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Locally symmetric homogeneous spaces

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1. Introduction and summary

We will give a necessary and sufficient condition for a locally symmetric RIEMANNian manifold to be globally homogeneous, extending the criterion ([20, Th.6], [21, Th.4]) for manifolds of constant curvature. This involves a study of the relations between symmetry, homogeneity, and a certain condition on the fundamental group.

Let Γ be a properly discontinuous group of isometries acting freely on a connected simply connected RIEMANNian symmetric manifold M, and consider the conditions:

(1) M/Γ is a RIEMANNian symmetric manifold.
(2) M/Γ is a RIEMANNian homogeneous manifold.
(3) Γ is a group of CLIFFORD translations (isometries of constant displacement) of M.

Our main result (Theorem 6.1) is that (2) *is equivalent to* (3). We also prove (Theorem 6.2) that (2) *implies* (1) *if, in É. CARTAN's decomposition of* M *as a product of EUCLIDEAN space and some irreducible symmetric spaces, none of the compact irreducible factors is a LIE group, an odd dimensional sphere, a complex projective space of odd complex dimension* > 1, $\mathbf{SU}(2n)/\mathbf{Sp}(n)$ *with* $n \geqq 2$, *or* $\mathbf{SO}(4n+2)/\mathbf{U}(2n+1)$ *with* $n \geqq 1$. It is known [20, Th.5] that (2) need not imply (1) if an irreducible factor of M is an odd dimensional sphere; the same is seen if a factor is a compact LIE group H by taking Γ to be the left translations by elements of a finite non-central subgroup of H; examples are given in § 5.5 to show that the other restrictions are necessary. These theorems are complemented by the result (Corollary 4.5.2 and [20, Th.4]) that *if* M *is a* LIE *group in 2-sided-invariant metric, then* (2) *is equivalent to* Γ *being conjugate, in the full group of isometries of* M, *to the left translations by the elements of a discrete subgroup* B *of* M, *and* (Theorem 4.6.3) (1) *is equivalent to* B *being in the center of* M. An interesting consequence of these theorems and their proof is (Theorem 6.4) that *the fundamental group of a* RIEMANNian *symmetric space is abelian.*

It is well known that (1) implies (2), and easily proved [20, Th.2] that (2) implies (3). Thus our results are obtained by studying the CLIFFORD translations of M. In § 3, that study is reduced to the case where M is EUCLIDEAN or irreducible. The EUCLIDEAN case is known [20, Th.4], and results of J. TITS

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allow us to dispose of the noncompact irreducible case. Then we need only determine the groups of CLIFFORD translations of compact LIE groups (this is done in § 4) and compact symmetric spaces with simple groups of isometries (this is known for spheres [21], and is done in § 5 for the other cases). It turns out that the most difficult case is when M is an odd dimensional sphere; this was treated in our work [21] on VINCENT's Conjecture.

In § 2 we establish definitions and notation, review some material on covering manifolds and symmetric spaces, and give a short exposition of É. CARTAN's method of determining the full group of isometries of a symmetric space. In § 6 we combine results from §§ 3–5 and obtain our main theorems.

I wish to thank Professors J. TITS, H. SAMELSON and C.T.C. WALL for helpful conversations. In particular, J. TITS showed me the results mentioned here in §§ 3.2.1–3.2.4, H. SAMELSON improved my proof of Theorem 6.4, and C.T.C. WALL provided the statement and proof of Lemma 5.5.10.

I am especially indebted to Professor H. FREUDENTHAL for showing me that diag. $\{a'; a, \ldots, a\}$ is a CLIFFORD translation of **SU** (2m) / **Sp** (m), and for confirming some of the results in § 5 by discovering an alternative method (to appear in Mathematische Annalen) for finding the CLIFFORD translations of certain compact RIEMANNian symmetric spaces.

2. Preliminaries on groups and symmetric spaces

2.1. Definitions and notation

2.1.1. We will assume as known: (1) the definition of a RIEMANNian manifold and elementary facts about the RIEMANNian metric, geodesics, completeness, isometries and the exponential map, (2) the notion of a LIE group and LIE algebra, and (3) the idea of a covering space.

Let M be a RIEMANNian manifold. The group of all isometries of M onto itself forms a LIE group $\mathbf{I}(M)$, the *full group of isometries* of M, whose identity component $\mathbf{I}_0(M)$ is called the *connected group of isometries* of M. M is *homogeneous* if $\mathbf{I}(M)$ is transitive on the points of M. Let $p \in M$. An element $s_p \in \mathbf{I}(M)$ of order 2 with p as isolated fixed point is called the (*global*) *symmetry* of M at p; if M is connected and s_p exists, then s_p is easily seen unique because it induces $-I\,(I = \text{identity})$ on the tangentspace M_p. M is (*globally*) *symmetric* if it has a symmetry at every point. If M is symmetric, then every component of M is homogeneous, and, if M is irreducible (see § 2.3.2), $\mathbf{I}_0(M)$ is the identity component of the subgroup of $\mathbf{I}(M)$ generated by the symmetries. M is complete if every component is homogeneous.

2.1.2. The *geodesic symmetry* to M at p is the map

$$\mathrm{Exp}_p(X) \to \mathrm{Exp}_p(-X) \quad (X \in M_p \text{ with } \mathrm{Exp}_p(X), \mathrm{Exp}_p(-X) \text{ defined}).$$

M is *locally symmetric* if, given $q \in M$, there is a neighborhood U of q such that the geodesic symmetry to M at q induces an isometry of U onto itself. If M is symmetric then it is locally symmetric, and M is locally symmetric if and only if the curvature tensor is invariant under parallel translation.

2.1.3. A CLIFFORD *translation* of a metric space X is an isometry $f: X \to X$ such that the distance $\varrho(x, f(x))$ is the same for every $x \in X$; thus a CLIFFORD translation is an isometry of constant displacement. Note that a CLIFFORD translation is the identity if it has a fixed point.

If an isometry $g: X \to X$ centralizes a transitive group H of isometries of X, then g is a CLIFFORD translation of X. For, given x and y in X, we choose $h \in H$ with $hx = y$, and have $\varrho(x, gx) = \varrho(hx, hgx) = \varrho(hx, ghx) = \varrho(y, gy)$.

2.1.4. The compact classical groups are the unitary group $\mathbf{U}(n)$ in n complex variables, the special unitary group $\mathbf{SU}(n)$ consisting of elements of determinant 1 in $\mathbf{U}(n)$, the orthogonal group $\mathbf{O}(n)$ in n real variables, the special orthogonal group $\mathbf{SO}(n)$ consisting of elements of determinant 1 in $\mathbf{O}(n)$, and the symplectic group $\mathbf{Sp}(n)$ in n quaternion variables, which is the quaternionic analogue of $\mathbf{U}(n)$. $\mathbf{SU}(n)$ is of type A_{n-1}, $\mathbf{SO}(2n+1)$ of type B_n, $\mathbf{Sp}(n)$ of type C_n, and $\mathbf{SO}(2n)$ of type D_n in the CARTAN-KILLING classification. We will use boldface to denote the compact connected simply connected group of a given CARTAN-KILLING type. Thus $\mathbf{SU}(n+1) = \mathbf{A}_n$, $\mathbf{Sp}(n) = \mathbf{C}_n$, and $\mathbf{E}_6$ is the compact connected simply connected group of type E_6.

2.2. RIEMANNian coverings

A RIEMANNian *covering* is a covering $\pi: M \to N$ of connected RIEMANNian manifolds, where π is a local isometry. The group Γ of deck transformations of the covering (homeomorphisms $\gamma: M \to M$ with $\pi \cdot \gamma = \pi$) is then a subgroup of $\mathbf{I}(M)$. If N is homogeneous, then the centralizer of Γ in $\mathbf{I}_0(M)$ is transitive on the points of M, and every $\gamma \in \Gamma$ is a CLIFFORD translation of M [20]; if the covering is normal and the centralizer of Γ in $\mathbf{I}(M)$ is transitive on (the points of) M, then that centralizer induces a transitive group of isometries of N, so N is homogeneous. If N is symmetric, then M is symmetric and every symmetry of M normalizes Γ, whence products of symmetries centralize Γ, for every symmetry of N lifts to a π-fibre preserving symmetry of M; if M is symmetric, the covering is normal and products of symmetries centralize Γ, then N is symmetric. If M is symmetric, then N is complete and locally symmetric. If N is complete and locally symmetric and M is simply connected, then M is symmetric [3].

2.3. É. CARTAN's classification of symmetric spaces

2.3.1. CARTAN decomposition. Let M be a connected simply connected RIEMANNian symmetric manifold, $p \in M$, $G = \mathbf{I}_0(M)$, K the isotropy subgroup of G at p, and σ the involutive automorphism of the LIE algebra $\mathfrak{G}$ of G induced from conjugation of $\mathbf{I}(M)$ by the symmetry s_p. This gives the CARTAN *decomposition* $\mathfrak{G} = \mathfrak{K} + \mathfrak{P}$ where $\mathfrak{P}$ is the eigenspace of -1 for σ and $\mathfrak{K}$, the LIE algebra of K, is the eigenspace of $+1$; we have the well-known $[\mathfrak{K}, \mathfrak{K}] \subset \mathfrak{K}$, $[\mathfrak{K}, \mathfrak{P}] \subset \mathfrak{P}$ and $[\mathfrak{P}, \mathfrak{P}] \subset \mathfrak{K}$, which is just another way of expressing the CARTAN decomposition and specifying σ.

The adjoint action of K on $\mathfrak{G}$ induces an action of K on $\mathfrak{P}$; K also acts on the tangentspace M_p as linear isotropy subgroup of G, and the identification $gK \to g(p)$ of the coset space G/K with M gives a K-equivariant identification of $\mathfrak{P}$ with M_p. Thus the RIEMANNian metric on M can be viewed as an $Ad(K)$-invariant positive definitive bilinear form on $\mathfrak{P}$.

2.3.2. Product structure. M is EUCLIDEAN if $[\mathfrak{P}, \mathfrak{P}] = 0$; then M is isometric to a EUCLIDEAN space. M is *irreducible* if K acts irreducibly on $\mathfrak{P}$; then $[\mathfrak{P}, \mathfrak{P}] = \mathfrak{K}$ so $\mathfrak{G}$ is semisimple. In any case, M is isometric to a product $M_0 \times M_1 \times \ldots \times M_t$ of RIEMANNian symmetric manifolds, where M_0 is EUCLIDEAN and the $M_j (j > 0)$ are irreducible. This decomposition is due to É. CARTAN [8, § 1], will be called CARTAN's *symmetric-space decomposition* of M, and is a special case of G. DE RAHM's decomposition [17] of a complete connected simply connected RIEMANNian manifold under the holonomy group. $\mathbf{I}_0(M) = \mathbf{I}_0(M_0) \times \ldots \times \mathbf{I}_0(M_t)$, and $\mathbf{I}(M)$ is generated by $\mathbf{I}(M_0) \times \ldots \times \mathbf{I}(M_t)$ and permutations of isometric factors M_j.

2.3.3. Duality. M is determined by $(\mathfrak{G}, \sigma, B)$ where B is the positive definite K-invariant bilinear form on $\mathfrak{P}$ determined by the inner product on M_p. $\mathfrak{G}^* = \mathfrak{K} + \mathfrak{P}^*$, $\mathfrak{P}^* = \sqrt{-1}\,\mathfrak{P}$, is a real subalgebra of the complexification of $\mathfrak{G}$; σ induces an involutive automorphism σ^* of $\mathfrak{G}^*$; $B^*(\sqrt{-1}\,X, \sqrt{-1}\,Y) = B(X, Y)$ is a positive definite K-invariant form on $\mathfrak{P}^*$. The *dual symmetric space* to M is the connected simply connected RIEMANNian manifold M^* determined by $(\mathfrak{G}^*, \sigma^*, B^*)$.

2.3.4. Types of irreducible spaces. Now assume M irreducible. There are 4 possibilities:

1. $\mathfrak{G} = \mathfrak{G}' \oplus \mathfrak{G}'$ with $\mathfrak{G}'$ compact simple and $\sigma(X, Y) = (Y, X)$.
2. $\mathfrak{G}$ is compact and simple.
3. $\mathfrak{G}$ is complex simple and σ is conjugation with respect to a compact real form $\mathfrak{K}$.

4. $\mathfrak{G}$ is noncompact simple with simple complexification, and $\mathfrak{K}$ is a maximal compact subalgebra.

Under the duality, type (1) corresponds to type (3), and type (2) corresponds to type (4). By irreducibility of K on $\mathfrak{P}$, the metric is essentially induced by the KILLING form of $\mathfrak{G}$.

2.3.5. Classification of irreducible spaces. In view of the duality, one need only list the irreducible spaces of types (1) and (2), i.e., the compact irreducible M. The spaces of type (1) are just the compact simple LIE groups, and will be described more fully in § 4.1. É. CARTAN has treated them in detail [10]. The spaces of type (2) were first classified by É. CARTAN by considering possible holonomy groups [8], then by using the duality and his classification [7] of the real simple LIE groups [9], and finally by using involutions of LIE algebras [12]. The classification is readily accessible from F. GANTMACHER's account [16] of involutions of LIE algebras, and CARTAN's list [9, §§ 58–68] is well known.

