Variation Diminishing Transformations and STURM-LIOUVILLE Systems.

Variation Diminishing Transformations and Sturm-Liouville Systems

by I. I. Hirschman, Jr.[1])

1. Introduction. This paper is the fourth in a series; however the following material may be considered as an introduction not only to this paper, but to the preceding papers [3], [4], [5], as well. Given a matrix $M = [m(i, j)]$ $i, j = 1, 2, \ldots, n$ let us denote by $M\begin{bmatrix} i_1, \ldots, i_k \\ j_1, \ldots j_k \end{bmatrix}$ where $1 \leq i_1 < i_2 < \ldots < i_k \leq n$, $1 \leq j_1 < j_2 < \ldots < j_k \leq n$, the submatrix of M formed from the elements in the rows $i_1, \ldots, i_k$ and the columns $j_1, \ldots, j_k$. A matrix M is said to be totally non-negative if all of the quantities det $M\begin{bmatrix} i_1, \ldots, i_k \\ j_1, \ldots, j_k \end{bmatrix}$ are non negative. Matrices with this property and with related properties play an essential role in several fields of mathematics, see [2] and [7]. Here we cite only the following result. Let H_n be the space of real column vectors with n entries

$$u = \begin{bmatrix} u(1) \\ u(2) \\ \vdots \\ u(n) \end{bmatrix}.$$

We denote by $V[u]$ the number of changes of sign of the sequence $u(1), u(2), \ldots, u(n)$.

Theorem 1a. A necessary and sufficient condition that for every $u \in H_n$, $V[Mu] \leq V[u]$, and that if $V[Mu] = V[u]$ the first non-zero components of u and Mu have the same sign, is that M be totally non-negative.

For a demonstration of this result, due to Schoenberg and Krein, see [2, p. 291].

A matrix of the form

$$T = \begin{bmatrix} r(1) & s(1) & 0 & \cdots 0 & 0 \\ t(1) & r(2) & s(2) & \cdots 0 & 0 \\ 0 & t(2) & r(3) & \cdots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots t(n-1) & r(n) \end{bmatrix} \tag{1}$$

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with $t(k)s(k) > 0$ $k = 1, \ldots, n-1$ is called a normal JACOBI matrix. We will deal with JACOBI matrices satisfying the additional condition

$$t(k) < 0, s(k) < 0, k = 1, 2, \ldots, n-1 . \tag{2}$$

Consider the following problem: what matrix functions $\Phi(T)$ are totally non-negative. We begin by recalling the definition of $\Phi(T)$. It is not hard, see [2, p. 80], to show that the characteristic values of a normal JACOBI matrix are real and distinct, so that the matrix is diagonizable. Let $\lambda(1), \ldots, \lambda(n)$ be the characteristic values, and let $y(k)$,

$$y(k) = \begin{bmatrix} y(1, k) \\ y(2, k) \\ \vdots \\ y(n, k) \end{bmatrix}$$

be the eigen vector of T corresponding to $\lambda(k).y(k)$ is uniquely determined up to multiplication by an arbitrary non-zero constant, and the matrix $Y = [y(i, k)]$ is non-singular. Let L be the diagonal matrix with entries $\lambda(1), \ldots, \lambda(n)$. We then have $T = Y L Y^{-1}$. For $\varphi(\lambda)$ a real function defined for $\lambda \in S(T)$, the spectrum of T, $S(T) = \{\lambda(1), \ldots, \lambda(n)\}$, we first define $\varphi(L)$ to be the diagonal matrix with entries $\varphi[\lambda(1)], \ldots, \varphi[\lambda(n)]$ and then put

$$\varphi(T) = Y \varphi(L) Y^{-1} .$$

We note for future reference the following (familiar) properties:

$$\begin{aligned} &\text{i. if } \varphi(\lambda) = \lambda, \text{ then } \varphi(T) = T ; \\ &\text{ii. if } \varphi(\lambda) = c_1\varphi_1(\lambda) + c_2\varphi_2(\lambda) \text{ then } \varphi(T) = c_1\varphi_1(T) + c_2\varphi_2(T); \\ &\text{iii. if } \varphi(\lambda) = \varphi_1(\lambda)\cdot\varphi_2(\lambda) \text{ then } \varphi(T) = \varphi_1(T)\cdot\varphi_2(T); \\ &\text{iv. if } \varphi(\lambda) = \lim_{n\to\infty}\varphi_n(\lambda) \text{ then}^{2)}\ \varphi(T) = \lim_{n\to\infty}\varphi_n(T) . \end{aligned} \tag{3}$$

Let us define

$$\begin{aligned} \Lambda^+(T) &= \text{l.u.b.}\ \{\lambda \mid \lambda \in S(T)\} , \\ \Lambda^-(T) &= \text{l.u.b.}\ \{-\lambda \mid \lambda \in S(T)\} . \end{aligned} \tag{4}$$

Theorem 1b. Let T be a JACOBI matrix satisfying (2) and let us assume[3]) that $\Lambda^+(T) \geq 0$, $\Lambda^-(T) \geq 0$. Let a_k $k = 1, 2, \ldots$, b_k $k = 1, 2, \ldots$, and c satisfy the conditions[4])

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[2]) This means that the i, j-th entry of $\varphi(T)$ is the limit as $n \to \infty$ of the i, j-th entry of $\varphi_n(T)$ for $i, j = 1, 2, \ldots, n$.

[3]) This involves no real loss of generality since if $T_1 = T - \mu I$ then $\Lambda^+(T_1) = \Lambda^+(T) - \mu$ and $\Lambda^-(T) = \Lambda^-(T) + \mu$.

[4]) There may be finitely many or even no a_k's and finitely many or no b_k's.

$$\begin{aligned} &\Lambda^+(T) \leq a_1 \leq a_2 \leq \ldots, \qquad \sum_k a_k^{-1} < \infty, \\ &\Lambda^-(T) < b_1 \leq b_2 \leq \ldots, \qquad \sum_k b_k^{-1} < \infty, \\ &\qquad\qquad c \geq 0, \end{aligned} \tag{5}$$

and let

$$\varphi(\lambda) = e^{c\lambda} \frac{\prod_k \left(1 - \frac{\lambda}{a_k}\right)}{\prod_k \left(1 + \frac{\lambda}{b_k}\right)};$$

then $\varphi(T)$ is totally non-negative.

In order to carry out the (simple) demonstration of this theorem we need several elementary results concerning not necessarily normal JACOBI matrices. It follows from [2, p. 93] that a JACOBI matrix is totally non-negative if and only if:

i. the entries not on the main diagonal are non-negative;

ii. the characteristic values are non-negative.

It follows that if $\varphi(\lambda) = 1 - a^{-1}\lambda$ where $a \geq \Lambda^+(T)$ then $\varphi(T) = I - a^{-1}T$ is totally non-negative. The matrix M is said to be sign regular if and only if

$$(-1)^{i_1+\cdots+i_k+j_1+\cdots+j_k} \det M \begin{bmatrix} i_1, \ldots, i_k \\ j_1, \ldots, j_k \end{bmatrix} \geq 0$$

for all $1 \leq i_1 < i_2 < \ldots < i_k \leq n$, $1 \leq j_1 < j_2 < \ldots < j_k \leq n$. It is shown in [2, p. 93] that a JACOBI matrix is sign regular if and only if:

i. the entries not on the main diagonal are non-positive;

ii. the characteristic values are non-negative.

