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**Autor:** Hirsch, Morris W.  
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# On Combinatorial Submanifolds of Differentiable Manifolds<sup>1)</sup>

by MORRIS W. HIRSCH, California (USA)

## § 1. Introduction

The purpose of this work is to prove the following results relating combinatorial and differentiable manifolds.

(A) A combinatorial submanifold  $V$  of a differentiable manifold  $M$  of the same dimension possesses a compatible differentiable structure.

(B) Every compact and contractible combinatorial manifold  $V$  possesses a compatible differentiable structure.<sup>2)</sup>

(A differentiable structure on a combinatorial manifold  $M$  is called *compatible* if  $M$  has a rectilinear subdivision, each simplex of which is differentiably imbedded.)

A. M. GLEASON has announced (unpublished) that a contractible *unbounded* combinatorial manifold has a compatible differentiable structure. Theorem (B) follows easily from this and Theorem (A). The proof of (B) given here is derived from JOHN STALLINGS' proof [11] of the generalized POINCARÉ conjecture.

(C) The sequence

$$\dots \xrightarrow{d} \Gamma^n \xrightarrow{j} \Theta^n \xrightarrow{k} \Lambda^n \xrightarrow{d} \Gamma^{n-1} \longrightarrow \dots \quad (1)$$

is well defined and exact.

This result was announced in [3]. Here  $\Gamma^n$  is the group of differentiable structures on  $S^n$  compatible with the usual combinatorial structure;  $\Theta^n$  is the group of differentiable homotopy  $n$ -spheres modulo  $J$ -equivalence, and  $\Lambda^n$  the combinatorial analogue of  $\Omega^n$ . Using powerful intrinsic methods, STEPHEN SMALE has shown [9, 10] that for  $n \geq 8$  or  $n = 5$ , the map  $j: \Gamma^n \rightarrow \Theta^n$  is an isomorphism, and using (B) (but not (C)) that  $\Lambda^n = 0$ , for  $n \geq 8$  or  $n = 6$ .

In [8] SMALE proves that  $\Gamma^3 = 0$ , (this was proved independently by J. MUNKRES, and J. H. C. WHITEHEAD.) The fact that every combinatorial 3-manifold possess a unique (up to a diffeomorphism) compatible differentiable structure [6] implies that  $k: \Theta^3 \rightarrow \Lambda^3$  is an isomorphism. Thus only the

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<sup>1)</sup> Presented at the International Colloquium on Differential Geometry and Topology, Zürich, June 1960.

<sup>2)</sup> Added in proof; The hypothesis of compactness is unnecessary.

subsequences  $0 \rightarrow \Gamma^7 \rightarrow \Theta^7 \rightarrow \Lambda^7 \rightarrow \Gamma^6 \rightarrow 0$  and  $0 \rightarrow \Lambda^5 \rightarrow \Gamma^4 \rightarrow \Theta^4 \rightarrow \Lambda^4 \rightarrow 0$  remain. In proving (C), SMALE's results are not used.

## § 2. Proof of (A)

In order to prove (A) it suffices to establish the stronger result 2.5 below.

If  $K$  is a subcomplex of complex  $N$ , the  $n$ th *simplicial neighborhood* of  $K$  is the union of the closed simplexes of the  $n$ 'th barycentric subdivision of  $N$  that meet  $K$ .

**Lemma 2.1.** *Let  $K$  be the boundary of a combinatorial manifold  $M$ . The second simplicial neighborhood  $A$  of  $K$  is combinatorially equivalent to  $K \times I$ , where  $I$  is the unit interval.*

*Proof.* This is a well known result. It follows e. g. from theorems 22 and 23 of [14], which state that any two "regular neighborhoods" of  $M$  (in the sense of [14]) in the same manifold are combinatorially equivalent and that  $A$  is a regular neighborhood of  $K$  in  $M$ . If  $K$  is identified with  $K \times 0$ , then  $M' = M \cup K \times I$  is a manifold, and in  $M'$  both  $A$  and  $K \times I$  are regular neighborhoods of  $K$ .

Now let  $V$  be a bounded combinatorial  $n$ -manifold imbedded as a subcomplex of an unbounded combinatorial  $n$ -manifold  $M$ . Assume  $M$  has a metric  $d(x, y)$ .