2.4. The full group of isometries of a symmetric space

2.4.1. Let M be a connected simply connected irreducible RIEMANNian symmetric manifold. É. CARTAN has given a technique for calculating $\mathbf{I}(M)$ from (we retain the notation of § 2.3) $G = \mathbf{I}_0(M)$ and an isotropy subgroup K of G; see [11] for the general theory and the cases where M is of type (2) or (4), and see [10] and § 4.1.2 if M is of type (3) or (1). Nevertheless, this material is somewhat inaccessible, so we will give a short exposition for the convenience of the reader.

2.4.2. Choose a CARTAN decomposition $\mathfrak{G} = \mathfrak{K} + \mathfrak{P}$. K acts irreducibly on $\mathfrak{P}$, for M was assumed irreducible, whence SCHUR's Lemma says that the centralizer F of $Ad(K)|_{\mathfrak{P}}$ in the algebra of linear endomorphisms of $\mathfrak{P}$ is a real division algebra. Thus F is one of the three fields $\mathbf{R}$ (real), $\mathbf{C}$ (complex) or $\mathbf{H}$ (quaternion). Actually, $\mathbf{H}$ is excluded:

2.4.3. Lemma. (1) $F = \mathbf{R}$ *if and only if* K *is semisimple.* (2) $F \neq \mathbf{H}$. (3) *These are equivalent:* (a) K *has a* 1*-dimensional center,* (b) K *is not semisimple,* (c) M *has a* G*-invariant* KÄHLER *structure* (*and is thus a so called "Hermitian symmetric space"*) *which induces the original* RIEMANNian *structure, and* (d) $F = \mathbf{C}$.

Proof. The results are essentially due to CARTAN [11, 13]; part of the treatment is taken from CARTAN [11] and BOREL [3].

We identify K with $Ad(K)|_{\mathfrak{P}}$, and let Z denote the center of K. As K is compact, Z is a subgroup of the multiplicative group F' of unimodular elements of F. Thus $F = \mathbf{R}$ or $\mathbf{H}$ implies that Z is finite, whence K is semisimple.

Now suppose that F has a complex subfield, i.e., that F' has an element J with $J^2 = -I$. We define real subspaces of the complexified algebra $\mathfrak{G}^C$ by $\mathfrak{V} = (I + \sqrt{-1}\,J)\,\mathfrak{P}$ and $\overline{\mathfrak{V}} = (I - \sqrt{-1}\,J)\,\mathfrak{P}$, whence $\mathfrak{L} = \mathfrak{K}^C + \mathfrak{V}$ gives $\overline{\mathfrak{L}} = \mathfrak{K}^C + \overline{\mathfrak{V}}$. One easily checks that $[\mathfrak{K}, \mathfrak{P}] \subset \mathfrak{P}$ implies $[\mathfrak{K}^C, \mathfrak{V}] \subset \mathfrak{V}$ and $[\mathfrak{K}^C, \overline{\mathfrak{V}}] \subset \overline{\mathfrak{V}}$, whence $[\mathfrak{V}, \mathfrak{V}] \subset [\mathfrak{P}^C, \mathfrak{P}^C) \subset \mathfrak{K}^C \subset \mathfrak{L}$, $[\overline{\mathfrak{V}}, \overline{\mathfrak{V}}] \subset [\mathfrak{P}^C, \mathfrak{P}^C] \subset \mathfrak{K}^C \subset \overline{\mathfrak{L}}$, and $[\mathfrak{K}^C, \mathfrak{K}^C] \subset \mathfrak{K}^C = \mathfrak{L} \cap \overline{\mathfrak{L}}$, show that $\mathfrak{L}$ and $\overline{\mathfrak{L}}$ are complex conjugate real subalgebras of $\mathfrak{G}^C$ with $\mathfrak{L} + \overline{\mathfrak{L}} = \mathfrak{G}^C$ and $\mathfrak{L} \cap \overline{\mathfrak{L}} = \mathfrak{K}^C$. This is precisely A. FROLICHER's criterion [14, § 20] that J define a G-invariant complex structure on $M = G/K$. This G-invariant complex structure and our original G-invariant RIEMANNian metric give us a G-invariant HERMITIAN metric on M which induces the RIEMANNian metric. K is the holonomy group of M because M is connected, simply connected, and irreducible RIEMANNian symmetric; thus the complex structure is invariant under parallel translation, so the HERMITIAN metric is KÄHLER. This implies (replace M by its dual if M is noncompact, and thus assume M compact) $\mathbf{H}^2(M; \mathbf{R}) \neq 0$ because of the fundamental 2-form of the KÄHLER metric, whence $\pi_2(M)$ is infinite. The homotopy sequence

$$0 = \pi_2(G) \to \pi_2(M) \to \pi_1(K) \to \pi_1(G) = \text{finite}$$

now shows that K is not semisimple, so Z is infinite; thus F' is infinite and so $F = \mathbf{C}$. This shows the equivalence of the conditions of (3) and proves $F \neq \mathbf{H}$. (1) follows. *Q.E.D.*

2.4.4. Let K' be the isotropy subgroup of $\mathbf{I}(M)$ at the point at which K is the isotropy subgroup of $G = \mathbf{I}_0(M)$; our conditions on M show that K is the identity component of K', and $\mathbf{I}(M) = G \cdot K'$. Thus, in order to find $\mathbf{I}(M)$ from G and K, it suffices to find K'.

We identify K' with $Ad(K')|_{\mathfrak{P}}$, and observe that the symmetry $s \in K'$ is identified with the endomorphism $-I$ of $\mathfrak{P}$. We know that $s \in K$ if and only if K has a central element of order 2, for s centralizes K, and the irreducibility shows that s is the only involutive element of K' which centralizes K. Let K'' denote the subgroup $K \cup s \cdot K$ of K'; to find K', it suffices to find $\{k_1 = 1, k_2, \ldots, k_m\} \subset K'$ such that K' is the disjoint union of the $k_j \cdot K''$; we then define $G'' = G \cup s \cdot G$ and have $\mathbf{I}(M)$ as the disjoint union of the $k_j \cdot G''$. CARTAN's main idea for finding the k_j is:

2.4.5. Lemma. *Each* $k_j (j > 1)$ *induces an outer automorphism of* K *which is induced by an automorphism of* G. *For distinct* $k_j (j \geqq 1)$, *these automorphism of* K *do not differ by an inner automorphism of* K.

Remark. This gives a sharp bound on the number of components of G.

Proof. Note that, retaining the notation of § 2.4.2, $F' \subset K''$. For if $F = \mathbf{R}$, then $F' = \{I, -I\} = \{1, s\}$. If $F = \mathbf{C}$, then Lemma 2.4.3 shows that the center of K' contains a circle, whence $F' = \mathbf{C}' \subset K = K''$. Lemma 2.4.3 shows F to be $\mathbf{R}$ or $\mathbf{C}$. Two elements of K' are identical if they give the same linear transformation of $\mathfrak{P}$, for two isometries of a connected manifold are identical if they have the same tangent map at a point. Thus $k \in K'$ lies in K'' if and only if it induces an inner automorphism of K. The Lemma now follows. *Q.E.D.*

2.4.6. CARTAN's use of this Lemma to determine $\mathbf{I}(M)$ [11] does not always give results which are sufficiently explicit for our purposes. Thus we will occasionally have to repeat some of his determinations in order to obtain the k_j explicitly as isometries of M. This will be done in § 5, as it is needed.

3. Reduction to the case of a compact irreducible symmetric space

Let Γ be the group of deck transformations of a RIEMANNian covering $\pi: M \to N$ where M is complete and simply connected. We consider the DE RAHM decomposition $M = M_0 \times M_1 \times \ldots \times M_t$, where M_0 is EUCLIDEAN and $M_j (j > 0)$ is non-EUCLIDEAN and irreducible, and recall that $\mathbf{I}(M)$ is generated by $\mathbf{I}(M_0) \times \ldots \times \mathbf{I}(M_t)$ and permutations of isometric M_j. We wish to prove that Γ is, under certain conditions on N, a subgroup of $\mathbf{I}(M_0) \times \ldots \times \mathbf{I}(M_t)$, and thus reduce our considerations to the case where M is EUCLIDEAN or irreducible; this is done in § 3.1. In § 3.2, we consider the case where M is noncompact and irreducible. The final reduction is made in § 3.3.

3.1. Splitting of CLIFFORD translations

3.1.1. Let f be a self-homeomorphism of a topological product space

$$X = X_1 \times \ldots \times X_n . \tag{1}$$

We say that f *preserves the product structure* (1) if we have a permutation i of $\{1, \ldots, n\}$ and homeomorphisms $f_j : X_{i_j} \to X_j$ such that $f(x_1, \ldots, x_n\} = (f_1(x_{i_1}), \ldots, f_n(x_{i_n}))$. We say that f *decomposes under* (1) if it preserves the product structure (1) and the permutation i is trivial.

3.1.2. Theorem. *Let $(X, \varrho) = (X_1, \varrho_1) \times \ldots \times (X_n, \varrho_n)$ be a metric product of metric spaces where each X_j has at least two points, and let f be a* CLIFFORD *translation of (X, ϱ) which preserves the product structure* (1). *Then f decomposes into $f_1 \times \ldots \times f_n$ under* (1) *where f_j is a* CLIFFORD *translation of (X_j, ϱ_j).*

Proof. $f(x_1, \ldots, x_n) = (f_1(x_{i_1}), \ldots, f_n(x_{i_n}))$. We need only show $i = 1$, for then f decomposes into $f_1 \times \ldots \times f_n$ under (1), and we see that f_j is a CLIFFORD translation of (X_j, ϱ_j) by holding all x_k $(k \neq j)$ fixed and varying x_j over X_j.

The Theorem is trivial if $n = 1$; we proceed by induction on n. Writing i as a product of disjoint cycles, we obtain a decomposition

$$X = Y_1 \times \ldots \times Y_s \tag{2}$$

such that each Y_j is a product of some of the X_k and f decomposes under (2). So by our induction hypothesis we need only consider the case

$$i = (1, 2, \ldots, n).$$

Now suppose $i = (1, 2, \ldots, n)$. Each f_j is an isometry, so we may identify (X_{j+1}, ϱ_{j+1}) and (X_j, ϱ_j) under f_j, $1 \leq j < n$, and assume that $f(x_1 \ldots, x_n) = (f_1(x_n), x_1, \ldots x_{n-1})$. As f is CLIFFORD,

$$\varrho(x, f(x))^2 = \varrho_1(x_1, f_1(x_n))^2 + \varrho_1(x_2, x_1)^2 + \ldots + \varrho_1(x_n, x_{n-1})^2$$

is some constant c, for $x \in X$. The choice $x = (x_1, x_1, \ldots, x_1)$ would give us

$$c = \varrho_1(x_1, f_1(x_1))^2 + \varrho_1(x_1, x_1)^2 + \ldots + \varrho_1(x_1, x_1)^2,$$

whence $c = \varrho_1(x_1, f_1(x_1))^2$. As X_j has at least two points, $n > 2$ would give the possibility of $x_2 \neq x_1 = x_n$, whence

$$c = \varrho_1(x_1, f_1(x_1))^2 + \varrho_1(x_2, x_1)^2 + \ldots + \varrho_1(x_n, x_{n-1})^2 > c;$$

thus $n \leq 2$. But $n = 2$ and $x = (x_1, f_1(x_1))$ would give

$$c = \varrho_1(f_1(f_1(x_1)), f_1(x_1))^2 + \varrho_1(f_1(x_1), x_1)^2 = 2c,$$

whence $n = 1$. *Q.E.D.*

3.1.3. Corollary. *Let $M = M_0 \times M_1 \times \ldots \times M_t$ be the* DE RAHM *decomposition of a complete connected simply connected* RIEMANNIAN *manifold M, and let γ be a* CLIFFORD *translation of M. Then $\gamma = \gamma_0 \times \gamma_1 \times \ldots \times \gamma_t$ where γ_j is a* CLIFFORD *translation of M_j.*

For γ is an isometry of M, so it preserves the product structure of the DE RAHM decomposition.

3.1.4. Corollary. *Let $M = M_0 \times M_1 \times \ldots \times M_t$ be* CARTAN*'s symmetric-space decomposition of a connected simply connected* RIEMANN*ian symmetric manifold M, and let γ be a* CLIFFORD *translation of M. Then*

$$\gamma = \gamma_0 \times \gamma_1 \times \ldots \times \gamma_t$$

where γ_j is a CLIFFORD *translation of M_j.*

For M is complete, and CARTAN's symmetric-space decomposition of M coincides with the DE RHAM decomposition of M.

3.2. CLIFFORD translations of irreducible noncompact symmetric spaces

It is known [20, Th. 3] that a CLIFFORD translation of hyperbolic space is necessarily trivial,because distinct geodesics diverge. We will describe J. TITS' extension of that result to strictly non-EUCLIDEAN (no EUCLIDEAN factor) strictly noncompact (no compact factor) RIEMANNian symmetric spaces.