Finally if M is non-singular then M is totally non-negative if and only if M^{-1} is sign regular, see [2, p. 87]. These remarks taken together show that if $\varphi(\lambda) = [1 + b^{-1}\lambda]^{-1}$ then $\varphi(T) = [I + b^{-1}T]^{-1}$ is totally non-negative provided $b > \Lambda^-(T)$. By the CAUCHY-BINET formula the product of totally non-negative matrices is again totally non-negative. Thus if $a_k \geq \Lambda^+(T)$ $k = 1, \ldots, m$ and if $b_j > \Lambda^-(T)$ $j = 1, \ldots, n$, and if

$$\varphi(\lambda) = \frac{\prod_k (1 - \lambda/a_k)}{\prod_j (1 + \lambda/b_j)}$$

then $\varphi(T)$ is totally non-negative. Here we have used iii. of (3). Using property iv. of (3) the demonstration of our theorem is easily completed.

We now turn to the specific subject matter of the present paper. Let H be the real HILBERT space of functions $u(x)$ defined and measurable on $0 \leq x < \infty$, and such that

$$||u|| = [\int_0^\infty u(x)^2 dx]^{\frac{1}{2}}$$

is finite. Let T be the differential operator

$$Tu(x) = q(x)u(x) - u''(x) \tag{7}$$

with the boundary condition

$$u(0)\cos\alpha + u'(0)\sin\alpha = 0. \tag{8}$$

Here $q(x)$ is a continuous real function defined for $0 \leq x < \infty$, and α is a parameter $-\pi < \alpha \leq 0$. More specifically the domain of T consists of those functions $u(x) \in H$ such that: i. $u(x)$ and $u'(x)$ are absolutely continuous for $0 \leq x < \infty$, and $Tu \in H$; and ii. the condition (8) is satisfied. Under suitable assumptions on q, T as defined above is self-adjoint, see [13, Chapter III]. Let $S(T)$ be the spectrum of T and $E(d\lambda)$ the resolution of the identity in H associated with T. Then if $\varphi(\lambda)$ is any BOREL measurable function defined for $\lambda \in S(T)$ we may define $\varphi(T)$ by the formula

$$\varphi(T) = \int_{-\infty}^{\infty} \varphi(\lambda)\, E(d\lambda)\, u$$

for all $u \in H$ for which

$$\int_{-\infty}^{\infty} |\varphi(\lambda)|^2 (u, E(d\lambda)u) < \infty.$$

If $\varphi(\lambda)$ is bounded then the domain of $\varphi(T)$ is H. Suitable analogues of the properties (3) are valid. It is natural to conjecture that the evident analogue of Theorem 1b is true. In the present paper we assume that:

i. $q(x)$ is continuous and bounded for $0 \leq x < \infty$;

ii. $\int_0^\infty (1 + x^2)\, |q(x)|\, dx < \infty$.

Under these assumptions it can be shown that $S(T)$ consists of the points $0 \leq \lambda < \infty$, all belonging to the continuous spectrum, together with a finite number of points $\lambda_m < \lambda_{m-1} < \ldots < \lambda_1 < 0$ all belonging to the point spectrum. We will show that if

$$\varphi(\lambda) = \left[e^{c\lambda} \prod_k \left(1 + \frac{\lambda}{b_k}\right)\right]^{-1}, \quad \lambda \in S(T),$$

where $c \geq 0$, $\Lambda^-(T) < b_1 \leq b_2 \leq \ldots$, $\sum_k b_k^{-1} < \infty$, then $\varphi(T)$ is variation diminishing, i.e. $V[\varphi(T)u] \leq V[u]$ for all $u \in H$. Note that for $u \in H$, $V[u]$

is defined as the minimum of the number of changes of sign of all the functions $u^*(x)$ which are almost everywhere equal to $u(x)$ for $0 \le x < \infty$. Conversely every bounded BOREL measurable φ defined for $\lambda \in S(T)$ for which $\varphi(T)$ is variation diminishing is essentially of the above form. The demonstration of this second statement, which is the more difficult and the more interesting, is carried out using the asymptotic argument developed in [5]. However the discrete spectrum which may be present here causes difficulties not encountered in the previous application of this method.

2. The spectral structure of T. The results of this section are taken from TITCHMARSH [13, Chapter V], and to a lesser extent from LEVINSON [8]. In the few cases in which the results are not stated in the form we require this is pointed out and the necessary modifications are indicated. We assume throughout that:

$$\text{i. } q(x) \text{ is continuous and bounded for } 0 \le x < \infty; \tag{1}$$

$$\text{ii. } \int_0^\infty (1 + x^2)\,|\,q(x)\,|\,dx < \infty .$$

Let $y(x, \lambda)$ be the (unique) solution of the differential equation

$$Ty = \lambda y, \qquad (0 \le x < \infty), \tag{2}$$

where $Ty = qy - y''$, under the initial conditions

$$y(0, \lambda) = \sin\alpha, y'(0, \lambda) = -\cos\alpha \ (-\pi < \alpha \le 0). \tag{3}$$

From this definition it is easily proved that for each $x, 0 \le x < \infty, y(x, \lambda)$ is an entire function of λ. Moreover setting $\lambda = s^2$ we find that for $0 \le x < \infty$

$$|\,y(x, s^2)\,| \le K(1 + x)\,e^{|\tau|x} \qquad s = \sigma + i\tau . \tag{4}$$

Here and throughout we use K for any finite positive constant depending only on $q(\cdot)$. TITCHMARSH proves a slightly different inequality, but his method implies (4). For the special case $\alpha = 0$ we have the stronger inequality;

$$|\,y(x, s^2)\,| \le K\,x\,e^{|\tau|x}/(1 + |\,s\,|\,x), \tag{5}$$

see LEVINSON [8]. Let us define $M(s)$ by the formula

$$2sM(s) = s\sin\alpha - i\cos\alpha + i\int_0^\infty e^{isx} q(x)\,y(x, s^2)\,dx . \tag{6}$$

It is evident from (4) that if $s = \sigma + i\tau$ then $2sM(s)$ is analytic for $\tau > 0$, and continuous for $\tau \ge 0$. Moreover we have for s fixed, $\tau > 0$,

$$y(x, s^2) = e^{-isx} M(s) + o(e^{-isx}) \quad x \to +\infty, \tag{7}$$

and for $\tau = 0, \sigma > 0$,

$$y(x, \sigma^2) = e^{-i\sigma x} M(\sigma) + e^{i\sigma x} \overline{M(\sigma)} + o(1) \quad x \to \infty . \tag{8}$$

It follows from (6) that

$$\begin{aligned} 2s\, M(s) &= s \sin\alpha + o(s) \quad |s| \to \infty, \ -\pi < \alpha < 0, \\ &= -i + o(1) \quad |s| \to \infty, \quad \alpha = 0, \end{aligned} \tag{9}$$

uniformly for $\tau \geq 0$. For the case $\alpha = 0$ see [8]. It can also be shown that

$$2s\, M(s) = a + bs + o(1) \quad |s| \to 0 \tag{10}$$

uniformly for $\tau \geq 0$. Here a and b are not both 0. The relation (10) is proved in [8] for the case $\alpha = 0$. The general case is entirely similar. It is to be noted that it is only in the demonstration of (10) that the full force of condition ii. of (1) is needed. It follows from (9) and (10) that $M(s)$ has at most finitely many zeros on the line $s = i\tau, 0 < \tau < \infty$. Let these zeros be $iL_1, iL_2, \ldots, iL_m$ where $0 < L_1 < L_2 < \ldots < L_m$. These zeros of $M(s)$ are simple and $M(s)$ has no other zeros in $\operatorname{Im} s \geq 0$.

