On Combinatorial Submanifolds of Differentiable Manifolds.

On Combinatorial Submanifolds of Differentiable Manifolds[1])

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§ 1. Introduction

The purpose of this work is to prove the following results relating combinatorial and differentiable manifolds.

(A) *A combinatorial submanifold* V *of a differentiable manifold* M *of the same dimension possesses a compatible differentiable structure.*

(B) *Every compact and contractible combinatorial manifold* V *possesses a compatible differentiable structure.*[2])

(A differentiable structure on a combinatorial manifold M is called *compatible* if M has a rectilinear subdivision, each simplex of which is differentiably imbedded.)

A. M. GLEASON has announced (unpublished) that a contractible *unbounded* combinatorial manifold has a compatible differentiable structure. Theorem (B) follows easily from this and Theorem (A). The proof of (B) given here is derived from JOHN STALLINGS' proof [11] of the generalized POINCARÉ conjecture.

(C) *The sequence*

$$\cdots \xrightarrow{d} \Gamma^n \xrightarrow{j} \Theta^n \xrightarrow{k} \Lambda^n \xrightarrow{d} \Gamma^{n-1} \longrightarrow \cdots \qquad (1)$$

is well defined and exact.

This result was announced in [3]. Here Γ^n is the group of differentiable structures on S^n compatible with the usual combinatorial structure; Θ^n is the group of differentiable homotopy n-spheres modulo J-equivalence, and Λ^n the combinatorial analogue of Ω^n. Using powerful intrinsic methods, STEPHEN SMALE has shown [9, 10] that for $n \geqslant 8$ or $n = 5$, the map $j : \Gamma^n \to \Theta^n$ is an isomorphism, and using (B) (but not (C)) that $\Lambda^n = 0$, for $n \geqslant 8$ or $n = 6$.

In [8] SMALE proves that $\Gamma^3 = 0$, (this was proved independently by J. MUNKRES, and J. H. C. WHITEHEAD.) The fact that every combinatorial 3-manifold possess a unique (up to a diffeomorphism) compatible differentiable structure [6] implies that $k : \Theta^3 \to \Lambda^3$ is an isomorphism. Thus only the

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[2]) *Added in proof;* The hypothesis of compactness is unnecessary.

subsequences $0 \to \Gamma^7 \to \Theta^7 \to \Lambda^7 \to \Gamma^6 \to 0$ and $0 \to \Lambda^5 \to \Gamma^4 \to \Theta^4 \to \Lambda^4 \to 0$ remain. In proving (C), SMALE's results are not used.

§ 2. Proof of (A)

In order to prove (A) it suffices to establish the stronger result 2.5 below.

If K is a subcomplex of complex N, the nth *simplicial neighborhood of* K is the union of the closed simplexes of the n'th barycentric subdivision of N that meet K.

Lemma 2.1. *Let* K *be the boundary of a combinatorial manifold* M. *The second simplicial neighborhood* A *of* K *is combinatorially equivalent to* $K \times I$, *where* I *is the unit interval.*

Proof. This is a well known result. It follows e. g. from theorems 22 and 23 of [14], which state that any two "regular neighborhoods" of M (in the sense of [14]) in the same manifold are combinatorially equivalent and that A is a regular neighborhood of K in M. If K is identified with $K \times 0$, then $M' = M \cup K \times I$ is a manifold, and in M' both A and $K \times I$ are regular neighborhoods of K.

Now let V be a bounded combinatorial n-manifold imbedded as a subcomplex of an unbounded combinatorial n-manifold M. Assume M has a metric $d(x, y)$.

Lemma 2.2. *Let* U *be a neighborhood of* V *in* M, *and* ϵ *a positive continuous function on* M. *There is a semi-linear homeomorphism* $h: M \to M$ *with the following properties:*

a) $h(V)$ *is the second simplicial neighborhood of* V *in a subdivision of* M;
b) $h(V) \subset U$;
c) $h(x) = x$ *if* $x \in M - U$;
d) $d(x, h(x)) < \epsilon(x)$ *for all* $x \in M$.