**Lemma 2.2.** *Let  $U$  be a neighborhood of  $V$  in  $M$ , and  $\epsilon$  a positive continuous function on  $M$ . There is a semi-linear homeomorphism  $h: M \rightarrow M$  with the following properties:*

- a)  $h(V)$  is the second simplicial neighborhood of  $V$  in a subdivision of  $M$ ;
- b)  $h(V) \subset U$ ;
- c)  $h(x) = x$  if  $x \in M - U$ ;
- d)  $d(x, h(x)) < \epsilon(x)$  for all  $x \in M$ .

*Proof.* By 2.1, the boundary of  $K$  of  $V$  has a neighborhood combinatorially equivalent to  $K \times I$  in  $V$ , and another in  $cl(M - V)$ . The union  $B_0$  of these two neighborhoods is again equivalent to  $K \times I$ . Moreover, we can take  $B_0$  to be the second simplicial neighborhood of  $K$  in a subdivision of  $M$ ; if this subdivision is sufficiently fine, the second simplicial neighborhood  $B$  of  $B_0$  will be in  $U$ . It will be clear that if the subdivision is sufficiently fine, d) will be satisfied. There is a combinatorial equivalence  $u: B \rightarrow K \times I$  such that  $u(B_0) = K \times [1/4, 3/4]$  and  $u(K) = K \times 1/2$ . We may assume that  $h(B_0 \cap V) = K \times [0, 1/2]$ . Let  $f: I \rightarrow I$  be a semi-linear homeomorphism

such that  $f(x) = x$  for  $x$  in a neighborhood of 0 and 1, and  $f(1/2) = 3/4$ . Define  $g: K \times I \rightarrow K \times I$  by  $g(x, t) = (x, f(t))$ . Now define  $h: M \rightarrow M$  by  $h(x) = \begin{cases} x & \text{if } x \in M - B \\ u^{-1}gu(x) & \text{if } x \in B. \end{cases}$

Then  $h$  is the desired homeomorphism.

Now let  $M$  be a differentiable manifold. A combinatorial manifold  $A$  which is a subcomplex of a smooth triangulation of  $M$  is called a *combinatorial submanifold of  $M$* . A vector field  $\Phi$  on  $A$  in  $M$  is *transverse* if it is transverse to  $A$  in every coordinate system, in the sense of [13]. The following lemma is well known.

**Lemma 2.3.** *Let  $A$  be the boundary of the second simplicial neighborhood  $B$  of a subcomplex  $K$  of  $M$ . Then there is a transverse field on  $A$ .*

*Proof.* Each simplex  $\rho$  of the first simplicial neighborhood  $B'$  of  $K$  is the join  $\sigma * \tau$  of unique simplices  $\sigma \subset K$  and  $\tau \subset M - K$ . Each closed simplex  $\alpha$  of  $A$  lies in such a join  $\sigma * \tau$ , disjoint from  $\sigma$  and  $\tau$ , and each  $x \in \alpha$  lies on a unique line  $p * q$  with  $p \in \sigma$ ,  $q \in \tau$ . It is easily seen that the unit tangent  $\Phi(x)$  to  $p * q$ , directed from  $p$  to  $q$ , is transverse to  $\alpha$  at  $x$ , and that  $\Phi$  is continuous. Thus  $\Phi$  is a transverse field on  $A$ .

**Lemma 2.4.** *Let  $M$  be an unbounded differentiable  $n$ -manifold, and  $V \subset M$  a combinatorial submanifold, also of dimension  $n$ . Let  $U$  be a neighborhood of the boundary  $A$  of  $V$ ,  $d$  a metric on  $M$ , and  $\epsilon$  a positive continuous function on  $M$ . There is a homeomorphism  $h: M \rightarrow M$  such that*

- a)  $h$  is a diffeomorphism on each closed simplex of a subdivision of  $M$ ;
- b)  $h(A)$  has a transverse field;
- c)  $h(x) = x$  if  $x \in M - U$ ;
- d)  $d(x, h(x)) < \epsilon(x)$  for all  $x \in M$ .