3.2.1. A *bounded isometry* $f: M \to M$ of a metric space (M, ϱ) is an isometry f such that the displacement function $d_f(x) = \varrho(x, f(x))$ is bounded on M. A *bounded automorphism* $\alpha: G \to G$ of a topological group G is an automorphism α such that G has a compact subset K with $\alpha(g)\cdot g^{-1} \in K$ for every $g \in G$.

3.2.2. Lemma (J. TITS). *Let f be an isometry of a connected* RIEMANN*ian homogeneous manifold M, and let α be the induced automorphism of $\mathbf{I}_0(M)$. Then f is a bounded isometry of M if and only if α is a bounded automorphism of $\mathbf{I}_0(M)$.*

3.2.3. Lemma (J. TITS). *Let G be a connected semisimple* LIE *group without compact factor*[2]*), and let α be a bounded automorphism of G. Then α is the identity.*

3.2.4. Theorem (J. TITS). *Let M be a connected* RIEMANN*ian homogeneous manifold such that $\mathbf{I}_0(M)$ is a semisimple group without compact factor. Then $\mathbf{I}_0(M)$ centralizes every bounded isometry of M.*

3.2.5. Corollary. *Let M be a connected simply connected* RIEMANN*ian symmetric manifold, product of irreducible noncompact non-*EUCLIDEAN *symmetric spaces. Let Γ be a group of* CLIFFORD *translations of M. Then $\Gamma = \{1\}$.*

Proof. A CLIFFORD translation of M is a bounded isometry, and the assumptions on M imply that $\mathbf{I}_0(M)$ is a centerless semisimple LIE group without compact factor; it follows from Theorem 3.2.4 that $\Gamma \cap \mathbf{I}_0(M) = \{1\}$. Now a CLIFFORD translation $\neq 1$ has no fixed point, and É. CARTAN's famous

[2]) The universal covering group of G is a product of noncompact simple LIE groups.

argument [12, §16] shows that an isometry of M of finite order must have a fixed point; it follows that every element $\neq 1$ of Γ has infinite order. But $\Gamma \cap \mathbf{I}_0(M) = \{1\}$ and $\mathbf{I}(M)$ has only finitely many components; thus $\Gamma = \{1\}$. Q.E.D.

3.2.6. Corollary. *Let M be a connected simply connected RIEMANNian homogeneous manifold such that $\mathbf{I}_0(M)$ is a semisimple group without compact factor, and let Γ be the group of deck transformations of a RIEMANNian covering $\pi: M \to N$. Then N is RIEMANNian homogeneous if and only if Γ is a group of CLIFFORD translations of M.*

As π is a normal covering, this follows directly from Theorem 3.2.4 and from [20, Th. 1 and 2].

3.3. The final reduction

Theorem. *Let $\pi: M \to N$ be a RIEMANNian covering where M is a connected simply connected RIEMANNian symmetric manifold, let Γ be the group of deck transformations of the covering, and let $M = M_0 \times M_1 \times \ldots \times M_t$ be CARTAN's symmetric-space decomposition of M. Suppose that Γ is a group of CLIFFORD translations of M, so that* (Cor. 3.1.4) *every $\gamma \in \Gamma$ is of the form $\gamma_0 \times \gamma_1 \times \ldots \times \gamma_t$ where γ_j is a CLIFFORD translation of M_j, let Γ_j be the subgroup $\{\gamma_j : \gamma \in \Gamma\}$ of $\mathbf{I}(M_j)$, and let G_j be the identity component of the centralizer of Γ_j in $\mathbf{I}(M_j)$. Then N is homogeneous if and only if G_j is transitive on M_j whenever M_j is compact, and N is symmetric if and only if $G_j = \mathbf{I}_0(M_j)$ whenever M_j is compact.*

Proof. Let G be the identity component of the centralizer of Γ in $\mathbf{I}(M)$;

$$G = G_0 \times G_1 \times \ldots \times G_t .$$

Now Γ_0 is a group of CLIFFORD translations of the EUCLIEDAN space M_0, so Γ_0 is a group of ordinary translations of M_0 [20, Th. 4]; thus G_0 contains the full group of translations of M_0. Also, $G_j = \mathbf{I}_0(M_j)$ if $j > 0$ and M_j is noncompact, by Theorem 3.2.4 or by Corollary 3.2.5. Thus G is transitive on M if and only if G_j is transitive on M_j for each compact M_j. As N is homogeneous if and only if G is transitive on M [20. Th. 1], the first part of the Theorem is proven.

Let Γ' be the subgroup of $\mathbf{I}(M)$ generated by the Γ_j, and let $M' = M/\Gamma'$. If $G_j = \mathbf{I}_0(M_j)$ whenever M_j is compact, then our remarks above on G_j for M_j noncompact show that every symmetry of M normalizes Γ'; thus M' is symmetric, whence its covering manifold $N = M/\Gamma$ is symmetric. Conversely, if N is symmetric, then evey symmetry of M normalizes Γ, for every symmetry of N lifts to a symmetry of M; it follows that every symmetry of M_j normalizes Γ_j. The symmetries of M_j form a connected subset of $\mathbf{I}(M_j)$, that set being all gsg^{-1} with $g \in \mathbf{I}_0(M_j)$ where s is some fixed symmetry of M_j; it follows that Γ_j is centralized by the subgroup of $\mathbf{I}(M_j)$ generated by all products of two symmetries of M_j; this implies $G_j = \mathbf{I}_0(M_j)$ for $j > 0$, so $G_j = \mathbf{I}_0(M_j)$ whenever M_j is compact. *Q.E.D.*

4. Clifford translations of group spaces

4.1. Group spaces as symmetric spaces

4.1.1. A *2-sided-invariant metric* on a compact Lie group G is a Riemannian metric on the underlying manifold of G such that the group $\mathbf{T}(G)$ of all transformations $(h, k): g \to hgk^{-1}$ of G for $h, k \in G$ is a group of isometries of G, i.e., such that left translations and right translations are isometries. Given a 2-sided-invariant metric on G, the map $s: g \to g^{-1}$ is an isometry; thus G is a Riemannian symmetric space, the symmetry at $h \in G$ being given by $g \to hg^{-1}h$, and s being the symmetry at $1 \in G$.

4.1.2. For simplicity, we will now assume G to be connected. Then $\mathbf{I}_0(G) = \mathbf{T}(G)$. If G is semisimple, then $\mathbf{I}(G)$ is generated by $\mathbf{T}(G)$, those products actually defined on G of symmetries at the identity to the simple factors of G, and the outer automorphisms of G; in any case, $\mathbf{I}(G)$ is contained in the group generated by $\mathbf{T}(G)$, those products actually defined on G of symmetries at the identity to the simple factors and the connected center of G, and the outer automorphisms of G. Neither $\mathbf{I}(G)$ nor the Levi-Civita connection on G depend on the choice of 2-sided-invariant metric. The geodesics of G are the left and right translates of the 1-parameter subgroups.

4.1.3. If G is simply connected (hence semisimple), then Cartan's symmetric-space decomposition of G is just the decomposition of G into a product of simple Lie groups. In any case, we identify the Lie algebra $\mathfrak{G}$ with the tangentspace at $1 \in G$, and observe the one-to-one correspondence between 2-sided-invariant metrics on G and $Ad(G)$-invariant positive definite bilinear forms on $\mathfrak{G}$. Decomposing $\mathfrak{G}$ into a sum $\mathfrak{A} \oplus \mathfrak{G}_1 \oplus \ldots \oplus \mathfrak{G}_t$ of the center $\mathfrak{A}$ and semisimple ideals $\mathfrak{G}_j$, we note that the $Ad(G)$-invariant positive definite bilinear forms on $\mathfrak{G}$ are just the forms $B_0 \oplus B_1 \oplus \ldots \oplus B_t$ where B_0 is a positive definite bilinear form on $\mathfrak{A}$ and $B_j (j > 0)$ is a negative real multiple of the Killing form on $\mathfrak{G}_j$.

For a detailed analysis of these symmetric spaces, we refer to É. Cartan's memoir [10]. In the sequel, the 2-sided-invariant metric on G will not be mentioned explicitly.

4.2. CLIFFORD translations in $\mathbf{T}(G)$

4.2.1. Lemma. *Let* $(u, v) \in \mathbf{T}(G)$ *be a* CLIFFORD *translation of* G. *Then* u *commutes with every conjugate of* v.

Proof. Let ϱ be the distance function on G; given $h \in G$, we have

$$\varrho(1, uv^{-1}) = \varrho(h, uhv^{-1}) = \varrho(1, u \cdot hv^{-1}h^{-1}).$$

Thus the distance from any conjugate of u to any conjugate of v is the constant $c = \varrho(1, uv^{-1}) = \varrho(u, v)$. Now take a minimizing geodesic $\varrho(t) = u \cdot \exp(tX)$ $(0 \leqq t \leqq 1, X \in \mathfrak{G})$ from u to hvh^{-1}. $\sigma'(0) = \sigma_* \left(\frac{d}{dt}\Big|_{t=0}\right)$ is orthogonal to $ad(G)u = \{gug^{-1} : g \in G\}$, for $ad(G)u$ lies in the sphere of radius c about hvh^{-1}. Orthogonal to some $ad(G)$ — orbit on G, σ is orthogonal to every $ad(G)$-orbit, whence $\sigma'(1)$ is tangent to the centralizer of hvh^{-1} [5]. That centralizer being totally geodesic in G, it contains $u = \sigma(0)$.

Q.E.D.

4.2.2. The LIE algebra $\mathfrak{G}$ is a sum $\mathfrak{A} \oplus \mathfrak{G}_1 \oplus \ldots \oplus \mathfrak{G}_m$ where the $\mathfrak{G}_j$ are simple ideals and $\mathfrak{A}$ is an abelian ideal. Let $A = \exp(\mathfrak{A})$ and $G_j = \exp(\mathfrak{G}_j)$. Every element $g \in G$ has expression $g_A g_1 \ldots g_m$ with $g_A \in A$, $g_j \in G_j$.

Lemma. *Let* $(u, v) \in \mathbf{T}(G)$ *be a* CLIFFORD *translation of* G. *Then, for each* j, *either* u_j *or* v_j *is central in* G_j.

Proof. Suppose that u_j is not central in G_j, so the centralizer Z of u_j in G_j is a proper subgroup of G_j. Lemma 4.2.1 says that Z contains $ad(G_j)v_j$, so the closed normal subgroup H of G_j, which is the closed subgroup of G_j generated by $ad(G_j)v_j$, lies in Z; thus H is a proper closed normal subgroup of G_j. As G_j is connected and simple, it follows that H is a discrete central subgroup of G_j. Thus v_j is central in G_j. Q.E.D.

4.2.3. Lemma. *Every element of* $s \cdot \mathbf{T}(G)$ *has a fixed point on* G.

Proof. Let $\gamma = s \cdot (u, v) \in s \cdot \mathbf{T}(G)$. We may assume $v = 1$ because

$$(1, v) \cdot \gamma \cdot (1, v)^{-1} = (1, v) \cdot s \cdot (u, v) \cdot (1, v^{-1}) = s \cdot (v, 1) \cdot (u, v) \cdot (1, v^{-1}) = s \cdot (vu, 1).$$

Now assume $\gamma = s \cdot (u, 1)$. Choose $h \in G$ with $h^2 = u^{-1}$, and notice that $\gamma(h) = s(uh) = h^{-1}u^{-1} = h$. Q.E.D.

4.2.4. Theorem. *Let* $G = A \times G_1 \times \ldots \times G_m$ *be a compact connected* LIE *group where* A *is a torus and the* G_j *are simple. Let* $\mathbf{I}'(G)$ *be the group of isometries of* G *generated by* $\mathbf{T}(G)$ *and the symmetries* s_A, s_j *to* A *and* G_j *at* *their identities. Let $\Gamma \subset \mathbf{I}'(G)$ be a group of* CLIFFORD *translations of G. Then Γ is conjugate in $\mathbf{I}'(G)$ to a group of left translations of G.*

Proof. $G = A \times G_1 \times \ldots \times G_m$ is essentially CARTAN's symmetric-space decomposition of G, and it follows from Theorem 3.1.2 or Corollary 3.1.4 that every $\gamma \in \Gamma$ is of the form $\gamma_A \times \gamma_1 \times \ldots \times \gamma_m$ where (write $\gamma_0 = \gamma_A$, $A = G_0$) γ_j is a CLIFFORD translation of G_j. Thus Lemma 4.2.3 shows $\Gamma \subset \mathbf{T}(G)$, so every $\gamma \in \Gamma$ is of the form

$$(a, 1) \times (u_1, v_1) \times \ldots \times (u_m, v_m) \text{ with } (u_j, v_j) \in \mathbf{T}(G_j)\,.$$

Now suppose, for some j with $1 \leqq j \leqq m$, that we have $\gamma, \gamma' \in \Gamma$ with neither u_j nor v_j' central in G_j; then u_j' and v_j are central in G_j by Lemma 4.2.2, whence neither $u_j' u_j$ nor $v_j' v_j$ is central in G_j. As $\gamma'\gamma$ is a CLIFFORD translation of G, this contradicts Lemma 4.2.2. In other words, given $1 \leqq j \leqq m$, either u_j is central in G_j for every $\gamma \in \Gamma$ or v_j is central in G_j for every $\gamma \in \Gamma$; in the former case, we may conjugate Γ by s_j, and assume every v_j is central in G_j. Thus, possibly after having been conjugated by some of the s_j, Γ is a group of left translations of G. *Q.E.D.*

We remark that the particular form of G was required only in order that the symmetries s_j act on G. As our primary interest is the case where G is simple, in view of Theorem 3.3, this restriction will cause no difficulty.

