A remark on the moduli of rings.

A remark on the moduli of rings

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1. Modulus of a ring. A finite doubly-connected plane domain is called a ring. Given any ring R we let C_0 and C_1 denote, respectively, the bounded and unbounded components of the complement of R in the extended plane. We further let $B_0 = \partial C_0$ and $B_1 = \partial C_1$, where ∂E denotes the boundary of the set E. B_0 and B_1 are then the components of ∂R.

Each ring R in the z-plane can be mapped conformally by $w(z)$ onto an annulus $0 \leq a < | w | < b \leq \infty$ so that B_0 corresponds to $| w | = a$ and B_1 to $| w | = b$ [1]). The conformal invariant

$$\operatorname{mod} R = \log \frac{b}{a} \tag{1}$$

is called the modulus of R. When B_0 and B_1 are both non-degenerate, $0 < a < b < \infty$ and the function

$$u(z) = \frac{\log \left| \frac{w(z)}{a} \right|}{\log \frac{b}{a}} \tag{2}$$

is harmonic in R with boundary values 0 on B_0 and 1 on B_1. It is also easy to verify that

$$\frac{2\pi}{\operatorname{mod} R} = \iint_R | \nabla u |^2 d\sigma .$$

If we now appeal to the DIRICHLET principle, we obtain

$$\frac{2\pi}{\operatorname{mod} R} = \inf_v \iint_R | \nabla v |^2 d\sigma , \tag{3}$$

where v is any function which is continuously differentiable in R and has boundary values 0 on B_0 and 1 on B_1. When B_0 or B_1 reduces to a point, $\operatorname{mod} R = \infty$ and the infimum on the right hand side of (3) is 0. Hence (3) yields an alternative definition for the modulus of a ring which does not depend upon conformal mapping.

It is sometimes convenient to work with a slightly larger class of competing functions v. For example if v satisfies a LIPSCHITZ condition on each compact

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[1]) See p. 203 of [4].

subset of R, then v is differentiable almost everywhere in R. If, in addition, v has boundary values 0 on B_0 and 1 on B_1, we can apply GREEN's theorem to show that

$$\frac{2\pi}{\operatorname{mod} R} \leq \iint_R |\nabla v|^2 d\sigma \text{ [2]}. \tag{4}$$

Hence the infimum in (3) can be taken over the class of all such functions v.

2. Continuity of the modulus. A sequence of sets $\{E_n\}$ is said to *converge uniformly* to a set E if, for each $\varepsilon > 0$, there exists an N such that $n > N$ implies each point of E_n lies within distance ε of E and each point of E lies within distance ε of E_n.

In the preceding paper [2] VÄISÄLÄ and I appealed to the following continuity property of mod R.

Theorem. *Let $\{R_n\}$ be a sequence of rings and let R be a bounded ring. If each of the components of ∂R_n converges uniformly to the corresponding component of ∂R, then*

$$\operatorname{mod} R = \lim_{n\to\infty} \operatorname{mod} R_n \text{ [3]}. \tag{5}$$

It is easy to establish this result by a direct argument in the special cases we required, that is when R is bounded by concentric rectangles or by concentric ellipses. I present here an elementary proof for the general case where R is an arbitrary bounded ring.

3. The proof depends upon an equicontinuity property for the harmonic functions u defined in (2). (See also p. 386 in [5].)

Lemma 1. *Let $0 < a < b$, let R be a bounded ring and let the diameter of B_0 exceed b. Let u be the harmonic function defined in (2) and extend u so that u is 0 on C_0 and 1 on C_1* [4]*. Then*

$$|u(z_1) - u(z_2)| \leq c \tag{6}$$

whenever $|z_1 - z_2| \leq a$, *where*

$$c = 2\pi\left(\operatorname{mod} R \cdot \log\frac{b}{a}\right)^{-\frac{1}{2}}. \tag{7}$$

Proof. Fix z_1 and z_2 so that $|z_1 - z_2| \leq a$. Since $0 \leq u(z_1)$, $u(z_2) \leq 1$, we may clearly assume that $c < 1$ for otherwise (6) follows trivially.