It can be verified that the spectrum $S(T)$ of T consists of $0 \leq \lambda < \infty$, all points of which belong to the continuous spectrum, together with the points $\lambda = -L_1^2, \ldots, -L_m^2$, all belonging to the point spectrum. Let us denote, as in § 1, by H the space of all real LEBESGUE measurable functions $u(x)$ on $0 \leq x < \infty$ for which

$$\| u \|^2 = \int_0^\infty u(x)^2\, dx$$

is finite. Let us also denote by $H^\wedge$ the space of all real LEBESGUE measurable functions $g(\lambda)$ defined for $\lambda \in S(T)$ and such that

$$\| g \|^2 = \frac{1}{\pi} \int_0^\infty [g(\lambda)]^2 \lambda^{-\frac{1}{2}} \, | M(\sqrt{\lambda}) |^{-2} d\lambda + \sum_{k=1}^m P_k\, g(-L_k^2)$$

is finite where

$$P_k \int_0^\infty [y(x, -L_k^2)]^2 dx = 1 \qquad k = 1, \ldots, m .$$

With these definitions it follows that the mappings $f \to f^\wedge$ and $g \to g^\vee$ defined[5]) by

$$f^\wedge(\lambda) = \int_0^\infty y(x, \lambda) f(x)\, dx \qquad \lambda \in S(T), \tag{11}$$

[5]) Both integrals here must be interpreted as limits in the mean of order two; that is, $f^\wedge(\lambda)$ is the limit in $H^\wedge$ of the sequence $f_A^\wedge(\lambda) = \int_0^A y(x, \lambda) f(x)\, dx$ as $A \to +\infty$, etc.

and

$$g^{\vee}(x) = \frac{1}{\pi}\int_0^\infty y(x,\lambda)\,g(\lambda)\,\lambda^{-\frac{1}{2}}\,|\,M(\sqrt{\lambda})\,|^{-2} d\lambda + \sum_{k=1}^{m} y(x,-L_k^2)\,g(-L_k^2)\,P_k, \quad (12)$$

are isometric mappings of H onto $H^{\wedge}$, and of $H^{\wedge}$ onto H. Moreover these mappings are inverse; that is $(f^{\wedge})^{\vee} = f$, $(g^{\vee})^{\wedge} = g$. Finally $f \in H$ is in the domain $D(T)$ of T if and only if $\lambda f^{\wedge}(\lambda) \in H^{\wedge}$, and in this case we have $(Tf)^{\wedge}(\lambda) = \lambda f^{\wedge}(\lambda)$.

The operational calculus of § 1 now takes the form

$$\begin{aligned}\varphi(T)\,f\cdot(x) = \frac{1}{\pi}\int_0^\infty y(x,\lambda) f^{\wedge}(\lambda)\,\varphi(\lambda)\,\lambda^{-\frac{1}{2}}\,|\,M(\sqrt{\lambda})\,|^{-2}\,d\lambda \\ + \sum_{k=1}^{m} y(x,-L_k^2) f^{\wedge}(-L_k^2)\,\varphi(-L_k^2)\,P_k\end{aligned} \quad (13)$$

where $\varphi(\lambda)$ is any bounded LEBESGUE measurable function on $S(T)$. Note that $\varphi_1(T) = \varphi_2(T)$ if and only if $\varphi_1(\lambda) = \varphi_2(\lambda)$ almost everywhere for $0 \le \lambda < \infty$, and $\varphi_1(-L_k^2) = \varphi_2(-L_k^2)$ for $k = 1, \ldots, m$. Our problem is, of course, to determine those functions φ for which $\varphi(T)$ is a variation diminishing transformation of H into itself.

For future use it is necessary for us to rewrite the inversion formula (12) in a form suitable for the application of the calculus of residues. For each s in the half plane $\tau > 0$ there exists a function $y_1(x, s)$ such that $Ty_1(x, s) = s^2 y_1(x, s)$, and

$$|\,y_1(x,s)\,| \le K\,e^{-\tau x}, \quad (14)$$

$$|\,y_1(x,s) - e^{isx}\,| \le K\,e^{-\tau x}\,|\,s\,|^{-1}\int_x^\infty |\,q(\xi)\,|\,d\xi, \quad (15)$$

for $0 \le x < \infty$. For this result see LEVINSON [8]. Note that $T - \lambda I$ annihilates $y(x, \lambda)$ while $T - s^2 I$ annihilates $y_1(x, s)$. We also have

$$|\,y_1'(x,s) - is\,e^{isx}\,| \le K\,e^{-\tau x}\int_x^\infty |\,q(\xi)\,|\,d\xi. \quad (16)$$

This is easily deduced from the relation 5.9 of [8]. If a series of analytic functions converges uniformly in a region to a (necessarily) analytic limit then the derivatives also converge to the derivatives of the limit function. Applying this principle to $e^{-isx}y_1(x, s)$ one easily deduces that for $\tau > 0$

$$\left(\frac{\partial}{\partial s}\right)^n y_1(x,s) = (ix)^n e^{isx}[1 + o(1)] \quad x \to +\infty. \quad (17)$$

The functions $y_1(x, \sigma)$ and $\overline{y_1(x, \sigma)}$ are annihilated by $T - \sigma^2 I$, and they are independant as can be seen from (15). $y(x, \sigma^2)$ is also annihilated by $T - \sigma^2 I$, and is therefore a linear combination of $y_1(x, \sigma)$ and $\overline{y_1(x, \sigma)}$. Using (8) we see that for $\sigma > 0$

$$y(x, \sigma^2) = M(\sigma)\,\overline{y_1(x, \sigma)} + \overline{M(\sigma)}\,y_1(x, \sigma)\,. \tag{18}$$

If in (12) we first make the change of variables $\lambda = \sigma^2$ and then use (18) we obtain

$$g^{\vee}(x) = \frac{2}{\pi}\int_0^\infty [M(\sigma)\overline{y_1(x, \sigma)} + \overline{M(\sigma)}y_1(x, \sigma)]\,g(\sigma^2)\,|\,M(\sigma)\,|^{-2}d\sigma$$
$$+ \sum_{k=1}^{m} y(x, -L_k^2)\,g(-L_k^2)\,P_k\,.$$