4.3. CLIFFORD translations and automorphisms

4.3.1. Lemma. *Let $\alpha : h \to h^\alpha$ be an automorphism of a compact connected* LIE *group G, let $g \in G$, and let $\gamma : h \to g h^\alpha$ be a* CLIFFORD *translation of G. Then $(ugu^{-\alpha})^\alpha = ugu^{-\alpha}$ for every $u \in G$.*

Note: $u^{-\alpha}$ denotes $(u^{-1})^\alpha$.

Proof. Let S be the sphere $\{v \in G : \varrho(1, v) = \varrho(1, g)\}$ about $1 \in G$, where ϱ is the distance function on G. Given $u \in G$, we have $\varrho(1, g) = \varrho(1, \gamma(1)) = \varrho(u, \gamma(u)) = \varrho(u, gu^\alpha) = \varrho(1, gu^\alpha u^{-1})$, so $gu^\alpha u^{-1} \in S$. Now choose $X \in \mathfrak{G}$ such that $\exp(tX)$, $0 \leqq t \leqq 1$, is a minimizing geodesic in G from 1 to g, so the left translation $(g, 1)_* X$ is orthogonal to S at g. Let $Y \in \mathfrak{G}$, define $u_t = \exp(tY)$ and $q_t = g u_t^\alpha u_t^{-1}$, and note that $q_t \in S$. Thus $Z = q_*\left(\dfrac{d}{dt}\bigg|_{t=0}\right) = (g, 1)_* (Y^\alpha - Y)$ is tangent to S at g. It follows that X is orthogonal to $(\alpha - 1)\,\mathfrak{G}$. As α is an orthogonal linear transformation of $\mathfrak{G}$, we conclude that $X^\alpha = X$, whence $g = \exp(X)$ tells us that $g^\alpha = g$.

Let $u \in G$ and note that $(u, 1) \cdot \gamma \cdot (u, 1)^{-1} = (u, 1) \cdot (g, 1) \cdot \alpha \cdot (u^{-1}, 1) = (u, 1) \cdot (g, 1) \cdot (u^{-\alpha}, 1) \cdot \alpha = (ugu^{-\alpha}, 1) \cdot \alpha$ is a CLIFFORD translation of G. Now the Lemma follows from the above paragraph. *Q.E.D.*

4.3.2. Lemma. *Retain the hypothesis of* Lemma 4.3.1 *and let* B *be the identity component of the centralizer of* g *in* G. *Then* $B^\alpha = B$, *and, if the restriction* $\alpha|_B$ *is an inner automorphism of* B, *then* α *is an inner automorphism of* G.

Proof. $B^\alpha = B$ because, by Lemma 4.3.1 with $u = 1$, we have $g^\alpha = g$. If α is inner on B, then it preserves each element of a maximal torus of B; then α preserves each element of a maximal torus of G, for B and G have the same rank. This implies [15, Th. 19] that α is inner on G. *Q.E.D.*

4.3.3. Theorem. *Let* α *be an automorphism of a compact connected* LIE *group* G, *let* $(g, h) \in \mathrm{T}(G)$ *and* $\gamma = (g, h)\cdot\alpha$, *and suppose that both* γ *and* γ^2 *are* CLIFFORD *translations of* G. *Then* α *is an inner automorphism of* G.

Proof. $\gamma = (g, h)\cdot\alpha = (gh^{-1}, 1)\cdot\{(h, h)\cdot\alpha\}$. (h, h) is an inner automorphism of G, so α is inner if and only if $(h, h)\cdot\alpha$ is inner. Therefore we may assume $h = 1$, i.e., $\gamma = (g, 1)\cdot\alpha$. This puts us in a position to use our Lemmas.

We assume the Theorem for groups whose dimension is less than dim. G. Thus, by induction hypothesis and Lemma 4.3.2, we need only consider the case where g is central in G. Thus the α-invariance of $ugu^{-\alpha}$ given by Lemma 4.3.1 can be expressed

$$u(u^{-\alpha})^2 = u^\alpha u^{-\alpha^2} u^{-\alpha} \tag{1}$$

which gives

$$(u^\alpha)^2 = uu^{\alpha^2} \text{ if } u \text{ commutes with } u^\alpha . \tag{2}$$

We write the LIE algebra $\mathfrak{G}$ as a sum $\mathfrak{A} \oplus \mathfrak{G}_1 \oplus \ldots \oplus \mathfrak{G}_m$ where $\mathfrak{A}$ is an abelian ideal and the $\mathfrak{G}_j$ are simple ideals. α preserves $\mathfrak{A}$ and $\mathfrak{G}_1 \oplus \ldots \oplus \mathfrak{G}_m$, and induces a permutation of the $\mathfrak{G}_j$, because it is an automorphism of $\mathfrak{G}$. Let $A = \exp(\mathfrak{A})$ and $G_j = \exp(\mathfrak{G}_j)$. If α sends $\mathfrak{G}_i$ to $\mathfrak{G}_j$ with $i \neq j$, we could choose $u \in G_i$ very near 1, and then $u(u^{-\alpha})^2$ would not lie in any G_k; but $u^\alpha u^{-\alpha^2} u^{-\alpha}$ would lie in some G_k, in violation of (1). Thus $A^\alpha = A$, and $(G_j)^\alpha = G_j$ for every j. Now every element $u \in G$ has expression $u = u_A u_1 \ldots u_m$ with $u_A \in A$ and $u_j \in G_j$. We pick some such decomposition $g = g_A g_1 \ldots g_m$ and observe that g_A and g_j are central in G, whence

$$\gamma(u_A u_1 \ldots u_m) = (g_A u_A^\alpha)\ (g_1 u_1^\alpha) \ldots (g_m u_m^\alpha) .$$

Thus γ is decomposed into $(\gamma_A, \gamma_1, \ldots, \gamma_m)$ with

$$\gamma(u_A u_1 \ldots u_m) = \gamma_A(u_A)\cdot\gamma_1(u_1)\cdot \ldots \cdot\gamma_m(u_m) .$$

To see that γ_A is a CLIFFORD translation of A and γ_j is a CLIFFORD translation of G_j, we fix all but one component of u and vary that component. Thus, if the decomposition $\mathfrak{G} = \mathfrak{A} \oplus \mathfrak{G}_1 \oplus \ldots \oplus \mathfrak{G}_m$ is nontrivial, our induction hypothesis shows that α is inner on $\mathfrak{A}$ and on $\mathfrak{G}_j$, so α is inner on $\mathfrak{G}$.

Now we may assume that G is simple or is a torus, and g is central in G. If G is a torus T^n, then, by analyticity, equation (2) lifts to the universal covering group $\mathbf{R}^n$. In additive notation, α is an orthogonal linear transformation of $\mathbf{R}^n$ such that $2\alpha(\vec{u}) = \vec{u} + \alpha^2(\vec{u})$, whence $(\alpha - I)^2 = 0$. This implies $\alpha = I$ because α is fully reducible on $\mathbf{R}^n$, so α is inner. We now assume G simple. We may assume $g \neq 1$, as $g = 1$ implies $\gamma(1) = 1$, whence $\gamma = 1$, so $\alpha = 1$ is inner. We may also assume $g^2 \neq 1$, for $g^2 = 1$ implies (γ^2 was assumed to be a CLIFFORD translation of G, and Lemma 4.3.1 gives $g^\alpha = g$) $\alpha^2 = 1$, whence either $\alpha = 1$ and is inner, or there is a 1-parameter subgroup u_t of G with $u_t^\alpha = u_t^{-1}$; in the latter case equation (2) gives $u_{-2t} = u_{2t}$, which is impossible for small nonzero t. We may also assume that G has an outer automorphism; in fact, by Lemma 4.3.1 and the assumption $g^2 \neq 1$, we may assume that G has an outer automorphism which leaves invariant some central element with square $\neq 1$. There is no such simple LIE group. *Q.E.D.*

4.4. CLIFFORD translations and symmetries

4.4.1. Lemma. *Let α be an automorphism of a compact connected* LIE *group G, let s be the symmetry $h \to h^{-1}$ at $1 \in G$, let $g \in G$, and suppose that $\gamma = (g, 1)\cdot\alpha\cdot s$ is a* CLIFFORD *translation of G. Then $(ugu^{-\alpha^2})^{-1} = (ugu^{-\alpha^2})^\alpha$ for every $u \in G$.*

Proof. Let S be the sphere $\{v \in G : \varrho(1,v) = \varrho(1,g)\}$ about $1 \in G$. Given $u \in G$, we have $\varrho(1,g) = \varrho(1,\gamma(1)) = \varrho(u,\gamma(u)) = \varrho(u, gu^{-\alpha}) = \varrho(1, gu^{-\alpha}u^{-1})$, so $gu^{-\alpha}u^{-1} \in S$. Let $\exp(tX)$ $(0 \leqq t \leqq 1, X \in \mathfrak{G})$ be a minimizing geodesic in G from 1 to g, so $(g,1)_*X$ is orthogonal to S at g. Let $Y \in \mathfrak{G}$, define $u_t = \exp(tY)$ and $q_t = gu_t^{-\alpha}u_t^{-1}$, and note that $q_t \in S$. Thus $q_*\left(\left.\frac{d}{dt}\right|_{t=0}\right) = (g,1)_*(-Y^\alpha - Y)$ is tangent to S at g, hence orthogonal to $(g,1)_*X$. It follows that X is orthogonal to $(\alpha + 1)\,\mathfrak{G}$, whence $X^\alpha = -X$ because α is an orthogonal linear transformation of $\mathfrak{G}$, so $g^\alpha = g^{-1}$ because $g = \exp(X)$.

The fact that γ is a CLIFFORD translation of G implies that

$$(u, u^\alpha)\cdot\gamma\cdot(u, u^\alpha)^{-1} = (u, u^\alpha)\cdot(g, 1)\cdot\alpha\cdot s\cdot(u^{-1}, u^{-\alpha}) =$$
$$= (u, u^\alpha)\cdot(g,1)\cdot\alpha\cdot(u^{-\alpha}, u^{-1})\cdot s = (u, u^\alpha)\cdot(g,1)\cdot(u^{-\alpha^2}, u^{-\alpha})\cdot\alpha\cdot s = (ugu^{-\alpha^2}, 1)\cdot\alpha\cdot s$$

is a CLIFFORD translation of G. The Lemma now follows from the above paragraph. *Q.E.D.*

4.4.2. Theorem. *Let α be an automorphism of a compact connected* LIE *group G, let s be the symmetry at $1 \in G$, let $(g, h) \in \mathbf{T}(G)$, and suppose that $\gamma = (g, h) \cdot \alpha \cdot s$ is a* CLIFFORD *translation of G. Then G is a torus and γ is a left translation.*

Proof. As in the proof of Theorem 4.3.3, we may assume $\gamma = (g, 1) \cdot \alpha \cdot s$. Now assume the Theorem true for compact connected LIE groups of dimension less than dim. G, and let B be the identity component of the centralizer of g in G. The restriction of s to B is the symmetry $b \to b^{-1}$ of B at 1, and $g^\alpha = g^{-1}$ (by Lemma 4.4.1 with $u = 1$) implies $B^\alpha = B$, whence γ preserves B. As γ is a CLIFFORD translation of G and B is totally geodesic in G, we see that γ is a CLIFFORD translation of B.