Proof. By 2.1, the boundary of K of V has a neighborhood combinatorially equivalent to $K \times I$ in V, and another in $cl(M - V)$. The union B_0 of these two neighborhoods is again equivalent to $K \times I$. Moreover, we can take B_0 to be the second simplicial neighborhood of K in a subdivision of M; if this subdivision is sufficiently fine, the second simplicial neighborhood B of B_0 will be in U. It will be clear that if the subdivision is sufficiently fine, d) will be satisfied. There is a combinatorial equivalence $u: B \to K \times I$ such that $u(B_0) = K \times [1/4, 3/4]$ and $u(K) = K \times 1/2$. We may assume that $h(B_0 \cap V) = K \times [0, 1/2]$. Let $f: I \to I$ be a semi-linear homeomorphism such that $f(x) = x$ for x in a neighborhood of 0 and 1, and $f(1/2) = 3/4$. Define $g: K \times I \to K \times I$ by $g(x,t) = (x, f(t))$. Now define $h: M \to M$ by $h(x) = \begin{cases} x \text{ if } x \in M - B \\ u^{-1}gu(x) \text{ if } x \in B. \end{cases}$

Then h is the desired homeomorphism.

Now let M be a differentiable manifold. A combinatorial manifold A which is a subcomplex of a smooth triangulation of M is called a *combinatorial submanifold of* M. A vector field Φ on A in M is *transverse* if it is transverse to A in every coordinate system, in the sense of [13]. The following lemma is well known.

Lemma 2.3. *Let A be the boundary of the second simplicial neighborhood B of a subcomplex K of M. Then there is a transverse field on A.*

Proof. Each simplex ϱ of the *first* simplicial neighborhood B' of K is the join $\sigma * \tau$ of unique simplices $\sigma \subset K$ and $\tau \subset M - K$. Each closed simplex α of A lies in such a join $\sigma * \tau$, disjoint form σ and τ, and each $x \in \alpha$ lies on a unique line $p * q$ with $p \in \sigma$, $q \in \tau$. It is easily seen that the unit tangent $\Phi(x)$ to $p * q$, directed from p to q, is transverse to α at x, and that Φ is continuous. Thus Φ is a transverse field on A.

Lemma 2.4. *Let M be an unbounded differentiable n-manifold, and $V \subset M$ a combinatorial submanifold, also of dimension n. Let U be a neighborhood of the boundary A of V, d a metric on M, and ϵ a positive continuous function on M. There is a homeomorphism $h: M \to M$ such that*

a) *h is a diffeomorphism on each closed simplex of a subdivision of M;*

b) *$h(A)$ has a transverse field;*

c) *$h(x) = x$ if $x \in M - U$;*

d) *$d(x, h(x)) < \epsilon(x)$ for all $x \in M$.*

Proof. Apply 2.2 and 2.3.

Let V be a combinatorial submanifold of a differentiable unbounded n-manifold M. Assume either

1. *V has dimension n*; or
2. *V has dimension $n-1$, is unbounded, and admits a transverse field.*

Let U be a neighborhood of bdV and ϵ a positive continuous function on M.

Theorem 2.5. *There is a homeomorphism $h: M \to M$ such that:*

a) *$h(V)$ is a differentiable submanifold of M, combinatorially equivalent to V;*

b) *M has a smooth triangulation in which* V *is a subcomplex, every closed simplex of which is mapped diffeomorphically by* h;

c) and d) *as in* 2.4.

Proof. Case 1) follows from 2.1 and case 2); Thus we assume 2).

By standard approximation methods, it may be assume that there is a differentiable non-zero vector field Φ on a neighborhood W of V contained in U, such that $\Phi|V$ is transverse field. A generalization of the CAIRNS-WHITEHEAD theory of transverse fields [1, 13] shows that there is a submanifold C of dimension $n-1$ differentiably imbedded in an arbitrary neighborhood of V, such that $\Phi|C$ is transverse. (The CAIRNS-WHITEHEAD theory applies to a q-dimensional submanifold of EUCLIDean $(q+p)$-space endowed with a transverse p-plane field. The present case follows, e. g., by imbedding M in R^{n+k}, and assigning to each point $x \in V$ the $k+1$ plane generated by $\Phi(x)$ and the k-plane normal to M at x. Alternatively, the methods of [13] simplify considerably in the special case where the submanifold has codimension 1, if transverse lines are replaced by the integral curves of a transverse vector field.[3]) We can assume that each integral curve of Φ meets V in a unique point and C in a unique point. This establishes a map $f: V \to C$ which is a diffeomorphism on each closed simplex of V. Thus C is combinatorially equivalent to V. Let A be the region bounded by V and C which is fibered by the integral curves of Φ. Define $G: V \times I \to A$ by $G(x, o) = x, G(x, 1) = f(x)$ and $G(x, t)$ is the point dividing the length of the integral curve joining x and $f(x)$ in the ratio $t \mid (1-t)$, for $0 < t < 1$. Then if Δ is a closed simplex of $V, G \mid \Delta \times I$ is a diffeomorphism. Hence $G: K \times I \to M$ is a non-degenerate C^∞ subcomplex of M in the sense of [15]. By [15, p. 822, *addendum*] this triangulation of A can be extended to a smooth triangulation of M, after possible subdivision. An easy application of 2.1 (cf. proof of 2.2) establishes the desired extension $h: M \to M$ of $f: V \to C$. This completes the proof.