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[2]) This follows, for example, from the proof of Theorem 4.3 in [3].

[3]) The restriction that R be bounded can be omitted by redefining the notion of uniform convergence in terms of the metric on the RIEMANN sphere.

[4]) The fact that R is bounded implies that B_1 is also non-degenerate.

Now let $\gamma = \gamma(r)$ denote the circle with center $\frac{1}{2}(z_1 + z_2)$ and radius r, $r > 0$. By the SCHWARZ inequality

$$\Big(\int_{\gamma \cap R} |\nabla u| ds\Big)^2 \leq 2\pi r \int_{\gamma \cap R} |\nabla u|^2 ds$$

and, with FUBINI's theorem, we obtain

$$\int_{\frac{1}{2}a}^{\frac{1}{2}b} \Big(\int_{\gamma \cap R} |\nabla u| ds\Big)^2 \frac{dr}{r} \leq 2\pi \iint_R |\nabla u|^2 d\sigma = \frac{(2\pi)^2}{\operatorname{mod} R} .$$

Hence there exists an r, $\frac{1}{2} a \leq r \leq \frac{1}{2} b$, for which

$$\int_{\gamma \cap R} |\nabla u| ds \leq c .$$

Suppose that the corresponding circle γ lies in R. Since the diameter of γ is less than that of B_0, R contains Δ, the disk bounded by γ. Thus u satisfies the maximum principle in Δ and

$$|u(z_1) - u(z_2)| \leq \operatorname*{osc}_{\gamma} u \leq \int_{\gamma} |\nabla u| ds \leq c ,$$

as desired.

Next suppose that γ meets both R and C_0. Then $\gamma \cap R$ is the union of open arcs α and, since

$$\operatorname*{osc}_{\alpha} u \leq \int_{\gamma \cap R} |\nabla u| ds \leq c < 1 ,$$

it follows that γ does not meet C_1. Thus each α has its endpoints in B_0 and hence

$$\sup_{\gamma} u \leq c .$$

We see $\partial(\Delta \cap R) \subset \gamma \cup B_0$ and, since u satisfies the maximum principle in $\Delta \cap R$ and vanishes on $\Delta \cap C_0$ and B_0,

$$|u(z_1) - u(z_2)| \leq \sup_{\Delta \cap R} u = \sup_{\gamma} u \leq c .$$

A slight modification of the above argument handles the case where γ meets R and C_1.

Finally suppose that γ is contained in a component of the complement of R. The fact that the diameter of γ is less than that of B_0 implies Δ lies in the same component. Hence $u(z_1) = u(z_2)$ and the proof is complete.

4. We now use Lemma 1 to establish the following result.

Lemma 2. *Let $0 < a < b$ and let R and R' be bounded rings with boundary components B_0, B_1 and B_0', B_1', respectively. Let all points of B_0' and B_1' lie within distance a of B_0 and B_1, respectively, and let the diameter of B_0 exceed b. Then*

$$\operatorname{mod} R' \geq (1 - 4c) \operatorname{mod} R \,, \tag{8}$$

where c is as in (7).

Proof. We may assume that $c < \frac{1}{4}$ for otherwise (8) follows trivially. Now let u be the harmonic function defined in (2) and extend u so that u is 0 on C_0 and 1 on C_1. Then define v as follows:

$$v = \begin{cases} 0 & \text{if } \; u < c \,, \\ \dfrac{u - c}{1 - 2c} & \text{if } \; c \leq u \leq 1 - c \,, \\ 1 & \text{if } \; 1 - c < u \,. \end{cases}$$