*Proof.* Apply 2.2 and 2.3.

Let  $V$  be a combinatorial submanifold of a differentiable unbounded  $n$ -manifold  $M$ . Assume either

- 1.  $V$  has dimension  $n$ ; or
- 2.  $V$  has dimension  $n - 1$ , is unbounded, and admits a transverse field.

Let  $U$  be a neighborhood of  $\partial V$  and  $\epsilon$  a positive continuous function on  $M$ .

**Theorem 2.5.** *There is a homeomorphism  $h: M \rightarrow M$  such that:*

- a)  $h(V)$  is a differentiable submanifold of  $M$ , combinatorially equivalent to  $V$ ;



b)  $M$  has a smooth triangulation in which  $V$  is a subcomplex, every closed simplex of which is mapped diffeomorphically by  $h$ ;

c) and d) as in 2.4.

*Proof.* Case 1) follows from 2.1 and case 2); Thus we assume 2).

By standard approximation methods, it may be assumed that there is a differentiable non-zero vector field  $\Phi$  on a neighborhood  $W$  of  $V$  contained in  $U$ , such that  $\Phi|V$  is transverse field. A generalization of the CAIRNS-WHITEHEAD theory of transverse fields [1, 13] shows that there is a submanifold  $C$  of dimension  $n - 1$  differentiably imbedded in an arbitrary neighborhood of  $V$ , such that  $\Phi|C$  is transverse. (The CAIRNS-WHITEHEAD theory applies to a  $q$ -dimensional submanifold of EUCLIDEAN  $(q + p)$ -space endowed with a transverse  $p$ -plane field. The present case follows, e. g., by imbedding  $M$  in  $R^{n+k}$ , and assigning to each point  $x \in V$  the  $k + 1$  plane generated by  $\Phi(x)$  and the  $k$ -plane normal to  $M$  at  $x$ . Alternatively, the methods of [13] simplify considerably in the special case where the submanifold has codimension 1, if transverse lines are replaced by the integral curves of a transverse vector field.<sup>3)</sup> We can assume that each integral curve of  $\Phi$  meets  $V$  in a unique point and  $C$  in a unique point. This establishes a map  $f: V \rightarrow C$  which is a diffeomorphism on each closed simplex of  $V$ . Thus  $C$  is combinatorially equivalent to  $V$ . Let  $A$  be the region bounded by  $V$  and  $C$  which is fibered by the integral curves of  $\Phi$ . Define  $G: V \times I \rightarrow A$  by  $G(x, 0) = x$ ,  $G(x, 1) = f(x)$  and  $G(x, t)$  is the point dividing the length of the integral curve joining  $x$  and  $f(x)$  in the ratio  $t | (1 - t)$ , for  $0 < t < 1$ . Then if  $\Delta$  is a closed simplex of  $V$ ,  $G| \Delta \times I$  is a diffeomorphism. Hence  $G: K \times I \rightarrow M$  is a non-degenerate  $C^\infty$  subcomplex of  $M$  in the sense of [15]. By [15, p. 822, *addendum*] this triangulation of  $A$  can be extended to a smooth triangulation of  $M$ , after possible subdivision. An easy application of 2.1 (cf. proof of 2.2) establishes the desired extension  $h: M \rightarrow M$  of  $f: V \rightarrow C$ . This completes the proof.

*Remark.* It can be shown that if a neighborhood in  $V$  of a closed subset  $X \subset V$  is a differentiable submanifold of  $M$ ,  $h$  can be chosen so that for some neighborhood  $Y$  of  $X$  in  $M$ ,  $h(x) = x$  if  $x \in Y$ .

The following theorem was announced by S. S. Cairns [16].

**Theorem 2.6 (Cairns).** *If  $M$  is a combinatorial manifold and if for some  $p$ ,  $M \times R^p$  has a compatible differentiable structure, then so has  $M$ .*

*Proof.* By induction on  $p$ . The case  $p = 0$  is trivial. Let  $F^p \subset R^p$  be a closed half-space. First assume  $M$  is unbounded. If  $p > 0$  and if  $M \times R^p$

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<sup>3)</sup> Cf. [18].

has a compatible differentiable structure, then so has  $M \times F^p$ , by (A). So therefore does its boundary  $M \times R^{p-1}$ , completing the induction. The case where  $M$  is bounded follows easily now from (A).