Using the relation $\overline{y_1(x, \sigma)} = y_1(x, -\sigma)$ and the fact that $y(x, -L_k^2) = c_k y_1(x, i\,L_k)$ this becomes

$$g^{\vee}(x) = \frac{2}{\pi}\int_{-\infty}^{\infty} y_1(x, \sigma)\,g(\sigma^2)\,M(\sigma)^{-1}\,d\sigma \tag{19}$$
$$+ \sum_{k=1}^{m} y_1(x, i\,L_k)\,g(-L_k^2)\,P'_k$$

where $P_k' = c_k P_k$. We require an alternative expression for P_k'. To this end we set

$$y_1^{\wedge}(\lambda, i\tau) = \int_0^\infty y(x, \lambda)\,y_1(x, i\tau)\,dx\,.$$

Using the differential equations satisfied by y and y_1 and integrating by parts twice it is easy to show that for $k(\tau)$ correctly chosen and $\tau \neq L_1, \ldots, L_m$,

$$y_1^{\wedge}(\lambda, i\tau) = k(\tau)\,/\,(\lambda + \tau^2)\,. \tag{20}$$

Since $(y_1^{\wedge})^{\vee} = y_1$ we have

$$y_1(x, i\tau) = \frac{2}{\pi}\int_{-\infty}^{\infty} k(\tau)\,(\tau^2 + \sigma^2)^{-1}\,y_1(x, \sigma)\,M(\sigma)^{-1}\,d\sigma \tag{21}$$
$$+ \sum_{k=1}^{m} y_1(x, iL_k)\,k(\tau)\,(\tau^2 - L_k^2)^{-1}\,P_k'.$$

We deform the line of integration in the integral on the right from $\operatorname{Im} s = 0$ to $\operatorname{Im} s = r$, where r is large and positive. This is easily justified using (9), (10), and (14). We find that if $-iM_k$ is the residue of $1/M(s)$ at L_k (M_k is then real because $M(s)$ is real on the positive imaginary axis) we have

$$y_1(x, i\tau) = 2\pi i \left\{\frac{2}{\pi} \sum_{k=1}^{m} k(\tau)\,(\tau^2 - L_k^2)^{-1} y_1(x, iL_k)(-iM_k)\right\}$$
$$+ 2\pi i \left\{\frac{2}{\pi} k(\tau)\,(2i\tau)^{-1}\, y_1(x, i\tau)\, M(i\tau)^{-1}\right\}$$
$$+ \sum_{k=1}^{m} y_1(x, iL_k)\, k(\tau)\,(\tau^2 - L_k^2)^{-1} P_k'$$
$$+ R(x)\,,$$

where

$$R(x) = \frac{2}{\pi} \int_{-\infty + ri}^{\infty + ri} k(\tau)\,(\tau^2 + \sigma^2)^{-1}\, y_1(x, \sigma)\, M(\sigma)^{-1}\, d\sigma\,.$$

It is easy to see that $|R(x)| \leq K e^{-rx}$ and since by CAUCHY's theorem $R(x)$ is independant of r it follows that $R(x) \equiv 0$. It is now evident that $P_k' = -4M_k$ $k = 1, \ldots, m$, and also that $2k(\tau) = \tau M(i\tau)$. Using this we now obtain the inversion formula in the required form

$$g^{\vee}(x) = \frac{2}{\pi} \int_{-\infty}^{\infty} y_1(x, \sigma)\, g(\sigma^2)\, M(\sigma)^{-1}\, d\sigma \tag{22}$$
$$- 4 \sum_{k=1}^{m} y_1(x, iL_k)\, g(-L_k^2)\, M_k\,.$$

3. Sufficient Conditions. We assume that conditions i. and ii. of (1) § 2 hold.

Theorem 3a. If $b > \Lambda^-(T)$ and if $\varphi(\lambda, b) = (1 + \lambda/b)^{-1}$ then $\varphi(T, b)$ is variation diminishing; that is, for every $u \in H$ $V[\varphi(T, b)\, u] \leq V[u]$.

From § 2 we recall the following properties of $y_1(x, i\tau)$:

$$\begin{array}{lll} \text{i.} & Ty_1 + \tau^2 y_1 = 0\,; & \\ \text{ii.} & y_1(x, i\tau) = e^{-x\tau}[1 + o(1)] & \text{as } x \to \infty\,; \\ \text{iii.} & y_1'(x, i\tau) = -\tau e^{-x\tau}[1 + o(1)] & \text{as } x \to \infty\,. \end{array} \tag{1}$$

Let $\beta(x, \tau)$ be defined by

$$y_1(x, i\tau) = r \sin \beta(x, \tau)\,,$$
$$y_1'(x, i\tau) = r \cos \beta(x, \tau)\,,$$
$$r > 0\,.$$

Initially $\beta(x, \tau)$ is determined only up to an integral multiple of 2π; however once it has been fixed for any value of x it is to be determined for other values of x in such a way that it is continuous. We note that $\beta(x, \tau)$ is equal to an integral multiple of π only if $y_1(x, i\tau)$ vanishes and that if $\beta(x_1, \tau) = n_1\pi$

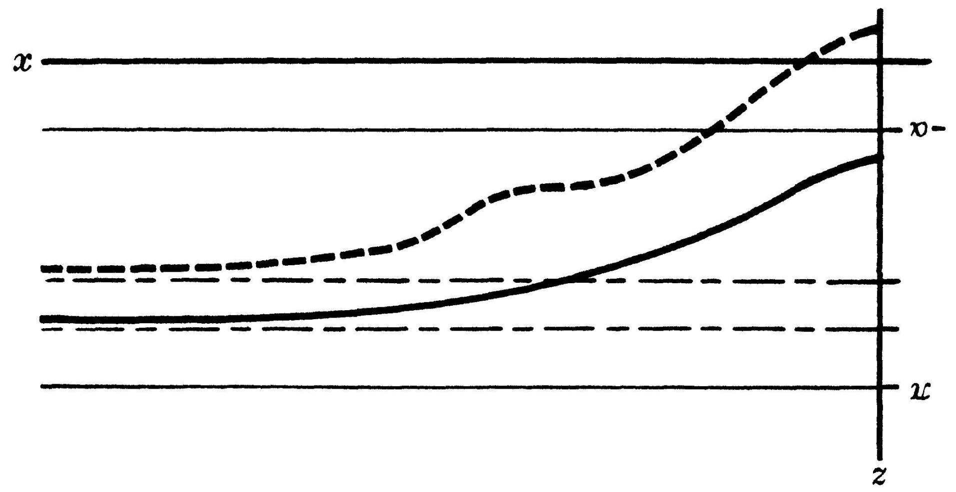
then $\beta(x, \tau) > n_1\pi$ for $x > x_1$. It is evident from (1) that we may take $\beta(\infty, \tau) = \pi - \operatorname{Arctan} \tau^{-1}$. We recall that $-\pi < \alpha \le 0$. We assert and will prove that if $\tau^2 > \Lambda^-(T)$ then

$$y_1(x, i\tau) > 0 \qquad (0 \le x < \infty), \tag{2}$$

and that

$$y_1(0, i\tau) \cos\alpha + y_1'(0, i\tau) \sin\alpha > 0. \tag{3}$$

Consider the graph of $\beta(x, \tau)$.