Remark. It can be shown that if a neighborhood in V of a closed subset $X \subset V$ is a differentiable submanifold of M, h can be chosen so that for some neighborhood Y of X in $M, h(x) = x$ if $x \in Y$.

The following theorem was announced by S. S. Cairns [16].

Theorem 2.6 (Cairns). *If M is a combinatorial manifold and if for some* p, $M \times R^p$ *has a compatible differentiable structure, then so has* M.

Proof. By induction on p. The case $p = 0$ is trivial. Let $F^p \subset R^p$ be a closed half-space. First assume M is unbounded. If $p > 0$ and if $M \times R^p$

[3]) Cf. [18].

has a compatible differentiable structure, then so has $M \times F^p$, by (A). So therefore does its boundary $M \times R^{p-1}$, completing the induction. The case where M is bounded follows easily now from (A).

§ 3. Proof of (B)

Let V be an n-dimensional compact combinatorial manifold which is contractible. Let $\tilde{V}$ be the double of V, obtained by identifying two disjoint copies of V along their boundary. Then $\tilde{V}$ is a closed combinatorial n-manifold of the same homotopy type as S^n, and V is a submanifold. J. Stallings proves in [11] that if x is any point of $\tilde{V}$, then $\tilde{V} - x$ is combinatorially equivalent to Euclidean n-space, provided $n \geqslant 7$, and states that E. C. Zeeman has extended the result to the case $n \geqslant 5$. (If $n \leq 4$, theorem (B) is a consequence of well known results of Cairns [1, 2].) Thus we can assume that V is a submanifold of the differentiable manifold R^n. (Alternatively, $\tilde{V} - x$ is an unbounded contractible manifold, and one can apply Gleason's theorem that $\tilde{V} - x$ has a compatible differentiable structure). Theorem (B) now follows from (A). Actually, this proves the following stronger result.

Theorem 3.1. *A compact, contractible, combinatorial n-manifold is combinatorially equivalent to a differentiable submanifold of* R^n.

Gleason's theorem is proved by showing that an *unbounded* combinatorial contractible n-manifold can be *immersed* in R^n. This follows from the mere existence of a compatible differentiable structure by observing that such a structure is necessarily parallelizable, and applying a theorem of [4].[4])

A plausible conjecture along these lines is that *any combinatorial manifold all of whose cohomology groups vanish has a compatible differentiable structure* J. Munkres [7] has proved this except for compatibility.

§ 4. Proof of (C)

We must show that the sequence

$$\xrightarrow{d} \Gamma^n \xrightarrow{j} \Theta^n \xrightarrow{k} \Lambda^n \xrightarrow{d} \Gamma^{n-1} \longrightarrow \qquad (1)$$

is well defined and exact.

The group Θ^n is defined as follows. An element of Θ^n is an equivalence class $[M]$ of oriented, closed differentiable manifolds M which have the homo-

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[4]) Gleason's theorem follows from 2.6 and [17], in which it is proved that if M^n is a contractible combinatorial unbounded manifold, then $M^n \times R^p$ is combinatorially equivalent to R^{n+p} for some p.

topy type of the n-sphere S^n, under the relation of J-equivalence. Two oriented differentiable manifolds M_0, M_1 are J-equivalent if there is an oriented differentiable manifold N whose boundary is (diffeomorphic to) the disjoint union of M_1 and $-M_0$ (where $-M_0$ means M_0 with the opposite orientation), and such that both M_0 and M_1 are deformation retracts of N. Addition in Θ^n is defined by $[A] + [B] = [A \# B]$ where $A \# B$ is the *connected sum* of A and B. This is defined by removing the interior of an n-ball from each of A and B and joining the two boundary $(n-1)$-spheres by an orientation reversing diffeomorphism which is extendable to the whole n-ball, and then smoothing the resulting corner. It can be shown that the diffeomorphism class of $A \# B$ is independent of the choices made, and the J-equivalence class $[A \# B]$ is independent of the representatives of $[A]$ and $[B]$ that are-chosen. See [5] for details. Define $-[A] = [-A]$, and Θ^n becomes an abelian group, with $[S^n]$ as identity element.