### § 3. Proof of (B)

Let  $V$  be an  $n$ -dimensional compact combinatorial manifold which is contractible. Let  $\tilde{V}$  be the double of  $V$ , obtained by identifying two disjoint copies of  $V$  along their boundary. Then  $\tilde{V}$  is a closed combinatorial  $n$ -manifold of the same homotopy type as  $S^n$ , and  $V$  is a submanifold. J. STALLINGS proves in [11] that if  $x$  is any point of  $\tilde{V}$ , then  $\tilde{V} - x$  is combinatorially equivalent to EUCLIDEAN  $n$ -space, provided  $n \geq 7$ , and states that E. C. ZEEMAN has extended the result to the case  $n \geq 5$ . (If  $n \leq 4$ , theorem (B) is a consequence of well known results of CAIRNS [1, 2].) Thus we can assume that  $V$  is a submanifold of the differentiable manifold  $R^n$ . (Alternatively,  $\tilde{V} - x$  is an unbounded contractible manifold, and one can apply GLEASON's theorem that  $\tilde{V} - x$  has a compatible differentiable structure). Theorem (B) now follows from (A). Actually, this proves the following stronger result.

**Theorem 3.1.** *A compact, contractible, combinatorial  $n$ -manifold is combinatorially equivalent to a differentiable submanifold of  $R^n$ .*

GLEASON's theorem is proved by showing that an *unbounded* combinatorial contractible  $n$ -manifold can be *immersed* in  $R^n$ . This follows from the mere existence of a compatible differentiable structure by observing that such a structure is necessarily parallelizable, and applying a theorem of [4].<sup>4</sup>)

A plausible conjecture along these lines is that *any combinatorial manifold all of whose cohomology groups vanish has a compatible differentiable structure*. J. MUNKRES [7] has proved this except for compatibility.

### § 4. Proof of (C)

We must show that the sequence

$$\xrightarrow{d} \Gamma^n \xrightarrow{j} \Theta^n \xrightarrow{k} \Lambda^n \xrightarrow{d} \Gamma^{n-1} \longrightarrow \quad (1)$$

is well defined and exact.

The group  $\Theta^n$  is defined as follows. An element of  $\Theta^n$  is an equivalence class  $[M]$  of oriented, closed differentiable manifolds  $M$  which have the homo-

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<sup>4</sup>) GLEASON's theorem follows from 2.6 and [17], in which it is proved that if  $M^n$  is a contractible combinatorial unbounded manifold, then  $M^n \times R^p$  is combinatorially equivalent to  $R^{n+p}$  for some  $p$ .

topology type of the  $n$ -sphere  $S^n$ , under the relation of  $J$ -equivalence. Two oriented differentiable manifolds  $M_0, M_1$  are  $J$ -equivalent if there is an oriented differentiable manifold  $N$  whose boundary is (diffeomorphic to) the disjoint union of  $M_1$  and  $-M_0$  (where  $-M_0$  means  $M_0$  with the opposite orientation), and such that both  $M_0$  and  $M_1$  are deformation retracts of  $N$ . Addition in  $\mathcal{O}^n$  is defined by  $[A] + [B] = [A \# B]$  where  $A \# B$  is the *connected sum* of  $A$  and  $B$ . This is defined by removing the interior of an  $n$ -ball from each of  $A$  and  $B$  and joining the two boundary  $(n-1)$ -spheres by an orientation reversing diffeomorphism which is extendable to the whole  $n$ -ball, and then smoothing the resulting corner. It can be shown that the diffeomorphism class of  $A \# B$  is independent of the choices made, and the  $J$ -equivalence class  $[A \# B]$  is independent of the representatives of  $[A]$  and  $[B]$  that are chosen. See [5] for details. Define  $-[A] = [-A]$ , and  $\mathcal{O}^n$  becomes an abelian group, with  $[S^n]$  as identity element.

Using combinatorial instead of differentiable manifolds,  $J$ -equivalence and connected sum are analogously defined, and  $\mathcal{A}^n$  is the group of  $J$ -equivalence classes  $\langle M \rangle$  of oriented combinatorial closed  $n$ -manifolds  $M$  which are homotopy spheres.