The assertions (2) and (3) are equivalent to the inequality $\beta(0, \tau) > -\alpha$. It is therefore sufficient to prove that $\beta(0, \tau) \le -\alpha$ is impossible. If $\beta(0, \tau) = -\alpha$, then $-\tau^2$ belongs to the point spectrum of T contrary to the assumption $\tau^2 > \Lambda^-(T)$. In order to proceed we need the following facts:

i. $\beta(0, \tau)$ is a continuous function of τ;
ii. $\tau_1 > \tau$ implies that $\beta(0, \tau_1) > \beta(0, \tau)$;
iii. $\lim_{\tau_1 \to \infty} \beta(0, \tau_1) = \pi$.

The first statement is easily deduced from the continuity of $y_1(x, s)$ for $\operatorname{Im} s \ge 0, 0 \le x < \infty$, see [8, p. 22]. The second and third statements are simple deductions from the STURM comparison theory applied to the interval $0 \le x < \infty$, see CODDINGTON and LEVINSON [1, p. 210]. It follows that if $\beta(0, \tau) < -\alpha$, then there will exist a value $\tau_1 > \tau$ for which $\beta(0, \tau_1) = -\alpha$. But then $-\tau_1^2$ belongs to the point spectrum of T which is impossible. Thus (2) and (3) are valid.

Now let $u \in D(T)$, the domain of T. We will show that $V[(bI + T)u] \ge V[u]$. Let $b = \tau^2$ (so that $\tau^2 > \Lambda^-(T)$).

A simple computation shows that

$$(\tau^2 I + T)\, u = -\, y_1(x, i\tau)^{-1} \left(\frac{d}{dx}\right) y_1\,(x, i\tau)^2 \left(\frac{d}{dx}\right) u(x) \,/\, y_1(x, i\tau)\,.$$

Since $u \in D(T)$, we have $u(0) = -r \sin\alpha,\ u'(0) = r \cos\alpha$.
Thus

$$\left(\frac{d}{dx}\right) u(x) \,/\, y_1(x, i\tau) \mid_{x=0} = r\,[y_1(0, i\tau)\cos\alpha + y_1'(0, i\tau)\sin\alpha]\, y_1(0, i\tau)^{-2}$$

and thus since $\beta(0, \tau) > -\alpha$ we have

$$sgn \left(\frac{d}{dx}\right) u(x) \,/\, y_1(x, i\tau) \mid_{x=0} = sgn\, u(0)\,.$$

For example if $u(0) > 0$ then the graph of $u(x)/y_1(x, i\tau)$ looks as follows.

It is evident that $\{u(x)/y_1(x, i\tau)\}'$ has a change of sign to the left of each change of sign of $u(x)$ and thus that

$$V[\{u(x)/y_1(x, i\tau)\}'] \geq V[u(x)]\,.$$

Next since, as is easily seen,

$$\varliminf_{x \to \infty} y_1(x, i\tau)^2\, \{u(x)/y_1(x, i\tau)\}' = 0$$

it follows that $\{y_1(x, i\tau)^2\, [u(x)/y_1(x, i\tau)]'\}'$ has a change of sign to the right of each change of sign of $\{u(x)/y_1(x, i\tau)\}'$ and thus that

$$V[\{y_1(x, i\tau)^2 \{u(x)/y_1(x, i\tau)\}'\}'] \geq V[\{u(x)/y_1(x, i\tau)\}']\,.$$

These results together show that $bI + T$ is variation increasing and so that $(bI + T)^{-1}$ is variation decreasing.

Theorem 3b. Assume that conditions (1) of § 2 hold.
If $\Lambda^-(T) < b_1 \leq b_2 \leq \ldots,\ \sum_k b_k^{-1} < \infty,\ c \geq 0$, and if

$$\varphi(\lambda) = [e^{c\lambda} \prod_k (1 + \lambda/b_k)]^{-1} \qquad \lambda \in S(T)\,, \tag{4}$$

then $\varphi(T)$ is variation decreasing on H.

Since the product of variation diminishing transformations is variation diminishing, we see from Theorem 3a that

$$\varphi(T, n/c)^n \prod_1^n \varphi(T, b_k) = \varphi_n(T)$$

is variation decreasing; i.e. $V[\varphi_n(T)u] \leq V[u]$. We have

$$\lim_{n\to\infty} \varphi_n(\lambda) = \varphi(\lambda) \qquad \lambda \in S(T).$$

Moreover there is a constant M such that for all $n = 1, 2, \ldots$

$$|\varphi_n(\lambda)| \leq M \qquad \lambda \in S(T).$$

Using (13) of § 2 this implies that

$$\lim_{n\to\infty} \| \varphi(T)u - \varphi_n(T)u \| = 0,$$

and thus by a simple general argument

$$V[\varphi(T)u] \leq \underline{\lim}_{n\to\infty} V[\varphi_n(T)u] \leq V[u],$$

as desired.

As we shall see in § 4, formula (4) gives all the bounded variation decreasing functions of T, if $\Lambda^-(T) = 0$. If however $\Lambda^-(T) > 0$ there is one further (trivial) variation decreasing function which is described in § 4.

4. Necessary Conditions. We require the following result, which is a special case of a theorem of SCHOENBERG [9].

Theorem 4a. Let $\psi(\sigma^2) \in L_1(0, \infty)$ and let $L(x) = \frac{4}{\pi}\int_0^\infty \psi(\sigma^2) \cos x\sigma d\sigma$. If for every $a(t) \in L_1(-\infty, \infty)$ we have $V[L*a] \leq V[a]$ then there exists a function[6])

$$\Psi(\sigma^2) = d[e^{c\sigma^2} \prod_k (1 + \sigma^2/b_k^2)]^{-1} \tag{1}$$

where $0 \leq c, 0 < b_1 \leq b_2 \leq \ldots, \sum_k b_k^{-2} < \infty$ such that $\psi(\sigma^2) = \Psi(\sigma^2)$ almost everywhere for $0 \leq \sigma < \infty$.

Here

$$L*a\cdot(x) = \int_{-\infty}^{\infty} L(x-t)\, a(t)\, dt \qquad -\infty < x < \infty,$$

and $V[a]$ counts the number of changes of sign of $a(x)$ for $-\infty < x < \infty$.

---

[6]) Note that b^2's «correspond to» b's.