Using combinatorial instead of differentiable manifolds, J-equivalence and connected sum are analogously defined, and Λ^n is the group of J-equivalence classes $\langle M \rangle$ of oriented combinatorial closed n-manifolds M which are homotopy spheres.

The elements of Γ^n are (diffeomorphism classes of) oriented differentiable manifolds which are combinatorially equivalent to the boundary of an $(n+1)$-simplex. Addition is defined using $\#$, and the inverse of $M \in \Gamma^n$ is $-M$. Using these definitions, Γ^n is an abelian group, with S^n for 0, although this is not obvious. It is a consequence of the following result.

Theorem 4.1. (MUNKRES-THOM). *There is at most one compatible differentiable structure on a contractible combinatorial n-manifold, up to a diffeomorphism.*

Proof. See [6, 12]

The only difficulty about proving that Γ^n is a group is showing

$$M \# (-M) = S^n.$$

We can obtain $M \# (-M)$ by removing the interior E of an n-ball from M, and taking the boundary V of $(M - E) \times I$ and smoothing the corner. But $(M - E) \times I$ is then a differentiable manifold combinatorially equivalent to $\Delta^n \times I$, where Δ^n is an n-simplex, and by 4.1 its boundary $V = M \# (-M)$ is diffeomorphic to S^n, which is the zero element of Γ^n.

The map $k : \Theta^n \to \Lambda^n$ is defined as follows. If M is a differentiable manifold, let kM be the corresponding combinatorial manifold, i. e., kM is a simplicial complex L such that there is a homeomorphism $t : L \to M$ which is a smooth triangulation of M. Such an L exists and is unique up to combinatorial equivalence [15].

Now define $k: \Theta^n \to A^n$ by $k[M] = \langle kM \rangle$. It is obvious that k preserves sums and J-equivalence, so k is a well defined homomorphism.

The map $j: \Gamma^n \to \Theta^n$ is defined by $jM = [M]$.

To define $d: A^n \to \Gamma^{n-1}$, let M represent an element of A^n, and let E be the interior of an n-simplex of M. Then $M - E$ is contractible, and by (B) possesses a compatible differentiable structure, which is unique by 4.1, up to diffeomorphism. The combinatorial structure of $M - E$ is independent of E, and dM is defined to be the boundary of $M - E$. We shall see shortly that if A and B are J-equivalent, then $dA = dB$.

Lemma 4.2. a) $d(M \# N) = dM + dN$
b) $d(-M) \quad = -dM$.

Proof. Let C and D be closed n-simplices in M and N respectively. In forming $M \# N$, remove the interiors of n-simplices disjoint from C and D. Now M-int C and N-int D have unique compatible differentiable structures by (B) and 4.1, and then $(M \# N) - (\text{int}\, C \cup \text{int}\, D) = (M\text{-int}\, C) \# (N\text{-int}\, D)$ has a compatible differentiable structure. Now join C to D in $M \# N$ by a simple differentiable arc, meeting C and D only at its end points. A tubular neighborhood Q of this arc can be chosen so that $C \cup D \cup Q$ is a combinatorial n-cell in $M \# N$. Then M-int $(C \cup D \cup Q)$ has $d(M \# N)$ for its boundary (after smoothing). On the other hand, this boundary is diffeomorphic to $\partial(M\text{-int}\, C) \# \partial(N\text{-int}\, D) = \mathrm{d}(M) \# d(M)$, which proves a). The proof of b) is obvious.

Lemma 4.3. *Let M be a closed oriented combinatorial homotopy n-sphere. If M has a compatible differentiable structure,* $dM = 0$.