The elements of  $\Gamma^n$  are (diffeomorphism classes of) oriented differentiable manifolds which are combinatorially equivalent to the boundary of an  $(n+1)$ -simplex. Addition is defined using  $\#$ , and the inverse of  $M \in \Gamma^n$  is  $-M$ . Using these definitions,  $\Gamma^n$  is an abelian group, with  $S^n$  for 0, although this is not obvious. It is a consequence of the following result.

**Theorem 4.1. (MUNKRES-THOM).** *There is at most one compatible differentiable structure on a contractible combinatorial  $n$ -manifold, up to a diffeomorphism.*

*Proof.* See [6, 12]

The only difficulty about proving that  $\Gamma^n$  is a group is showing

$$M \# (-M) = S^n.$$

We can obtain  $M \# (-M)$  by removing the interior  $E$  of an  $n$ -ball from  $M$ , and taking the boundary  $V$  of  $(M - E) \times I$  and smoothing the corner. But  $(M - E) \times I$  is then a differentiable manifold combinatorially equivalent to  $\Delta^n \times I$ , where  $\Delta^n$  is an  $n$ -simplex, and by 4.1 its boundary  $V = M \# (-M)$  is diffeomorphic to  $S^n$ , which is the zero element of  $\Gamma^n$ .

The map  $k: \mathcal{O}^n \rightarrow \mathcal{A}^n$  is defined as follows. If  $M$  is a differentiable manifold, let  $kM$  be the corresponding combinatorial manifold, i. e.,  $kM$  is a simplicial complex  $L$  such that there is a homeomorphism  $t: L \rightarrow M$  which is a smooth triangulation of  $M$ . Such an  $L$  exists and is unique up to combinatorial equivalence [15].

Now define  $k: \mathcal{O}^n \rightarrow \mathcal{A}^n$  by  $k[M] = \langle kM \rangle$ . It is obvious that  $k$  preserves sums and  $J$ -equivalence, so  $k$  is a well defined homomorphism.

The map  $j: \Gamma^n \rightarrow \mathcal{O}^n$  is defined by  $jM = [M]$ .

To define  $d: \mathcal{A}^n \rightarrow \Gamma^{n-1}$ , let  $M$  represent an element of  $\mathcal{A}^n$ , and let  $E$  be the interior of an  $n$ -simplex of  $M$ . Then  $M - E$  is contractible, and by (B) possesses a compatible differentiable structure, which is unique by 4.1, up to diffeomorphism. The combinatorial structure of  $M - E$  is independent of  $E$ , and  $dM$  is defined to be the boundary of  $M - E$ . We shall see shortly that if  $A$  and  $B$  are  $J$ -equivalent, then  $dA = dB$ .

**Lemma 4.2.** a)  $d(M \# N) = dM + dN$

b)  $d(-M) = -dM$ .

*Proof.* Let  $C$  and  $D$  be closed  $n$ -simplices in  $M$  and  $N$  respectively. In forming  $M \# N$ , remove the interiors of  $n$ -simplices disjoint from  $C$  and  $D$ . Now  $M$ -int  $C$  and  $N$ -int  $D$  have unique compatible differentiable structures by (B) and 4.1, and then  $(M \# N) - (\text{int } C \cup \text{int } D) = (M\text{-int } C) \# (N\text{-int } D)$  has a compatible differentiable structure. Now join  $C$  to  $D$  in  $M \# N$  by a simple differentiable arc, meeting  $C$  and  $D$  only at its end points. A tubular neighborhood  $Q$  of this arc can be chosen so that  $C \cup D \cup Q$  is a combinatorial  $n$ -cell in  $M \# N$ . Then  $M$ -int  $(C \cup D \cup Q)$  has  $d(M \# N)$  for its boundary (after smoothing). On the other hand, this boundary is diffeomorphic to  $\partial(M\text{-int } C) \# \partial(N\text{-int } D) = d(M) \# d(N)$ , which proves a). The proof of b) is obvious.