Suppose that g is not central in G; then our induction hypothesis implies that B is a maximal torus of G and γ is left translation by g on B. Thus $b \in B$ gives $gb = \gamma(b) = gb^{-\alpha}$, whence $b^\alpha = b^{-1}$. G is not abelian, and we use the fact that every element of G is conjugate to an element of the maximal torus B, and try to choose $a \in B$ such that the connected centralizer D of a in G is a nonabelian proper subgroup of G. But $a^\alpha = a^{-1}$ implies $D^\alpha = D$, whence $g \in D$ implies $\gamma(D) = D$; it follows that γ would be a CLIFFORD translation of D. In view of our induction hypothesis, we conclude that, given $a \in G$, the centralizer of a in G is either G or a maximal torus of G. It follows [4] that the semisimple part of G is a 3-dimensional simple LIE group. So the LIE algebra $\mathfrak{G} = \mathfrak{A} \oplus \mathfrak{G}'$ where $\mathfrak{A}$ is an abelian ideal and $\mathfrak{G}'$ is a 3-dimensional simple ideal. Let $A = \exp(\mathfrak{A})$ and $G' = \exp(\mathfrak{G}')$, and recall that every $u \in G$ has expression $u = u_A u'$ with $u_A \in A$ and $u' \in G'$. Choose some such expression $g = g_A g'$, observe that $A^\alpha = A$ and $(G')^\alpha = G'$ because α is an automorphism, and note that A is central in G; thus

$$\gamma(u_A u') = g_A g' \, u_A^{-\alpha} \, u'^{-\alpha} = (g_A u_A^{-\alpha}) \, (g' u'^{-\alpha}) \,,$$

and we easily check that $u' \to g' u'^{-\alpha}$ is a CLIFFORD translation of G'. This contradicts our induction hypothesis unless $G = G'$. It follows that we may assume G to be a 3-dimensional simple LIE group; either $G = \mathbf{SU}(2)$ or $G = \mathbf{SO}(3)$. This implies [21] that γ is conjugate in $\mathbf{I}(G)$ to a left translation, for G is isometric to the sphere S^3 or the projective space P^3; replacing γ by that conjugate, we assume that γ is a left translation. γ now assumes the form $(u, v) \cdot \beta \cdot s$ where $(u, v) \in \mathbf{T}(G)$ and β is conjugate to α in the group of automorphisms of G, and $\gamma(1) = uv^{-1}$ shows that $\gamma(h) = uv^{-1}h$ for every $h \in G$. Thus $v^{-1}hv = h^{-\beta}$ for every $h \in G$, so $h, k \in G$ gives $v^{-1}hv \cdot v^{-1}kv = v^{-1}hkv = (hk)^{-\beta} = k^{-\beta} h^{-\beta} = v^{-1}kv \cdot v^{-1}hv$; this implies that G is abelian, which is a contradiction.

We have shown that we may assume g central in G. But we may replace γ by $(v, 1)\cdot\gamma\cdot(v, 1)^{-1} = (v, 1)\cdot(g, 1)\cdot\alpha\cdot s\cdot(v^{-1}, 1) = (vg, 1)\cdot\alpha\cdot(1, v^{-1})\cdot s = (vg, v^{-\alpha})\cdot\alpha\cdot s = (vgv^{\alpha}, 1)\cdot\beta\cdot s$ where $\beta = (v^{-\alpha}, v^{-\alpha})\cdot\alpha$, for any $v \in G$, and assume vgv^{α} central in G. Thus vv^{α} is central for every $v \in G$, whence $v^{\alpha} = v^{-1}$ for every element v of the semisimple part of G. But $v \to v^{-1}$ is not an automorphism on the semisimple part of G. It follows that G is a torus.

Now G is a torus T^n. As G is abelian, Lemma 4.4.1 gives $u^{-1}u^{\alpha^2} = u^{\alpha}u^{-\alpha^3}$ for every $u \in G$. This lifts, by analyticity, to the universal covering group $\mathbf{R}^n$ of G; in vector notation, we have an orthogonal linear transformation α of $\mathbf{R}^n$ such that $(\alpha + I)(\alpha^2 - I) = 0$, because $\alpha^2(\vec{u}) - \vec{u} = \alpha(\vec{u}) - \alpha^3(\vec{u})$ for every $\vec{u} \in \mathbf{R}^n$; it follows that $\alpha^2 = I$ on $\mathbf{R}^n$, whence $\alpha^2 = 1$ on G. This shows that G is generated by its sub-tori G' and G'', where G' is the identity component of $\{h \in G : h^{\alpha} = h\}$ and G'' is the identity component of $\{h \in G : h^{\alpha} = h^{-1}\}$. Let $c = \varrho(1, \gamma(1)) = \varrho(1, g)$, so $c = \varrho(h, \gamma(h))$ for every $h \in G$. Given $h' \in G'$, we choose $h \in G'$ with $h^2 = h'$, and note that $c = \varrho(h, \gamma(h)) = \varrho(h, gh^{-1}) = \varrho(h^2, g) = \varrho(h', g)$. It follows that any minimizing geodesic in G from 1 to g is orthogonal to G' et 1, whence $g \in G''$. Similarly, given $h' \in G'$, any minimizing geodesic in G from h' to g is orthogonal to G' at h', whence $gh'^{-1} \in G''$. Thus $G' \subset G''$, so $G = G''$. Now we see $h^{\alpha} = h^{-1}$ for $h \in G$; it follows that $\gamma(h) = gh$ for $h \in G$, and the Theorem is proven. *Q.E.D.*

4.5. Clifford translations as left translations

4.5.1. Theorem. *Let $G = A \times G_1 \times \ldots \times G_m$ be a compact connected Lie group where A is a torus and the G_j are simple, and let Γ be a group of Clifford translations of G. Then Γ is conjugate in $\mathbf{I}(G)$ to a group of left translations of G.*

Proof. It follows from Theorems 4.3.3 and 4.4.2 that Γ is a subgroup of the group $\mathbf{I}'(G)$ of Theorem 4.2.4; the result now follows from Theorem 4.2.4. *Q.E.D.*

4.5.2. Corollary. *Let Γ be a group of Clifford translations of a compact connected Lie group G which is either centerless, simple or simply connected. Then Γ is conjugate in $\mathbf{I}(G)$ to a group of left translations of G.*

For G has the form required by Theorem 4.5.1.

4.5.3. Corollary. *Let Γ be the group of deck transformations of a Riemannian covering $\pi : G \to N$ where G is a compact simply connected Lie group. Then N is Riemannian homogeneous if and only if Γ is a group of Clifford translations of G.*

Proof. If N is homogeneous, then Γ is known to be a group of CLIFFORD translations [20, Th. 2]; as π is a universal covering, hence normal, we prove the converse by assuming Γ to be a group of CLIFFORD translations of G, and finding a subgroup B of $\mathbf{I}(G)$ which centralizes Γ and is transitive on G [20, Th. 1]. Since G is simply connected, Corollary 4.5.2 shows that we may assume Γ to be a group of left translations of G. Let B be the group of all right translations of G. *Q.E.D.*

4.6. Symmetric spaces as group spaces

4.6.1. We wish to consider the possibility of putting a 2-sided-invariant metric on a LIE group G which is not compact. If G is connected and is a covering group $\pi: G \to H$ of a compact LIE group H, then G carries a 2-sided-invariant metric such that π is a RIEMANNian covering, and the universal covering group of G is a product of compact simple groups and a vector group. This is the only possibility:

4.6.2. Lemma. *Let G be a connected LIE group. Then G admits a 2-sided-invariant metric*[3]) *if and only if the LIE algebra $\mathfrak{G}$ is a direct sum of compact simple ideals and an abelian ideal, and G is a RIEMANNian symmetric manifold in any 2-sided invariant metric.*

Proof. G has a 2-sided-invariant metric if and only if $\mathfrak{G}$ has a positive definite $Ad(G)$-invariant bilinear form. $\mathfrak{G}$ has the form, as constructed in § 4.1.3, if it is a sum of compact simple ideals and an abelian ideal. If $\mathfrak{G}$ has the form, then $\mathfrak{G}$ is a reductive LIE algebra, so $\mathfrak{G} = \mathfrak{G}_1 \oplus \ldots \oplus \mathfrak{G}_t \oplus \mathfrak{A}$ where the $\mathfrak{G}_j$ are simple ideals and $\mathfrak{A}$ is an abelian ideal; now each $\mathfrak{G}_j$ is compact because the form is definite. The last statement by observing that, if G is endowed with a 2-sided-invariant metric, then there is a RIEMANNian covering $\pi: G \to H$ where H is a compact LIE group. *Q.E.D.*

4.6.3. Theorem. *Let Γ be the group of deck transformations of a RIEMANNian covering $\pi: G \to N$ where G is a simply connected LIE group in 2-sided-invariant metric. Then these are equivalent:*

(1) *N is a RIEMANNian symmetric manifold.*

(2) *N is a LIE group and $\pi: G \to N$ is the universal covering group.*

(3) *G has a discrete central subgroup B such that Γ is conjugate in $\mathbf{I}(G)$ to the group of left translations of G by elements of B.*

Proof. The equivalence of (2) and (3) is clear, and Lemma 4.6.2 shows that (2) implies (1). Thus we need only prove that (1) implies (3).

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[3]) positive definitite, of course.

Now let N be RIEMANNian symmetric. Under CARTAN's symmetric-space decomposition, $G = A \times G_1 \times \ldots \times G_m$ where A is a vector group $\mathbf{R}^n$ and the G_j are compact simple groups. Γ is a group of CLIFFORD translations of G, whence every $\gamma \in \Gamma$ has form $\gamma_1 \times \gamma_2$, γ_1 being a CLIFFORD translation of A and γ_2 being a CLIFFORD translation of $G' = G_1 \times \ldots \times G_m$, by Corollary 3.1.4. Now $\Gamma_1 = \{\gamma_1 : \gamma \in \Gamma\}$ is a discrete subgroup of A acting by left translations, by [20, Th. 4]. Thus we need only prove that $\Gamma_2 = \{\gamma_2 : \gamma \in \Gamma\}$ is $\mathbf{I}(G')$-conjugate to a group of left translations of G' by central elements of G'. In other words, we may assume that G is compact and simply connected.

Now assume G compact and simply connected. Using Corollary 4.5.3, we may assume that Γ consists of left translations of G by elements of a discrete subgroup B. The symmetry to N at $\pi(1)$ lifts to the symmetry s of G at 1, whence s normalizes Γ. Now $b \in B$ gives $(b, 1) \in \mathbf{T}(G)$ with $s \cdot (b, 1) \cdot s^{-1} = (1, b)$, so we have $b' \in B$ such that $b'g = gb$ for every $g \in G$. This shows that B is a discrete normal subgroup of G; it follows that B is central in G.

Q.E.D.

5. CLIFFORD translations of spaces with simple group of isometries

5.1. Symmetric spaces with simple group of isometries

Let M be a compact connected simply connected irreducible RIEMANNian symmetric manifold. M is of type (1) if $\mathbf{I}_0(M)$ is not simple; CLIFFORD translations of these spaces were considered in § 4. We will now assume $\mathbf{I}_0(M)$ simple and study the CLIFFORD translations of M. For the most part, it turns out that $\Gamma \subset \mathbf{I}(M)$ is a group of CLIFFORD translations of M if and only if Γ centralizes $\mathbf{I}_0(M)$, and, in any case, M/Γ is RIEMANNian homogeneous if and only if Γ is a finite group of CLIFFORD translations of M.

The case where M is a sphere S^{n-1} has been treated [20; 21] in full. We take the field $F = \mathbf{R}, \mathbf{C}$ or $\mathbf{H}$ of maximal real dimension $r = 1, 2$ or 4 which divides $n, n = rm$. $M = S^{n-1}$ is viewed as the unit sphere in a left HERMITIAN vectorspace V of dimension m over F. If Γ is a subgroup of $\mathbf{I}(M)$, then these are equivalent:

(a) Γ is a discrete group of CLIFFORD translations of M.
(b) M/Γ is RIEMANNian homogeneous.
(c) Γ is conjugate in $\mathbf{I}(M)$ to a finite subgroup of the multiplicative group F' of unimodular elements of F, F' acting on M by scalar multiplication.

The finite subgroups of F' are well known. As M/Γ is RIEMANNian symmetric if and only if $\Gamma \subset \{\pm I\}$ (acting on V) [22, § 16], we can easily find Γ

such that M/Γ is RIEMANNian homogeneous but not RIEMANNian symmetric, provided that n-1 is odd.

5.2. CLIFFORD translations and inner automorphisms

5.2.1. Lemma. *Let K be a compact irreducible group of linear transformations of a real vectorspace V, and let $0 \neq \vec{x} \in V$. Then the orbit $K(\vec{x})$ does not lie in a half-space.*

Proof. Let $\mathfrak{H}$ be a half-space of V. This means that we have a basis $\{\vec{e}_i\}$ of V such that $\mathfrak{H} = \{\Sigma a_j \vec{e}_j : a_1 \geqq 0\}$. Now let H be the hyperplane $\{\Sigma a_j \vec{e}_j : a_1 = 0\}$ associated to $\mathfrak{H}$. The "center of gravity[4])" $\int_K k(\vec{x})\, d\mu(k)$ of K on $\vec{x}$ (μ is normalized Haar measure on K) is a K-invariant element of V, hence 0 by irreducibility of K.

Now suppose $K(\vec{x}) \subset \mathfrak{H}$. $K(\vec{x}) \not\subset H$ by irreducibility, so we may replace $\vec{x}$ by a K-image and assume $\vec{x} = \Sigma x_j \vec{e}_j$ with $x_1 > 0$. Given $\vec{y} = \Sigma y_j \vec{e}_j \in K(\vec{x})$, we have supposed $y_1 \geqq 0$. Thus $x_1 > 0$ contradicts $\int k(\vec{x})\, d\mu(k) = 0$. *Q.E.D.*

5.2.2. Theorem. *Let M be a connected RIEMANNian homogeneous manifold, and suppose that:*

(a) *The connected linear isotropy subgroups of $\mathbf{I}(M)$ are irreducible on the tangentspaces of M.*

(b) *$\beta \in \mathbf{I}(M)$ centralizes $\mathbf{I}_0(M)$, $g \in \mathbf{I}(M)$ has a fixed point on M, and $\gamma = g\beta$ is a CLIFFORD translation of M.*

Then $\gamma = \beta$, i.e., $g = 1$.