Proof. If E is the interior of an n-simplex Δ of M, and B the interior of an n-ball differentiably imbedded in E, then $M - E$ and $M - B$ are combinatorially equivalent. Assuming M has a compatible differentiable structure, take Δ to be a simplex of a smooth triangulation of M. Then $M - B$ is a differentiable submanifold of M and hence $\partial(M - B) = \partial B = S^{n-1} = 0 \in \Gamma^{n-1}$. By 4.1, compatible differentiable structures on $M - E$ and $M - B$ are diffeomorphic; hence $\partial(M - E) = d(M) = O$.

Corollary 4.4. *If M bounds a contractible manifold,* $d(M) = O$.

Proof. By (A), M has a compatible differentiable structure and 4.3 applies.

Theorem 4.5. $d: A^n \to \Gamma^{n-1}$ *defined* by $d\langle M \rangle = dM$ *is a well defined homomorphism.*

Proof. We must show first that if $\langle M \rangle = \langle N \rangle$, then $dM = dN$. If M is J-equivalent to N, then $M \# (-N)$ is J-equivalent to $\partial \Delta^{n+1}$. Since

$\partial \Delta^{n+1}$ bounds Δ^{n+1}, $M \# (-N)$ bounds a contractible manifold. By 4.4 $d(M \# (-N)) = O$, and by 4.2, $d(M \# (-N)) = d(M) - d(N)$. Thus $d(M) - d(N) = O$, so $d : \Lambda^n \to \Gamma^{n-1}$ is well defined, and 4.4 proves d to be homomorphism.

Now we prove that the sequence (1) is exact. We leave the proof that $jd = kj = dk = O$ to the reader as an exercise; the last equality, for example, follows from 4.3.

Let M be an element of Γ^n such that $j(M) = O$. This means M bounds a contractible differentiable manifold V. Since $\partial V = M$ is combinatorially equivalent to $\partial \Delta^{n+1}$, $V \cup \Delta^{n+1}$ is a combinatorial homotopy sphere and hence represents an element λ of Λ^{n+1}. It is obvious that $d(\lambda) = \partial(V\text{-int } \Delta^{n+1}) = M$. This establishes the exactness of the sequence jd.

Let $\langle M \rangle \in \Lambda^n$ be such that $d\langle M \rangle = O$. This means for some n-simplex Δ in M, M-int Δ has a compatible differentiable structure making $\partial(M\text{-int } \Delta)$ diffeomorphic to S^{n-1}. Choosing such a diffeomorphism, attach the n-ball D^n to M-int Δ to obtain a differentiable manifold N which is combinatorially equivalent to M. Thus $k[M] = \langle N \rangle$ and dk is exact.

Finally let M represent an element of Θ^n annihilated by k. This means M is J-equivalent to $\partial \Delta^{n+1}$ in the combinatorial sense. Let V be a combinatorial $(n+1)$-manifold realizing this J-equivalence. Let T be a «tube» joining M to $\partial \Delta^{n+1}$ in V, i.e., T is a equivalent to $I \times \Delta^n$ with $O \times \Delta^n \subset M$ and $1 \times \Delta^n \subset \partial \Delta^{n+1}$, and no other points of T in ∂V. (Such a T can easily be constructed by first putting a compatible differentiable structure on the contractible manifold $V \cup \Delta^{n+1}$.) Then V-int T is contractible if T is "unknotted", which is always true if $n + 1 > 3$ and which is the case for $n + 1 = 3$ provided T is chosen properly. (In fact, $\Theta^2 = O$, so this case is unnecessary.) Thus V-int T has a compatible differentiable structure, by (B). By 2.5, we can assume that M-int $(O \times \Delta^n)$ is a differentiable submanifold A of the boundary of V-int T. The closure of the complement of A is combinatorially equivalent to $\partial \Delta^{n+1}$, and hence is diffeomorphic to D^n, while A is combinatorially equivalent, and hence diffeomorphic to $\overline{M - E}$, where E is the interior of an n-ball, $\overline{E}$ differentiably imbedded in M. Thus there is a diffeomorphism $f : \partial(\overline{M - E}) \to S^{n-1}$ such that $\partial(V\text{-int } T)$ is diffeomorphic to $(\overline{M - E}) \cup_f D^n$. Let P be the manifold $\overline{E} \cup_f D^n$. Then P is an element of Γ^n, and $\partial(V\text{-int T})$ is the same as $M \# (-P)$. Since V-int T is contractible, $[M - P] = O$, and so $[M] = [P] = jP$. This establishes the exactness of the sequence (1).

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