The set where $c \leq u \leq 1 - c$ is a compact subset of R. Hence u satisfies a LIPSCHITZ condition at each point of this set. From this it follows that v satisfies a LIPSCHITZ condition at all points of the ring R'. Next let z' be a point of B_0'. By hypothesis z' lies within distance a of some point z of B_0 and Lemma 1 yields $u(z') = u(z') - u(z) \leq c$. Thus v is 0 on B_0'. Arguing similarly we see that v is 1 on B_1'. From (4) it follows that

$$\frac{2\pi}{\operatorname{mod} R'} \leq \iint_{R'} |\nabla v|^2 \, d\sigma \leq (1 - 2c)^{-2} \iint_{R} |\nabla u|^2 \, d\sigma = (1 - 2c)^{-2} \frac{2\pi}{\operatorname{mod} R} \,,$$

and, since $1 - 4c \leq (1 - 2c)^2$, we obtain (8) as desired.

5. Proof of the theorem. Suppose first that B_0 is non-degenerate and let $B_{0,\,n}$ and $B_{1,\,n}$ denote the boundary components of R_n. Then $\operatorname{mod} R < \infty$ and we can find a sequence $\{a_n\}$, $a_n > 0$, and a number $b > 0$ with the following properties. All points of $B_{0,\,n}$ and $B_{1,\,n}$ lie within a_n of B_0 and B_1 respectively, the diameter of B_0 exceeds b, and $a_n \to 0$. Lemma 2 then yields $\operatorname{mod} R_n \geq (1 - 4c_n) \operatorname{mod} R$ for large n, where

$$c_n = 2\pi \left(\operatorname{mod} R \cdot \log \frac{b}{a_n} \right)^{-\frac{1}{2}} ,$$

and, since $c_n \to 0$, we obtain

$$\lim_{n \to \infty} \inf \operatorname{mod} R_n \geq \operatorname{mod} R \,. \tag{9}$$

The uniform convergence next implies the existence of a second sequence $\{a_n'\}$, $a_n' > 0$, such that all points of B_0 and B_1 lie within a_n' of $B_{0,\,n}$ and $B_{1,\,n}$ respectively and $a_n' \to 0$. Since the diameter of $B_{0,\,n}$ eventually exceeds b, $\operatorname{mod} R \geq (1 - 4c_n') \operatorname{mod} R_n$ for large n, where

$$c_n' = 2\pi \left(\operatorname{mod} R_n \cdot \log \frac{b}{a_n'} \right)^{-\frac{1}{2}} .$$

Inequality (9) implies that $c_n' \to 0$. Hence

$$\lim_{n\to\infty} \sup \operatorname{mod} R_n \leq \operatorname{mod} R$$

and we obtain (5) for the case where B_0 is non-degenerate.

Now suppose that B_0 reduces to a point P, let r_n be the radius of the smallest closed disk with center at P and containing $B_{0,n}$, and let d_n be the distance from P to $B_{1,n}$. Then $\operatorname{mod} R = \infty$ and

$$\operatorname{mod} R_n \geq \log \frac{d_n}{r_n} \to \infty \ .$$

Hence we again obtain (5) and this completes the proof of the theorem.

6. Remarks. This result can also be proved directly using theorems on conformal mapping. On the other hand the above method can be used to establish the same continuity property for the moduli of rings in space [1].

Finally examining the above argument we can split up the hypotheses and conclusions for the theorem as follows.

If, for each $\varepsilon > 0$, *there exists an* N *such that* $n > N$ *implies the points of each component of* ∂R_n *lie within distance* ε *of the corresponding component of* ∂R, *then*

$$\operatorname{mod} R \leq \lim_{n\to\infty} \inf \operatorname{mod} R_n \ .$$

If, for each $\varepsilon > 0$, *there exists an* N *such that* $n > N$ *implies the points of each component of* ∂R *lie within distance* ε *of the corresponding component of* ∂R_n, *then*

$$\operatorname{mod} R \geq \lim_{n\to\infty} \sup \operatorname{mod} R_n \ .$$

It is clear that neither of the above inequalities can be replaced by equality.

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