Proof. As g has a fixed point, γ is a CLIFFORD translation by hypothesis and β is a CLIFFORD translation because it centralizes a transitive group $\mathbf{I}_0(M)$ of isometries of M, it follows that β and γ are isometries of the same constant displacement $b \geqq 0$. If $b = 0$, then $\gamma = \beta = g = 1$. We assume $b > 0$.

Now suppose $g \neq 1$. As g has a fixed point, judicious choice of $x \in M$ gives us the condition $0 < \varrho(\beta x, \gamma x) < \varepsilon$ on the distance, for any $\varepsilon > 0$. Furthermore, βx and γx each lies on the "sphere" of radius b about x. Thus we may choose $x \in M$ with $\beta x \neq \gamma x$ but $\varrho(\beta x, \gamma x)$ so small that (1) there is a unique minimizing geodesic $\exp(tX)$ $(0 \leqq t \leqq 1, X$ in the tangentspace $M_{\beta x})$ from βx to γx, and (2) given a minimizing geodesic $\sigma(t)$ $(0 \leqq t \leqq 1)$ from x to βx, the angle between $\sigma'(1)$ and X is at least $\pi/2$. Now make a choice of $\sigma(t)$, let H be the hyperplane $\sigma'(1)^{\perp}$ in $M_{\beta x}$, and let $\mathfrak{H}$ be the half-space $\langle Y, \sigma'(1)\rangle \leqq 0$ in $M_{\beta x}$.

[4]) This concept was brought to my attention by R.F. Williams.

As β centralizes $\mathbf{I}_0(M)$ and $\mathbf{I}_0(M)$ contains the *connected* isotropy subgroup K of $\mathbf{I}(M)$ at x, it follows that K is the connected isotropy subgroup of $\mathbf{I}(M)$ at βx. Let $K_* X = \{k_* X : k \in K\}$ be the orbit of $X \in M_{\beta x}$ (recall our choice of x) under the linear isotropy action of K at βx. Now $k \in K$ gives (1) $k \gamma k^{-1} = kgk^{-1}\beta$ is a CLIFFORD translation of displacement b, (2) $kx = x, k\beta x = \beta x$ and $k\gamma x = k\gamma k^{-1} x$, and (3) $\varrho(\beta x, \gamma x) = \varrho(k\beta x, k\gamma x) = \varrho(\beta x, k\gamma x)$. Thus $k_* X \in \mathfrak{H}$. It follows that $K_* X \subset \mathfrak{H}$, which contradicts Lemma 5.2.1. We conclude that $g = 1$. *Q.E.D.*

5.2.3. Corollary. *Let M be a compact connected RIEMANNian homogeneous manifold of EULER-POINCARÉ characteristic $\chi(M) \neq 0$ such that the connected linear isotropy subgroups of $\mathbf{I}(M)$ act irreducibly on the tangent-spaces of M, and suppose that $\mathbf{I}(M) = \bigcup \beta_j \cdot \mathbf{I}_0(M)$ where β_j centralizes $\mathbf{I}_0(M)$ (this is automatic if $\mathbf{I}_0(M)$ admits no outer automorphism). Let Γ be a group of CLIFFORD translations of M. Then $\Gamma \subset \{\beta_j\}, \Gamma$ centralizes $\mathbf{I}_0(M)$, and M/Γ is a RIEMANNian homogeneous manifold.*

Proof. $\chi(M) \neq 0$ implies [19, Th. 4] that every $g \in \mathbf{I}_0(M)$ has a fixed point on M, and that $\Gamma \cap \mathbf{I}_0(M) = \{1\}$. The Corollary now follows from Theorem 5.2.2. *Q.E.D.*

5.2.4. Corollary. *Let M be a compact connected simply connected RIEMANNian symmetric manifold with $\mathbf{I}_0(M)$ simple, but with $\mathbf{I}_0(M)$ not of type A_n $(n > 1)$, $D_n (n > 3)$ or E_6. Let Γ be a group of CLIFFORD translations of M. Then Γ centralizes $\mathbf{I}_0(M)$ and M/Γ is a RIEMANNian symmetric manifold.*

Proof. Let K be the isotropy subgroup of $G = \mathbf{I}_0(M)$ at $p \in M$, and let s be the symmetry to M at p. Conjugation by s is an inner automorphism of G, for our hypotheses ensure that G has no outer automorphism, whence K contains a maximal torus of G. Thus $\chi(G/K) \neq 0$. K acts irreducibly on the tangentspace M_p because M is irreducible. The other hypothesis of Corollary 5.2.3 is satisfied because G has no outer automorphism, so Γ centralizes $G = \mathbf{I}_0(M)$ by that Corollary. Thus M/Γ is symmetric. *Q.E.D.*

5.3. Fixed points and symmetries

We will use a result of J. DE SIEBENTHAL [18, p. 57]

Let G_0 be the identity component of a compact LIE group $G, x \in G$, and T a maximal torus of the centralizer of x in G. Then every element of the component $x \cdot G_0$ of x is $ad(G_0)$-conjugate to an element of $x \cdot T$

to show that certain components of certain $\mathbf{I}(M)$ contain no element without a fixed point, thus containing no CLIFFORD translation of M.

5.3.1. Lemma. *Let s be a symmetry of a compact connected* RIEMANN*ian symmetric space M. Then every element of $s \cdot \mathbf{I}_0(M)$ has a fixed point on M.*

Proof. s is the symmetry at $p \in M$. Let T be a maximal torus of the isotropy subgroup K' of $\mathbf{I}(M)$ at p. K' and the centralizer of s in $\mathbf{I}(M)$ have the same identity component. Thus DE SIEBENTHAL's Theorem shows that every element of $s \cdot \mathbf{I}_0(M)$ is conjugate to an element of $s \cdot T \subset K'$. The Lemma follows. *Q.E.D*

5.3.2. The proof of Lemma 5.3.1. proves

Lemma. *Let M be a compact connected* RIEMANN*ian homogeneous manifold, K' the isotropy subgroup of $\mathbf{I}(M)$ at $p \in M$, $k \in K'$ and $g \in k \cdot \mathbf{I}_0(M)$. Suppose that K' contains a maximal torus of the centralizer of k in $\mathbf{I}(M)$. Then g has a fixed point on M.*

5.3.3. Lemma. *Let M be a compact connected irreducible* RIEMANN*ian symmetric manifold such that a connected isotropy subgroup of $\mathbf{I}(M)$ admits no outer automorphism. Let γ be an isometry of M which has no fixed point. Then $\gamma \in \mathbf{I}_0(M)$.*

Proof. $\mathbf{I}(M) = \mathbf{I}_0(M) \cup s \cdot \mathbf{I}_0(M)$ *by* § 2.4.4. The Lemma now follows from Lemma 5.3.1. *Q.E.D.*

5.4. Rank and imbedding of symmetric spaces

5.4.1. Let M be a connected RIEMANNian symmetric manifold, $G = \mathbf{I}_0(M)$, K the isotropy subgroup of G at $x \in M$, and $\mathfrak{G} = \mathfrak{K} + \mathfrak{P}$ the CARTAN decomposition. Every subalgebra of $\mathfrak{G}$ contained in $\mathfrak{P}$ is abelian because $[\mathfrak{P}, \mathfrak{P}] \cap \mathfrak{P} = 0$, and is contained in a maximal such subalgebra. The maxima among the subalgebras of $\mathfrak{G}$ contained in $\mathfrak{P}$ are called *CARTAN subalgebras* of $(\mathfrak{G}, \sigma)$, are mutually conjugate under $Ad(K)|_{\mathfrak{P}}$, and thus all have the same dimension; this common dimension is called the *rank* of M. A CARTAN subalgebra of $(\mathfrak{G}, \sigma)$ contains a line and is contained in a CARTAN subalgebra of $\mathfrak{G}$; thus $1 \leqq \text{rank. } M \leqq \text{rank. } \mathfrak{G}$.

5.4.2. É. CARTAN has given a map of M into G which we will find useful for providing geometric interpretations of M. Given $g \in G$, define $g^* = \sigma(g^{-1})$. As $k \in K$ gives $kk^* = 1$, this defines a map $f: M \to G$ by $f(gK) = gg^*$, where M is identified with G/K. The image of f is $P = \exp(\mathfrak{P})$, and $f: M \to P$ is a covering with finite fibre K_σ/K where $K_\sigma = \{g \in G : \sigma(g) = g]$. If G is compact, we can endow G with a 2-sided-invariant RIEMANNian metric such that $\mathfrak{P}$ is a totally geodesic submanifold and $f: M \to P$ is a RIEMANNian covering; then f is G-equivariant, G acting by isometries on P by $\tau_g(p) = gpg^*$. If we write M as $\tilde{G}/\tilde{K}$ where $\tilde{G}$ is the universal covering group of $\mathfrak{G}$, and if M is compact and simply connected, then $\tilde{K}$ equals the fixed point set of σ on $\tilde{G}$ and the map corresponding to f is an isometry.

5.5. Clifford translations and symmetric quotients

We have reached the goal of § 5:

5.5.1. Theorem: *Let M be a compact connected simply connected irreducible Riemannian symmetric manifold with $\mathbf{I}_0(M)$ simple, and let Γ be a group of Clifford translations of M. If Γ is finite, then M/Γ is a Riemannian homogeneous manifold. If M is not an odd dimensional sphere, a space $\mathbf{SU}(2m)/\mathbf{Sp}(m)$ with $m > 1$, a complex projective space of odd complex dimension > 1, or a space $\mathbf{SO}(4n+2)/\mathbf{U}(2n+1)$ with $n > 0$, then Γ is finite and centralizes* $\mathbf{I}_0(M)$, and *M/Γ is a Riemannian symmetric manifold; the spaces excluded in this statement have finite groups of Clifford translations such that the quotient is Riemannian homogeneous but not Riemannian symmetric. If M is not an odd dimensional sphere or a space $\mathbf{SU}(2m)/\mathbf{Sp}(m)$ with $m > 1$, then Γ is finite; the spaces excluded in this statement have one parameter groups of Clifford translations.*

Proof. Let $G = \mathbf{I}_0(M)$ and let K be the isotropy subgroup at $x \in M$. By Corollary 5.2.4, we need only check the cases where G is of type $A_n (n > 1)$, $D_n (n > 3)$ or E_6. As the statements are known for spheres ([20; 21]; recalled in § 5.1), É. Cartan's classification [9] shows that we need only check the cases (AI) $\mathbf{SU}(n)/\mathbf{SO}(n)$ with $n > 2$, (AII) $\mathbf{SU}(2n)/\mathbf{Sp}(n)$ with $n > 1$,

$$\text{(AIII)}\ \mathbf{SU}(p+q)/\{\mathbf{SU}(p+q) \cap [\mathbf{U}(p) \times \mathbf{U}(q)]\}$$

with $pq > 1$, (DI) $\mathbf{SO}(p+q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$ with $p \geqq 2, q \geqq 2$ and $p+q > 4$ (this need only be checked when $p + q$ is even, i.e., when $\mathbf{SO}(p+q)$ is of type D), (DIII) $\mathbf{SO}(2n)/\mathbf{U}(n)$ with $n > 1$, (EI) $\mathbf{E}_6/\mathrm{ad}\,(\mathbf{C}_4)$, (EII) $\mathbf{E}_6/A_5 \times A_1$, (EIII) $\mathbf{E}_6/D_5 \times T^1$ where T^r denotes an r-torus, and (EIV) $\mathbf{E}_6/\mathbf{F}_4$.

5.5.2. If G is a compact matrix group, then $\mathfrak{G}$ is a Lie algebra of real or complex matrices and $B =$ -Real. trace (XY) is positive definite $Ad(G)$-invariant bilinear form on $\mathfrak{G}$. Thus B is proportional to the Killing form of $\mathfrak{G}$, $\mathfrak{P} = \mathfrak{K}^{\perp}$ relative to B where $\mathfrak{G} = \mathfrak{K} + \mathfrak{P}$ is a Cartan decomposition, and we may take any positive multiple of the restriction $B|_{\mathfrak{P}}$ for the Riemannian metric on M.

5.5.3. $M = \mathbf{SU}(n)/\mathbf{SO}(n)$. As in § 5.4.2, M can be represented as the symmetric matrices in $\mathbf{SU}(n)$. $\mathbf{SU}(n)$ acts by $g : x \to g x {}^t g$, K is the isotropy subgroup of G at I, and $\sigma(g) = {}^t g^{-1}$.

If n is odd, then $G = \mathbf{SU}(n)$ and $K = \mathbf{SO}(n)$, and $\mathbf{I}(M) = G \cup s \cdot G$ because K has no outer automorphism. If n is even, then $G = \mathbf{SU}(n)/\{\pm I\}$ and $K = \mathbf{SO}(n)/\{\pm I\}$; then $\mathbf{I}(M) = G \cup s \cdot G \cup \alpha \cdot G \cup s\alpha \cdot G$ where

$$\alpha : x \to a x {}^t a$$

for some $a \in \mathbf{U}(n)$ of determinant -1. In either case, $s : x \to x^{-1}$. We will use G' to denote G if n is odd and $G \cup \alpha \cdot G$ if n is even; $\mathbf{I}(M) = G' \cup s \cdot G'$.