We again assume that conditions (1) of § 2 are satisfied. We further assume that $\varphi(\lambda)$ is a bounded LEBESGUE measurable function defined on $S(T)$ such that $\varphi(T)$ is variation diminishing in H. Our objective in the present section is to prove that $\varphi(\lambda)$ must be essentially of the form (9) of § 1. Since the product of variation decreasing operators is again variation decreasing it follows from Theorem 3b that if $\psi(\lambda) = \varphi(\lambda)\, e^{-\lambda}$ then $\psi(\lambda)$ is variation decreasing. The advantage of working with $\psi(\lambda)$ is that it is small at ∞. Let us set

$$K(x,t) = \frac{2}{\pi} \int_{-\infty}^{\infty} y_1(x,\sigma)\, y(t,\sigma^2)\, \psi(\sigma^2)\, M(\sigma)^{-1}\, d\sigma \tag{2}$$
$$- 4 \sum_{k=1}^{m} y_1(x, iL_k)\, y(t, -L_k^2)\, \psi(-L_k^2)\, M_k\,.$$

Using (14) and (18) of § 2 it is easily seen that this formula defines $K(x,t)$ as a continuous bounded function of x and t for $0 \leq x, t < \infty$. It follows from FUBINI's theorem that if $u(x) \in L_1(0,\infty) \cap H$ then

$$\psi(T)u\cdot(x) = \int_0^{\infty} K(x,t)\, u(t)\, dt\,. \tag{3}$$

Consequently since $\psi(T)$ is variation diminishing

$$V\Big[\int_0^{\infty} K(x,t)\, u(t)\, dt\Big] \leq V[u(x)] \tag{4}$$

for every $u \in L_1(0,\infty) \cap H$. A simple approximation argument shows that in fact (4) holds for $u \in L_1(0,\infty)$.

Lemma 4b. Under the above assumptions there exists a function Ψ of the form (1) such that $\psi(\sigma^2) = \Psi(\sigma^2)$ almost everywhere for $0 \leq \sigma < \infty$.

Let us define

$$L(x) = \frac{4}{\pi} \int_0^{\infty} \cos x\sigma\, \psi(\sigma^2)\, d\sigma\,. \tag{5}$$

We will prove that for x and t fixed, $-\infty < x, t < \infty$

$$\lim_{r\to\infty} K(x+r, t+r) = L(x-t)\,. \tag{6}$$

Note for x and t fixed we will have $x+r$ and $t+r$ greater than or equal to 0 for all sufficiently large r, so that $K(x+r, t+r)$ is well defined. We have

$$K(x+r, t+r) = I_1 + I_2$$

where

$$I_1 = \frac{2}{\pi} \int_{-\infty}^{\infty} y_1(x+r,\sigma)\, y(t+r,\sigma^2)\, \psi(\sigma^2)\, M(\sigma)^{-1}\, d\sigma\,,$$

$$I_2 = -4 \sum_{k=1}^{m} y_1(x+r, iL_k)\, y(t+r, -L_k^2)\, \psi(-L_k^2)\, M_k\,.$$

By (14) and (15) of § 2

$$y_1(u, \sigma) = e^{iu\sigma} + \delta(u, \sigma)$$

where $|\delta(u, \sigma)| \leq K$ for $0 \leq u, \sigma < \infty$, and $\lim\limits_{u \to \infty} \delta(u, \sigma) = 0$ for $0 < \sigma < \infty$. By (18) of § 2

$$y(u, \sigma^2) = M(\sigma) e^{-iu\sigma} + \overline{M(\sigma)} e^{iu\sigma} + \epsilon(u, \sigma)$$

where

$$\epsilon(u, \sigma) = M(\sigma)\, \overline{\delta(u, \sigma)} + \overline{M(\sigma)}\, \delta(u, \sigma)\,.$$

Consequently

$$y_1(x+r, \sigma) y(t+r, \sigma^2) = e^{i(x-t)\sigma} M(\sigma) + e^{i(x+t+2r)\sigma}\, \overline{M(\sigma)} + R(x, t, r, \sigma)$$

where $|R| \leq K\,|M(\sigma)|$, and $\lim\limits_{r \to \infty} R = 0$. We have $I_1 = J_1 + J_2 + J_3$ where

$$J_1 = \frac{2}{\pi} \int_{-\infty}^{\infty} e^{i(x-t)\sigma}\, \psi(\sigma^2)\, d\sigma = L(x-t)\,,$$

$$J_2 = \frac{2}{\pi} \int_{-\infty}^{\infty} e^{i(x+t+2r)\sigma} \{\overline{M(\sigma)}/M(\sigma)\}\, \psi(\sigma^2)\, d\sigma\,,$$

$$J_3 = \frac{2}{\pi} \int_{-\infty}^{\infty} R(x, t, r, \sigma)\, \psi(\sigma^2)\, M(\sigma)^{-1}\, d\sigma\,.$$

By the RIEMANN-LEBESGUE theorem

$$\lim_{r \to \infty} J_2 = 0,$$

and by LEBESGUE's theorem on dominated convergence

$$\lim_{r \to \infty} J_3 = 0.$$

It follows from (14) of § 2 that $\lim\limits_{r \to \infty} I_2 = 0$. These results together establish (6). Let $a(x) \in L^1(-\infty, \infty)$; using (4) we find that

$$V_r\Big[\int_{-r}^{\infty} K(x+r, t+r)\, a(t)\, dt\Big] \leq V_r[a]$$

where by $V_r[a]$ we mean the number of changes of sign of a on $-r \leq x < \infty$. Letting $r \to \infty$ and using LEBESGUE's limit theorem we see that $V[L^* a] \leq V[a]$. We have now only to apply Theorem 4a to obtain our desired result.

Before proceding let us take account of how far we are from our goal. We have shown that for a function $\Psi(\lambda)$ defined (1) (with σ^2 replaced by λ) $\psi(\lambda) = \Psi(\lambda)$ almost everywhere for $0 \leq \lambda < \infty$. However we have not yet shown that $b_k \geq L_m$ $k = 1, 2, \ldots$, nor have we proved that $\psi(-L_k^2) = \Psi(-L_k^2)$ $k = 1, \ldots, m$.

We now divide our argument into two cases, as the constant d of (1) is zero or is not zero. The first case to which we will return at the end of our discussion is trivial. In the following lemmas 4c, 4d, and 4e we confine ourselves to the case $d \neq 0$. Without loss of generality we may assume that $d = 1$.

Lemma 4c. Let $K(x, t)$ be defined by (2). If $0 \leq x_1 < x_2 < \ldots < x_n$, $0 \leq t_1 < t_2 \ldots < t_n$, then

$$\det[K(x_i, t_j)] \geq 0 .$$

Let $0 \leq \xi_1 < \xi_2 < \ldots < \xi_N$, and $0 \leq \tau_1 < \tau_2 < \ldots < \tau_N$ include $\{x_i\}_1^n$ and $\{t_i\}_1^n$ respectively. The sets $\{\xi_i\}_1^N$ and $\{\tau_i\}_1^N$ will be specified more closely in a moment. An easy deduction from (4) shows that the matrix $[K(\xi_i, \tau_j)]$ $i, j = 1, \ldots, N$ is variation diminishing. Since $d = 1$ in (1) it is evident that $L(0) > 0$, while by the RIEMANN-LEBESGUE theorem $\lim_{x \to \infty} L(x) = 0$. Consequently if A is sufficiently large we have

$$\det[L(\{i - j\}A)]_{i,j=1}^{n} > 0 ,$$

$$\det[L(\{i - j\}A)]_{i,j=1}^{n+1} > 0 .$$

By (6) it follows that if r is sufficiently large

$$\det[K(r + iA, r + jA)]_{i,j=1}^{n} > 0 ,$$

$$\det[K(r + iA, r + jA)]_{i,j=1}^{n+1} > 0 .$$

Let $\{\xi_i\}_1^N$ include (in addition to $\{x_i\}_1^n$) $\{r + iA\}_1^{n+1}$, and $\{\tau_i\}_1^N$ include (in addition to $\{t_i\}_1^N$) $\{r + iA\}_1^{n+1}$. Since $[K(\xi_i, \tau_j)]_{i,j=1}^{N}$ is variation diminishing two minors of the same size have the same sign, whenever the size is less than the rank, see [6, Chapter V]. Our desired result now follows.