**Lemma 4.3.** Let  $M$  be a closed oriented combinatorial homotopy  $n$ -sphere. If  $M$  has a compatible differentiable structure,  $dM = 0$ .

*Proof.* If  $E$  is the interior of an  $n$ -simplex  $\Delta$  of  $M$ , and  $B$  the interior of an  $n$ -ball differentially imbedded in  $E$ , then  $M - E$  and  $M - B$  are combinatorially equivalent. Assuming  $M$  has a compatible differentiable structure, take  $\Delta$  to be a simplex of a smooth triangulation of  $M$ . Then  $M - B$  is a differentiable submanifold of  $M$  and hence  $\partial(M - B) = \partial B = S^{n-1} = 0 \in \Gamma^{n-1}$ . By 4.1, compatible differentiable structures on  $M - E$  and  $M - B$  are diffeomorphic; hence  $\partial(M - E) = d(M) = 0$ .

**Corollary 4.4.** If  $M$  bounds a contractible manifold,  $d(M) = 0$ .

*Proof.* By (A),  $M$  has a compatible differentiable structure and 4.3 applies.

**Theorem 4.5.**  $d: \mathcal{A}^n \rightarrow \Gamma^{n-1}$  defined by  $d\langle M \rangle = dM$  is a well defined homomorphism.

*Proof.* We must show first that if  $\langle M \rangle = \langle N \rangle$ , then  $dM = dN$ . If  $M$  is  $J$ -equivalent to  $N$ , then  $M \# (-N)$  is  $J$ -equivalent to  $\partial\Delta^{n+1}$ . Since

$\partial\Delta^{n+1}$  bounds  $\Delta^{n+1}$ ,  $M \# (-N)$  bounds a contractible manifold. By 4.4  $d(M \# (-N)) = O$ , and by 4.2,  $d(M \# (-N)) = d(M) - d(N)$ . Thus  $d(M) - d(N) = O$ , so  $d: \Lambda^n \rightarrow \Gamma^{n-1}$  is well defined, and 4.4 proves  $d$  to be homomorphism.

Now we prove that the sequence (1) is exact. We leave the proof that  $jd = kj = dk = O$  to the reader as an exercise; the last equality, for example, follows from 4.3.

Let  $M$  be an element of  $\Gamma^n$  such that  $j(M) = O$ . This means  $M$  bounds a contractible differentiable manifold  $V$ . Since  $\partial V = M$  is combinatorially equivalent to  $\partial\Delta^{n+1}$ ,  $V \cup \Delta^{n+1}$  is a combinatorial homotopy sphere and hence represents an element  $\lambda$  of  $\Lambda^{n+1}$ . It is obvious that  $d(\lambda) = \partial(V\text{-int } \Delta^{n+1}) = M$ . This establishes the exactness of the sequence  $jd$ .

Let  $\langle M \rangle \in \Lambda^n$  be such that  $d\langle M \rangle = O$ . This means for some  $n$ -simplex  $\Delta$  in  $M$ ,  $M\text{-int } \Delta$  has a compatible differentiable structure making  $\partial(M\text{-int } \Delta)$  diffeomorphic to  $S^{n-1}$ . Choosing such a diffeomorphism, attach the  $n$ -ball  $D^n$  to  $M\text{-int } \Delta$  to obtain a differentiable manifold  $N$  which is combinatorially equivalent to  $M$ . Thus  $k[M] = \langle N \rangle$  and  $dk$  is exact.