Let E_{ij} denote the $n \times n$ matrix with 1 in the (i, j)-place and zeros elsewhere. Then $\mathfrak{G}$ has a basis $\{X_{ij}, Y_{uv}, Z_e\}$ where $X_{ij} = E_{ij} - E_{ji}$ for $1 \leqq i < j \leqq n$, $Y_{uv} = \sqrt{-1}\,(E_{uv} + E_{vu})$ for $1 \leqq u < v \leqq n$, and $Z_e = \sqrt{-1}\,(E_{ee} - E_{e+1, e+1})$ for $1 \leqq e < n$. $\{X_{ij}\}$ is a basis for $\mathfrak{K}$ and $\{Y_{uv}, Z_e\}$ is a basis of $\mathfrak{P}$. Now

$$g = \text{diag.}\,\{\exp(\sqrt{-1}\,a_1), \ldots, \exp(\sqrt{-1}\,a_n)\} \in \mathbf{SU}(n)$$

with $|a_i|$ small and $\Sigma a_i = 0$ gives as distance $\varrho(I, g(I))^2 = \Sigma a_i^2$. It follows that we may take the distance on M to be given by $\varrho(x, y)^2 = \Sigma a_i^2$ where $y\bar{x}$ has eigenvalues $\exp(2\sqrt{-1}\,a_i)$ with $\Sigma a_i = 0$ and $|a_i| \leqq \pi$.

Let $g = \text{diag.}\,\{\exp(\sqrt{-1}\,a_1), \ldots, \exp(\sqrt{-1}\,a_n)\}$ represent an alement of $\Gamma \cap G'$ with the a_i chosen to minimize Σa_i^2. Then $c = \varrho(I, g(I))^2 = \Sigma a_i^2$, and $c = \varrho(x, g(x))^2$ for every $x \in M$. If we take $x = \text{diag.}\,\{\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right), -1; 1, \ldots, 1\}$, then $g(x) \cdot \bar{x} = \text{diag.}\,\{\exp(\sqrt{-1}\,(a_1 + a_2)), \exp(\sqrt{-1}\,(a_1 + a_2)); \exp(2\sqrt{-1}\,a_3), \ldots, \exp(2\sqrt{-1}\,a_n)\}$, whence $a_1^2 + a_2^2 = \frac{1}{2}(a_1 + a_2)^2$; it follows that $a_1 = a_2$. Similarly, $a_i = a_j$; thus g is scalar. It follows that $\Gamma \cap G'$ is represented by scalar matrices, and is thus central in G'.

$\Gamma \cap s \cdot G$ is empty by Lemma 5.3.1. Let $\gamma = sg \in \Gamma \cap s\alpha \cdot G$, and let g also denote the matrix in $a \cdot \mathbf{SU}(n)$. $\gamma^2 = sgsg = {}^t g^{-1} g$ lies in $\Gamma \cap G$, and is thus represented by a scalar matrix cI. Now ${}^t g = cg$, whence $g = {}^t({}^t g) = {}^t(cg) = c^2 g$; thus ${}^t g = \pm g$. If $g = {}^t g$, then $g = {}^t h^{-1} h^{-1} \in M$ for some $h \in \mathbf{SU}(n)$, and $x = h {}^t h \in M$; $\gamma(x) = s({}^t h^{-1} h^{-1} h {}^t h {}^t h^{-1} h^{-1}) = s({}^t h^{-1} h^{-1}) = h {}^t h = x$. This is impossible because γ has no fixed point; thus ${}^t g = -g$. That is impossible because $\det. g = -1$. We conclude that $\Gamma \cap s\alpha \cdot G$ is empty.

We have proved that Γ is a central subgroup of G'; in particular, Γ centralizes $\mathbf{I}_0(M)$.

5.5.4. $M = \mathbf{SU}(2m)/\mathbf{Sp}(m)$. M can be viewed as the skew (= antisymmetric) matrices in $\mathbf{SU}(2m)$ and we assume $m > 1$ because $\mathbf{SU}(2)/\mathbf{Sp}(1)$

is a single point. $G = \mathbf{SU}(2m)/\{\pm I\}$ acts by $g: x \to g x {}^t g$ and $K = \mathbf{Sp}(m)/\{\pm I\}$ is the isotropy subgroup at $J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} \in M$. $\mathbf{I}(M) = G \cup s \cdot G$ because K has no outer automorphism, and $s: x \to J x^{-1} \cdot {}^t J$; thus $s = J\alpha$ where $\alpha: x \to x^{-1}$, and conjugation of G by α is the operation of inverse transpose on representing elements of $\mathbf{SU}(2m)$.

$\Gamma \subset G$ by Lemma 5.3.1; we will see that every element of Γ is represented in $\mathbf{SU}(2m)$ by a matrix conjugate to some diag. $\{a'; a, \ldots, a\}$. For as in § 5.5.3, one can check that the distance on M is given by $\varrho(x, y)^2 = \frac{1}{2} \Sigma a_j^2$ where $-y\bar{x}$ has eigenvalues $\exp(2\sqrt{-1}\, a_j)$ with $|a_j| \leqq \pi$. Let $\gamma \in \Gamma$ and conjugate it so that γ is represented by a diagonal matrix $\begin{pmatrix} uv & 0 \\ 0 & u\bar{v} \end{pmatrix}$ with

$$u = \text{diag.}\{\exp(\sqrt{-1}\, u_1), \ldots, \exp(\sqrt{-1}\, u_m)\}$$
$$v = \text{diag.}\{\exp(\sqrt{-1}\, v_1), \ldots, \exp(\sqrt{-1}\, v_m)\}$$

for minimal $|u_j|$; then $c = \varrho(J, \gamma(J))^2 = \Sigma u_j^2$. The number c does not depend on our choice of conjugate of γ; conjugating by diag. $\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1, \ldots, 1\right\}$ if follows that $(u_1 - u_2)^2 = (v_1 - v_2)^2$. Similarly, $(u_i - u_j)^2 = (v_i - v_j)^2$ for every i and j. Another conjugation (exchanging the j^{th} and $(m + j)^{th}$ basis vectors) interchanges v_j and $-v_j$, resulting in $(v_i - v_j)^2 = (u_i - u_j)^2 = (v_i + v_j)^2$ whenever $i \neq j$. This proves that at most one of the v_i is nonzero, so we may assume $v_2 = v_3 = \ldots = v_m = 0$, and it follows that $u_2 = u_3 = \ldots = u_m$ and $u_1 - u_2 = \pm v_1$. This proves that the matrix representing γ is conjugate to some diag. $\{a'; a, \ldots, a\}$.

If Γ is cyclic, choose a generator γ, and let L be the connected centralizer of γ in G. If γ is not central, then G/L is complex projective space $\mathbf{P}^{2m-1}(\mathbf{C})$ of odd complex dimension. It is known, and not difficult to check by counting dimensions, that K acts transitively on G/L. Thus $G = KL$, which implies $G = LK$, so L acts transitively on $G/K = M$. If Γ is finite, so M/Γ is a manifold, then L induces a transitive group of isometries of M/Γ, and M/Γ is Riemannian homogeneous.

The argument above is valid even if Γ is not cyclic, provided that the image of Γ in the adjoint group of $\mathbf{SU}(2m)$ is cyclic. For then Γ is generated by central elements of G and at most one other element γ. Thus we need only prove:

Lemma. *Let Γ be a finite subgroup of* $\mathbf{SU}(k)$, $k > 2$, *in which every element is* $\mathbf{SU}(k)$*-conjugate to some* diag. $\{a'; a, \ldots, a\}$. *Then the image of Γ in ad*$(\mathbf{SU}(k))$ *is cyclic.*

Proof. Let $\beta : \mathbf{SU}(k) \to ad(\mathbf{SU}(k))$ be the projection. We will first prove that every abelian subgroup of $\beta(\Gamma)$ is cyclic. For this, it sufffces to show that $\beta(\Gamma)$ has no subgroup which is the product of two cyclic groups of prime order p. Let A be such a subgroup; we will derive a contradiction.

If p is prime to k, then it is easy to find elements γ and δ in $B = \beta^{-1}(A)$ such that the group Δ generated by γ and δ is mapped isomorphically onto A by β. Δ cannot act freely on the unit sphere in $\mathbf{C}^k$ because it is abelian and finite but not cyclic; thus Δ has an element $\neq I$ with an eigenvalue $+1$. Now we may assume that δ has an eigenvalue $+1$. Changing orthonormal basis in $\mathbf{C}^k$, we may assume that $\delta = \text{diag.}\,\{a'; a, \ldots, a\}$ and that either $\gamma = \text{diag.}\,\{b'; b, \ldots, b\}$ or $\gamma = \text{diag.}\,\{b, b'; b, \ldots, b\}$; this is possible because γ and δ commute, and because each is the product of an element of Γ with a scalar matrix. If $a = 1$, then $a' = \det.\,\delta = 1$ and $\delta = I$; thus $a \neq 1$ and $a' = 1$; it follows that $p \neq 2$. Now $b \neq b'$ because p is prime to k, $\det.\,\gamma = 1$ and γ has order p; as $p \neq 2$, so $\delta\gamma$ and $\delta\gamma^2$ each has precisely two distinct eigenvalues, the second possibility for the form of γ is eliminated and we have $\gamma = \text{diag.}\,\{b'; b, \ldots, b\}$. If $\varepsilon = \exp(2\pi\sqrt{-1}/p)$, this implies that Δ consists of matrices $\text{diag.}\,\{\varepsilon^u; \varepsilon^v, \ldots, \varepsilon^v\}$, whence Δ contains nontrivial scalar matrices. That is impossible because Δ has order prime to k. Thus A cannot exist.

If p divides k, we define

$$e = \text{diag.}\,\{\exp(2\pi\sqrt{-1}/k), \ldots, \exp(2\pi\sqrt{-1}/k)\} \in \mathbf{SU}(k)$$

and we choose elements γ and δ in $B = \beta^{-1}(A)$ which map onto generators of A. As $\beta(\gamma)$ and $\beta(\delta)$ commute, we have $\delta\gamma\delta^{-1} = \gamma e^u$ for some integer u. Looking at eigenvalues and using $k > 2$, we see that $\gamma\delta = \delta\gamma$. Thus the subgroup Δ of B generated by γ and δ is abelian. We may also assume that γ and δ were chosen of prime power order; then Δ is a p-group. Δ is not cyclic because it has the noncyclic group A as a homomorphic image; thus Δ does not act freely on the unit sphere in $\mathbf{C}^k$, and it follows that Δ has an element $\tau \neq I$ which has $+1$ for an eigenvalue. As before we may assume $\tau = \text{diag.}\,\{1; a, \ldots, a\}$. τ has some order $q = p^b$ because Δ is a p-group. As p divides k, p is prime to $k-1$; thus $\det.\,\tau \neq 1$. This contradicts the existence of A.

We have proved that every abelian subgroup of $\beta(\Gamma)$ is cyclic. Now suppose that $\beta(\Gamma)$ has elements $\beta(\gamma)$ and $\beta(\delta)$ such that

$$\beta(\delta)\cdot\beta(\gamma)\cdot\beta(\delta)^{-1} = \beta(\gamma)^r \neq \beta(\gamma)$$

for some integer r. Then $\delta\gamma\delta^{-1} = \gamma^r\alpha$ for some scalar matrix

$$\alpha = \text{diag.}\,\{a, \ldots, a\}.$$

We may assume $\gamma = \text{diag.}\{b'; b, \ldots, b\}$; thus $k > 2$ implies $b'^r a = b'$ and $b^r a = b$. This contradicts $\beta(\gamma)^r \neq \beta(\gamma)$. We conclude that $\beta(\Gamma)$ cannot have a subgroup E with generators x and y satisfying a relation of the form $xyx^{-1} = y^r \neq y$. As every abelian subgroup of $\beta(\Gamma)$ is cyclic, the odd Sylow subgroups are cyclic and the 2-Sylow subgroups are cyclic or generalized quaternionic (see [19]). A generalized quaternionic group being a group of the form E above, every Sylow subgroup of $\beta(\Gamma)$ is cyclic. It follows that $\beta(\Gamma)$ has generators x and y satisfying a relation $xyx^{-1} = y^r$ (see [19]); thus $\gamma(\Gamma)$ is commutative. This proves that $\beta(\Gamma)$ is cyclic. *Q.E.D.*

5.5.5. $M = \mathbf{SU}(n = p + q)/\{\mathbf{SU}(n) \cap [\mathbf{U}(p) \times \mathbf{U}(q)]\}$ with $pq > 1$. M is the GRASSMANN manifold of (complex) q-planes in $\mathbf{C}^n$ and is usually viewed as the coset space $\mathbf{U}(p+q)/\mathbf{U}(p) \times \mathbf{U}(q)$. If $q = 1$, then M is the complex projective space $\mathbf{P}^{n-1}(\mathbf{C})$ of (complex) dimension $p = n - 1$. In some orthonormal basis $\{e_j\}$ of $\mathbf{C}^n$, K is the isotropy subgroup of $G = \mathbf{SU}(n)$/scalars at the q-plane $x = e_{p+1} \wedge e_{p+2} \wedge \ldots \wedge e_n$ spanned by the last q basis vectors. $K \cong \{\mathbf{U}(p) \times \mathbf{U}(q)\}$/scalars and thus has outer automorphisms: only $k \to {}^t k^{-1}$ if $p \neq q$; $k \to {}^t k^{-1}$, exchange of the two factors $\mathbf{U}(p)$, and their product, if $p = q$. Let $\alpha : y \to \bar{y}$ be the transformation of M resulting from conjugation of $\mathbf{C}$ over $\mathbf{R}$ extended to $\mathbf{C}^n$ by means of $\{e_j\}$, and let $\beta : y \to y^\perp$ if $p = q$. It follows that $\mathbf{I}(M) = G \cup \alpha \cdot G$ if $p \neq q$, and $\mathbf{I}(M) = G \cup \alpha \cdot G \cup \beta \cdot G \cup \beta\alpha \cdot G$ if $p = q$. If $p = q$, β commutes with α and centralizes G, whence $\Gamma \cap (G \cup \beta \cdot G) \subset \{1, \beta\}$ if $p = q$, by Theorem 5.2.2, and $\Gamma \cap G = \{1\}$ in any case.