Lemma 4d. Under the above assumptions $b_1 > L_m$, and $\psi(-L_k^2) = \Psi(-L_k^2)$ $k = 1, 2, \ldots, m - 1$.

Although the demonstration of this lemma is very elementary it is nevertheless long and tedious. Also the case $\alpha = 0$ is in some degree special and to begin with we assume $-\pi < \alpha < 0$.

It is evident from the discussion in § 3 that if $-\pi < \alpha < 0$ then $y_1(x, i\tau) > 0$ for $0 \leq x < \infty$, $\tau > L_m - \delta$ if δ is a sufficiently small positive quantity.

If

$$Y_1(x, i\tau) = \psi(T) y_1(x, i\tau)$$

then

$$Y_1(x, i\tau) = \int_0^\infty K(x, t)\, y_1(t, i\tau)\, dt\,.$$

By Lemma 4c with $n = 1$ it follows that

$$Y_1(x, i\tau) \geq 0 \qquad 0 \leq x < \infty,\ \tau \geq L_m - \delta\,. \tag{7}$$

Alternatively we have the representation

$$Y_1(x, i\tau) = \frac{2}{\pi} \int_{-\infty}^{\infty} y_1(x, \sigma)\, k(\tau)\, (\sigma^2 + \tau^2)^{-1}\, \Psi(\sigma^2)\, M(\sigma)^{-1}\, d\sigma \tag{8}$$

$$- 4 \sum_{k=1}^{m} y_1(x, iL_k)\, k(\tau)\, (\tau^2 - L_k{}^2)^{-1}\, \psi(-L_k{}^2)\, M_k\,,$$

see (13) and (22) of § 2. We will show that if the conclusion of our lemma are not true we can use (8) to contradict (7).

Step 1. $b_1 \geq L_1$. Suppose this is false and that $b_1 < L_1$. Deform the line of integration in the integral on the right hand side of (8) from $\operatorname{Im} s = 0$, to $\operatorname{Im} s = r$, where $r > b_1$. We find that if $b_1 = b_2 = \ldots = b_N < b_{N+1}$ and if $\Psi_N(s^2) = (s^2 + b_1{}^2)^N\, \Psi(s^2)$ then

$$Y_1(x, i\tau) =$$

$$(2\pi i)\,(2/\pi)\,([N-1]!)^{-1} \left(\frac{\partial}{\partial s}\right)^{N-1} [y_1(x,s)\, k(\tau)\, (s^2 + \tau^2)^{-1}\, \Psi_N(s)\, (s + i b_1)^{-N}]_{s=ib_1} + R(x)\,,$$

where for some $\epsilon > 0$ $R(x) = O(e^{-(b_1+\epsilon)x})$ as $x \to \infty$. Using (17) of § 2 we now see that if $b_1 < L_1$

$$Y_1(x, i\tau) = c\, k(\tau)\, (\tau^2 - b_1{}^2)^{-1} x^{N-1}\, e^{-b_1 x} [1 + o(1)] \tag{9}$$

where c is a positive constant independant of τ. Now $k(\tau)$ changes sign as τ crosses L_m, since $k(\tau) = \frac{1}{2}\tau\, M(i\tau)$, see the end of § 2. Therefore if (9) holds it is possible to contradict (7).

Step 2. $b_1 > L_1$. The argument here is almost exactly as above. Assume $b_1 = L_1$. Choose $r > L_1$ and deform the line of integration in (8) to $\operatorname{Im} s = r$. We obtain

$$Y_1(x, i\tau) = (2\pi i)\, (N!)^{-1}\, (2/\pi) \left(\frac{\partial}{\partial s}\right)^{N} [y_1(x, s)\, k(\tau)\, (s^2 + \tau^2)^{-1}\, \Psi_N(s)$$

$$(s + iL_1)^{-N} \{(s - iL_1) M(s)^{-1}\}]_{s=iL_1} + R(x)$$

where $R(x) = O(e^{-(L_1+\epsilon)x})$, and from this it is apparent that

$$Y_1(x, i\tau) = c\, k(\tau)\, (\tau^2 - L_1^2)^{-1} M_1\, x^N e^{-L_1 x}\,[1 + o(1)]\,, \tag{10}$$

where here again c is a positive constant independant of τ, etc.

Step 3. $\psi(-L_1^2) = \Psi(-L_1^2)$. Choose $r > L_1$, etc. Let I_1 be the integral on the right of (8). Then the usual contour integration argument gives

$$I_1 = (2\pi i)(2/\pi)\, y_1(x, iL_1)\, k(\tau)\, (\tau^2 - L_1^2)^{-1}(-\, iM_1)\, \Psi(-L_k^2) + R(x)$$

where $R(x) = O(e^{-(L_1+\epsilon)x})$ for some $\epsilon > 0$. Let I_2 be the sum on the right of (8). Then

$$I_2 = -4\, y_1(x, iL_1)\, k(\tau)\, (\tau^2 - L_1^2)^{-1} M_1\, \psi(-L_1^2) + R(x)$$

where again $R(x) = O(e^{-(L_1+\epsilon)x})$ for some $\epsilon > 0$. Thus

$$Y_1(x, i\tau) = 4k(\tau)\, (\tau^2 - L_1^2)^{-1} M_1[\Psi(-L_1^2) - \psi(-L_1^2)]\, y_1(x, iL_1) + R(x)\,. \tag{11}$$

Since $k(\tau)$ changes sign across L_m (11) can be made to contradict (7) just as before, unless $\psi(-L_1^2) = \Psi(-L_1^2)$.

Repeated application of steps 1., 2., and 3. proves that $b_1 \geq L_m$ and that $\psi(-L_k^2) = \Psi(-L_k^2)$ $k = 1, \ldots, m-1$.

Step 4. $b_1 > L_m$. Arguing just as in step 2 we find that if $\tau > L_m$

$$Y_1(x, i\tau) = c\, k(\tau)\, (\tau^2 - L_m^2)^{-1} M_m\, x^N e^{-L_m x}[1 + o(1)] \tag{12}$$

where c is a *positive* constant independant of τ. Since by (9) of § 2 $M(i\tau) < 0$ if $\tau > L_m$, we have $M_m > 0$ and $k(\tau) < 0$. Thus (12) contradicts (7).

