of index two in D. $C_1 \approx \pi_1(\overline{M})$ is infinite. By Lemma (2.7) there is an element c of infinite order in C_1. D and c generate a subgroup of $\pi_1(M)$ isomorphic to $Z \times Z_2$. Lemma (6b.1) follows.

Lemma (6b.2). *If* $D_1 \neq 1$, *then* $\pi_1(M)$ *contains a subgroup isomorphic to* $Z \times Z_2$.

$\pi_1(\overline{M}) \approx C_1 \times D_1$ where C_1 is infinite and $D_1 \neq 1$. So C_1 and D_1 are torsion free, from Lemma (3.2). Let M have no projective plane boundaries; then $\overline{M}$ has no 2-sphere boundaries. By Lemma (3.1), $\overline{M}$ is aspherical and therefore M is aspherical. But we are assuming M is not aspherical. Therefore M has at least one projective plane boundary. Therefore $\pi_1(M) = C \times D$ contains an element of order two. Therefore either C or D contains an element of order two. Since C_1 and D_1 are torsion free, Lemma (6b.2) follows.

Lemma (6b.3). $\pi_1(M) \approx Z \times Z_2$.

From Lemmas (6b.1) and (6b.2) we see that $\pi_1(M)$ contains a subgroup isomorphic to $Z \times Z_2$. By Theorem (9.5) of [2],

$$C \times D \approx \pi_1(M) \approx (Z \times Z_2) * G,$$

for some group G. By Folgerung 4 of [1], $G = 1$. Therefore $\pi_1(M) \approx Z \times Z_2$.

Lemmas (6a.2) and (6b.3) imply Theorem (6.1).

Theorems (4.1), (5.1) and (6.1) imply the theorem stated at the beginning of the paper.

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