Finally let  $M$  represent an element of  $\Theta^n$  annihilated by  $k$ . This means  $M$  is  $J$ -equivalent to  $\partial\Delta^{n+1}$  in the combinatorial sense. Let  $V$  be a combinatorial  $(n+1)$ -manifold realizing this  $J$ -equivalence. Let  $T$  be a «tube» joining  $M$  to  $\partial\Delta^{n+1}$  in  $V$ , i. e.,  $T$  is equivalent to  $I \times \Delta^n$  with  $O \times \Delta^n \subset M$  and  $1 \times \Delta^n \subset \partial\Delta^{n+1}$ , and no other points of  $T$  in  $\partial V$ . (Such a  $T$  can easily be constructed by first putting a compatible differentiable structure on the contractible manifold  $V \cup \Delta^{n+1}$ .) Then  $V\text{-int } T$  is contractible if  $T$  is «unknotted», which is always true if  $n+1 > 3$  and which is the case for  $n+1 = 3$  provided  $T$  is chosen properly. (In fact,  $\Theta^2 = O$ , so this case is unnecessary.) Thus  $V\text{-int } T$  has a compatible differentiable structure, by (B). By 2.5, we can assume that  $M\text{-int } (O \times \Delta^n)$  is a differentiable submanifold  $A$  of the boundary of  $V\text{-int } T$ . The closure of the complement of  $A$  is combinatorially equivalent to  $\partial\Delta^{n+1}$ , and hence is diffeomorphic to  $D^n$ , while  $A$  is combinatorially equivalent, and hence diffeomorphic to  $M - E$ , where  $E$  is the interior of an  $n$ -ball,  $\overline{E}$  differentiably imbedded in  $M$ . Thus there is a diffeomorphism  $f: \partial(M - E) \rightarrow S^{n-1}$  such that  $\partial(V\text{-int } T)$  is diffeomorphic to  $(M - E) \cup, D^n$ . Let  $P$  be the manifold  $\overline{E} \cup, D^n$ . Then  $P$  is an element of  $\Gamma^n$ , and  $\partial(V\text{-int } T)$  is the same as  $M \# (-P)$ . Since  $V\text{-int } T$  is contractible,  $[M - P] = O$ , and so  $[M] = [P] = jP$ . This establishes the exactness of the sequence (1).

## REFERENCES

- [1] S. CAIRNS, *Homeomorphisms between topological manifolds and analytic manifolds*. Ann. of Math., 41 (1940), 796–808.
- [2] S. CAIRNS, *Introduction of a RIEMANNIAN geometry on a triangulable 4-manifold*. Ann. of Math. 45 (1944), 218–219.
- [3] M. HIRSCH, *An exact sequence in differential topology*. Bull. Amer. Math. Soc., 66 (1960), 322–323.
- [4] M. HIRSCH, *On imbedding differentiable manifolds in EUCLIDEAN space*. Ann. of Math., 73 (1960).
- [5] J. MILNOR, *Differentiable manifolds which are homotopy spheres*. Princeton University 1959 (mimeographed).
- [6] J. MUNKRES, *Obstructions to the smoothing of piecewise-differentiable homeomorphisms*. Ann. of Math., 72 (1960), 521–554.
- [7] J. MUNKRES, *Obstructions to imposing differentiable structures*. To appear.
- [8] S. SMALE, *Diffeomorphisms of the 2-sphere*. Proc. Amer. Math. Soc., 10 (1959), 621–626.
- [9] S. SMALE, *Generalized POINCARÉ's conjecture in dimensions greater than 4*. Ann. of Math., to appear.
- [10] S. SMALE, *Differentiable and combinatorial structures on manifolds*. Ann. of Math., to appear.
- [11] J. STALLINGS, *Polyhedral homotopy spheres*. Bull., Amer. Math. Soc., 66 (1960), 485–488.
- [12] R. THOM, *Des variétés triangulées aux variétés différentiables*. Proc. Int. Cong. Math., 1958, Cambridge Univ. Press, (1960), 248–255.
- [13] J. H. C. WHITEHEAD, *Manifolds with transverse fields in EUCLIDEAN space*. Ann. of Math., 73 (1960), 154–212.
- [14] J. H. C. WHITEHEAD, *Simplicial spaces, nuclei and  $m$ -groups*. Proc. Lond. Math. Soc., 45 (1939), 243–327.
- [15] J. H. C. WHITEHEAD, *On  $C^1$  complexes*. Ann. of Math. 41 (1940), 809–824.
- [16] S. CAIRNS, *The manifold smoothing problem*, Bull. Amer. Math. Soc. 67 (1961), 237–238.
- [17] D. McMILLAN, *Cartesian products of contractible manifolds*, Bull. Amer. Math. Soc. 67 (1961), 510–514.
- [18] D. MONTGOMERY and H. SAMELSON, *Examples for differentiable group actions on spheres* Proc. Nat. Acad. Sci. U.S.A. 47 (1961), 1202–1205.

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