It remains to indicate the modifications that are necessary if $\alpha = 0$. The difficulty is that the proof of (7) is in default, due to the fact that no matter how small $\delta > 0$ is chosen $y_1(x, i\tau)$ has one change of sign in $0 < x < \infty$, if $L_m - \delta < \tau < L_m$. We will prove that for $\delta > 0$ sufficiently small and for each τ, $L_m - \delta < \tau < L_m$, the inequality $Y_1(x, i\tau) \geq 0$ holds for all sufficiently large x, which is all we need. Choose a value x_0 for which $Y_1(x_0, iL_m) > 0$. Since, as is easily seen, $\lim_{\tau \to L_m} Y_1(x, i\tau) = Y_1(x, iL_m)$ there is a δ, $L_{m-1} < L_m - \delta < L_m$ such that $Y_1(x_0, i\tau) > 0$ if $L_m - \delta < \tau < L_m$. Now $y_1(x, i\tau)$ has one change of sign, and consequently $Y_1(x, i\tau)$ has either no changes of sign or one change of sign. If $Y_1(x, i\tau)$ has no changes of sign then since $Y_1(x_0, i\tau) > 0$ we must have $Y_1(x, i\tau) \geq 0$ for $0 < x < \infty$. If $Y_1(x, i\tau)$ has one change of sign we proceed as follows. Theorem 1a of § 1 together with Lemma 4c shows that since $y_1(x, i\tau)$ is first negative and then positive, $Y_1(x, i\tau)$ must be first negative and then positive, that is it must be non-negative for all sufficiently large x.

Lemma 4e. Under the above assumptions $\psi(-L_m^2) = \Psi(-L_m^2)$.

The argument here is somewhat different from that used to prove the preceding lemma. In what follows we shall for the sake of simplicity assume that $b_1 < b_2$, i.e. $N = 1$. This involves no loss of generality[7]). In the formula (2) let us deform the line of integration from $\mathrm{Im}\, s = 0$, to $\mathrm{Im}\, s = r$, where $b_1 < r < b_2$. Taking Lemma 4d into account we find that if $\Psi_1(s^2) = (s^2 + b_1^2)\, \Psi(s^2)$ then

$$K(x, t) = I_1(x, t) + I_2(x, t) + R(x, t)\,, \tag{13}$$

where

$$I_1(x, t) = 4M_m[\Psi(-L_m^2) - \psi(-L_m^2)]y_1(x, iL_m)y(t, -L_m^2)\,,$$
$$I_2(x, t) = 2[b_1M(ib_1)]^{-1}\Psi_1(-b_1^2)y_1(x, ib_1)\, y(t, -b_1^2)\,,$$

and where for some $\epsilon > 0$

$$|\, R(x, t)\, | \leq K\, e^{-(b_1+\epsilon)(x-t)}. \tag{14}$$

Let us examine the asymptotic behaviour of I_1 and I_2. We have $y(t, -L_m^2) = c_m y_1(x, iL_m)$. Thus if $4M_m c_m[\Psi(-L_m^2) - \psi(-L_m^2)] = A$, then

$$I_1(x, t) = A\, e^{-L_m(x+t)}[1 + o(1)] \text{ as } x, t \to \infty. \tag{15}$$

Similarly

$$I_2(x, t) = B\, e^{-b_1(x-t)}[1 + o(1)] \text{ as } x, t \to \infty\,. \tag{16}$$

In obtaining these results we have used formulas (7) and (15) from § 2. We note that $B > 0$, for the explicit expression for B shows that it is not 0, and $B < 0$ is impossible since it would imply $K(x, t) < 0$ for certain large values of x and t, which contradicts the case $n = 1$ of Lemma 4c. In order to prove our lemma it is enough to show that $A = 0$. It is trivial to see that $A \geq 0$, since the contrary assumption would imply $K(x, t) < 0$ for certain large values etc. By Lemma 4c if $0 < x_1 < x_2$, $0 \leq t_1 < t_2$, then

$$\Delta = \det \begin{bmatrix} K(x_1, t_1) & K(x_1, t_2) \\ K(x_2, t_1) & K(x_2, t_2) \end{bmatrix} \geq 0\,.$$

Let $\theta = (b_1 - L_m)/(b_1 + L_m)$ and choose $\theta < \lambda_1 < \lambda_2 < 1$. We set $x_1 = u$, $x_2 = u/\theta$, $t_1 = \lambda_1 u$, $t_2 = \lambda_2 u$. Using (13)–(16) we find that

[7]) It is always possible to replace $\psi(\lambda)$ by $\psi(\lambda)\,[1 + \lambda/p^2]^{-1}$ where $L_m < p^2 < b_1$.

$$\frac{K(x_1, t_2)}{K(x_1, t_1)} \sim e^{b_1(\lambda_2 - \lambda_1)u} \quad \text{as } u \to \infty ,$$

and if $A \neq 0$

$$\frac{K(x_2, t_2)}{K(x_2, t_1)} \sim e^{-L_m(\lambda_1 - \lambda_2)u} \quad \text{as } u \to \infty .$$

From this it is evident that for large u, $\Delta < 0$. This contradiction shows that $A = 0$.

Theorem 4f. Let the assumptions (1) of § 2 hold. If φ is a bounded measurable function on $S(T)$ such that $\varphi(T)$ is variation diminishing, then there exist constants d real, $c \geq 0$, $\Lambda^-(T) < b_1 \leq b_2 \leq \ldots$, $\sum_k b_k^{-1} < \infty$, such that[8])

$$\varphi(\lambda) = d[e^{c\lambda} \prod_k (1 + \lambda/b_k)]^{-1} \; \lambda \in S(T) , \tag{17}$$

or

$$\varphi(\lambda) = 0 \quad \text{for} \quad \lambda \in S(T) - \{L_m\} . \tag{18}$$

Suppose first that $\Psi(\sigma^2)$ in Lemma 4b is not identically zero; i.e. $d \neq 0$. In this case our preceding arguments prove that there exist constants $d \neq 0$, $\mathbf{c} \geq 0$, $\Lambda^-(T) < \mathbf{b}_1^2 \leq \mathbf{b}_2^2 \leq \ldots$, $\sum_k \mathbf{b}_k^{-2} < \infty$ such that

$$\psi(\sigma^2) = d[e^{\mathbf{c}\sigma^2} \prod_k (1 + \sigma^2/\mathbf{b}_k^2)]^{-1}$$

and thus that

$$\varphi(\lambda) = d[e^{c\lambda} \prod_k (1 + \lambda/b_k)]^{-1}$$

where $c = \mathbf{c} - 1$, $b_k = \mathbf{b}_k^2$, etc. Since by assumption ψ is bounded on $S(T)$ we must have $c \geq 0$.

There remains the case that $\Psi(\sigma^2)$ in Lemma 4b is identically zero; i.e. $d = 0$. In this case $\psi(\sigma^2) = 0$ for $0 \leq \sigma < \infty$. Using Step 3. of Lemma 4d we find that in this case $\psi(-L_k^2) = 0$ for $k = 1, \ldots, m - 1$. This gives (18). Since if $\psi(\lambda)$ has the form (18) every function in H is transformed into a multiple of $y_1(x, iL_m)$ it is evident that $\psi(T)$ is (trivially) variation decreasing.

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[8]) The relations (17) and (18) need hold only almost everywhere for $0 \leq \lambda < \infty$.

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