Let $\gamma = \alpha g \in \Gamma \cap \alpha \cdot G$. Then $\gamma^2 = \alpha g \alpha g = {}^t g^{-1} g = 1$, so ${}^t g^{-1} g$ is represented by a scalar matrix cI; the matrices ${}^t g = cg$, whence

$$g = {}^t({}^t g) = {}^t(cg) = c^2 g, \text{ so } c^2 = 1 \text{ and } {}^t g = \pm g .$$

If $g = {}^t g$, then $g = \exp(\sqrt{-1}\, Z)$ with Z real symmetric. There exists $h \in \mathbf{SO}(n)$ with hZh^{-1} diagonal; it follows that $d = hgh^{-1}$ is diagonal. Then $h\alpha g h^{-1} = h\alpha h^{-1} h g h^{-1} = h^t h \alpha d = \alpha d$ has $e_1 \wedge \ldots \wedge e_q$ as a fixed point, contrary to hypothesis on γ. In other words, $g = -{}^t g$. Thus we can find $h \in \mathbf{SU}(n)$ with $hg^t h = J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$ where $n = 2m$.

$${}^t h^{-1} \alpha g {}^t h = {}^t h^{-1} \alpha h^{-1} h g {}^t h = {}^t h^{-1} \cdot {}^t h \alpha J = \alpha J ,$$

so γ is conjugate to αJ. If p or q is even, then one easily produces a fixed point for αJ; thus p and q are odd. If $1 < q \leqq m = \dfrac{p+q}{2}$, so $q = 2u + 1$ with $0 < u$ and $u + 2 < m$, then we define

$$y = e_1 \wedge e_{m+1} \wedge e_2 \wedge e_{m+2} \wedge \cdots \wedge e_u \wedge e_{m+u} \wedge e_m$$
$$z = e_1 \wedge e_{m+1} \wedge \cdots \wedge e_{u-1} \wedge e_{m+u-1} \wedge e_{m+u} \wedge e_{m+u+1} \wedge e_{m+u+2}$$

and check that $\gamma = \alpha J$ gives

$$\gamma(y) = e_1 \wedge e_{m+1} \wedge \cdots \wedge e_u \wedge e_{m+u} \wedge e_{2m}$$
$$\gamma(z) = e_1 \wedge e_{m+1} \wedge \cdots \wedge e_{u-1} \wedge e_{m+u-1} \wedge e_u \wedge e_{u+1} \wedge e_{u+2}$$

resulting in $\varrho(z, \gamma(z))^2 = 3\,\varrho(y, \gamma(y))^2 \neq 0$. This contradicts the hypothesis that γ is a CLIFFORD translation; thus $q = 1$ if $q \leqq p$. We have now proved that $\Gamma \cap \alpha \cdot G$ is empty unless M is complex projective space $\mathbf{P}^{2m-1}(\mathbf{C})$, and that any element of $\Gamma \cap \alpha \cdot G$ is conjugate to αJ in that case.

Let $p = q$ and let $\gamma = \beta\alpha g \in \Gamma \cap \beta\alpha \cdot G$. Then $\gamma^2 = \beta\alpha g\beta\alpha g = \beta^2\alpha g\alpha g = \alpha g \alpha g = {}^t g^{-1} g = 1$ is represented by a scalar matrix, and an argument above shows that $g = \pm\, {}^t g$. If $g = -\,{}^t g$, then we may assume $g = J$ as above, and a fixed point for γ is given by $e_1 \wedge \ldots \wedge e_p$; thus $g = {}^t g$ and we may assume (as above) that g is diagonal. Examining q-planes spanned by subsets of $\{e_j\}$, q is odd because γ has no fixed point and then $q = 1$ because γ is a CLIFFORD translation. We have assumed $pq > 1$ because

$$\mathbf{U}(2)/\{\mathbf{U}(1) \times \mathbf{U}(1)\}$$

is a 2-sphere; thus $\Gamma \cap \beta\alpha \cdot G$ is empty. In particular, Γ cannot meet both $\beta \cdot G$ and $\alpha \cdot G$.

We have proved that $\Gamma = \{1\}$ or $\Gamma \subset \{1, \beta\}$ if M is not an odd dimensional complex projective space. In that case, then, Γ centralizes G. If M is an odd dimensional complex projective space, and if Γ does not centralize G, then we have shown that $\Gamma = \{1, \alpha J\}$; the centralizer of αJ in $G = \mathbf{SU}(2m)/\text{scalars}$ is $\mathbf{Sp}(m)/\{\pm I\}$, which acts transitively on M, so M/Γ is RIEMANNian homogeneous in any case.

5.5.6. $M = \mathbf{SO}(2n)/\mathbf{U}(n)$ with $n > 2$. M is the space of unitary structures on $\mathbf{R}^{2n}$ compatible with a given EUCLIDEAN structure. $G = \mathbf{SO}(2n)/\{\pm I\}$, $K = \mathbf{U}(n)/\{\pm I\}$, and the symmetry $s = \pm\,\mathrm{diag.}\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$. The only globally defined outer automorphism of K extends to G, and is induced from conjugation by $a = \pm\,\mathrm{diag.}\,\{1, -1; \ldots; 1, -1\}$. $\mathbf{I}(M) = G \cup \alpha \cdot G$ where conjugation by α is conjugation by a. This is inner on G if and only if n is even. Thus Γ centralizes G by Corollary 5.2.3 if n is even.

Now suppose that n is odd; then $\mathbf{I}(M) = \mathbf{O}(2n)/\{\pm I\}$ with α and a identified. Let $1 \neq \gamma \in \Gamma$; then γ is represented by a matrix of square $\pm I$

and determinant -1; using the fact that γ is not conjugate to s or to a, it follows that γ is conjugate to some $\pm\begin{pmatrix} I_{2n-v} & \\ & -I_v\end{pmatrix}$ with v odd and $v < n$. Let p be the point at which K is isotropy; if $v \geqq 3$, it follows that

$$\pm \text{diag.}\{1, -1; 1, \ldots, 1\} \text{ and } \pm \text{diag.}\{1, -1, 1, -1, 1, -1; 1, \ldots, 1\}$$

move p the same distance because they differ from a conjugate of γ by an element of K. This is easily seen impossible; thus $v = 1$ and the connected centralizer L of Γ in G is the $\mathbf{SO}(2n-1)$ acting on the first $2n-1$ coordinates. It is well known that K is transitive on the sphere $\mathbf{SO}(2n)/L$; thus $KL = G$. This proves $LK = G$, so L is transitive on M, and M/Γ is RIEMANNian homogeneous.

5.5.7. $M = \mathbf{SO}(p+q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$ with p even, q even, $p \geqq 2$, $q \geqq 2$ and $p + q > 4$. We set aside the case $p = 4 = q$ for § 5.5.8. M is the GRASSMANN manifold of oriented q-planes in an oriented $\mathbf{R}^n$ where $n = 2m = p + q$. $\mathbf{O}(n)$ acts on M in the obvious fashion with kernel $\{\pm I\}$. $G = \mathbf{SO}(n)/\{\pm I\}$ and we define $G' = \mathbf{O}(n)/\{\pm I\}$. $\alpha \in \mathbf{I}(M)$ is defined by $x \to -x$ (opposite orientation); if $p = q$, then $\beta \in \mathbf{I}(M)$ is defined by $x \to x^{\perp}$ with $x^{\perp}$ oriented so that $x \wedge x^{\perp}$ gives the orientation of $\mathbf{R}^n$. α centralizes G' and commutes with β, β centralizes G, and it is not difficult to verify from the following paragraph that $g\beta g^{-1} = \beta\alpha$ if $g \in G'$ has determinant -1.

$K = \{\mathbf{SO}(p) \times \mathbf{SO}(q)\}/\{\pm I\}$. If $p \neq q$, then the only outer automorphism of K which extends to G is induced by, say, $\pm$ diag. $\{-1; 1, \ldots, 1\} \in G'$. If $p = q$, there is this automorphism, the interchance of factors $\mathbf{SO}(p)$ induced by β, and their product. The only other possibility, in view of our exclusion of $p = 4 = q$, would be from triality automorphism on a factor $\mathbf{SO}(8)$ of K (excluded because triality is not well defined on $\mathbf{SO}(8)$) or a permutation of factors $\mathbf{SO}(3)$ of a factor $\mathbf{SO}(4)$ of K (excluded because it couldn't extend to G). Thus $\mathbf{I}(M) = G' \cup \alpha \cdot G'$ if $p \neq q$, and $\mathbf{I}(M) = G' \cup \alpha \cdot G' \cup \beta \cdot G' \cup \beta\alpha \cdot G'$ if $p = q$.

Both p and q being even, rank. $K =$ rank. G and $\Gamma \cap G = \{1\}$. If $\Gamma \subset G \cup \alpha \cdot G$ when $p \neq q$ and $\Gamma \subset G \cup \alpha \cdot G \cup \beta \cdot G \cup \beta\alpha \cdot G$ when $p = q$, then Γ centralizes $\mathbf{I}_0(M) = \mathrm{G}$ by Theorem 5.2.2; we will see that this is the case. If not, then we would have $\gamma = \delta g \in \Gamma$ with $\delta \in \{1, \alpha\}$ if $p \neq q$, $\delta \in \{1, \alpha, \beta, \alpha\beta\}$ if $p = q$, and det. $g = -1$. If δ is β or $\alpha\beta$, then $\gamma^2 = \alpha$, using Theorem 5.2.2, and it follows that g is conjugate in G to $\pm\begin{pmatrix} I_{n-v} & \\ & -I_v\end{pmatrix}$ with v odd. A short calculation shows then that γ cannot be a CLIFFORD translation because $q > 1$. Now let $\delta = \alpha$. $\gamma^2 = \delta g \delta g = \delta^2 g^2 = g^2 = \pm I$, so $g^2 = I$ because $\det. g = -1$. Conjugating by an element of G, we may assume $g = \pm\begin{pmatrix} I_u & \\ & -I_v \end{pmatrix}$ in an orthonormal basis $\{e_j\}$ with u and v odd; this comes from conjugation of γ because δ is central in $\mathbf{I}(M)$. It is easy to choose q of the e_j where the number of subscripts $j > u$ is even; these e_j span a fixed point for g on M. $g \neq \pm I$ now contradicts Theorem 5.2.2; thus $\Gamma \subset G \cup \alpha \cdot G$ if $p \neq q$ and $\Gamma \subset G \cup \alpha \cdot G \cup \beta \cdot G \cup \alpha\beta \cdot G$ if $p = q$. If follows that Γ centralizes in $\mathbf{I}_0(M)$.

5.5.8. $M = \mathbf{SO}(8)/\mathbf{SO}(4) \times \mathbf{SO}(4)$. The situation is much the same as in § 5.5.7, except that the triality automorphism of $G = \mathbf{SO}(8)/\{\pm I\}$ induces additional automorphisms of K. Retaining the notation of § 5.5.7, let G'' be the subgroup $G' \cup \alpha \cdot G' \cup \beta \cdot G' \cup \alpha\beta \cdot G'$ of $\mathbf{I}(M)$; $\mathbf{I}(M)$ has an element τ of order 3 such that conjugation by τ is the triality automorphism of G, and induces an outer automorphism of K not induced by an element of G''. The group of outer automorphisms of K is isomorphic to the symmetric group on 4 letters, being the group of permutations of the 4 local factors $\mathbf{SO}(3)$ of $K = \{\mathbf{SO}(4) \times \mathbf{SO}(4)\}/\{\pm I_8\}$, and thus has order 24; it follows that $\mathbf{I}(M) = G'' \cup \tau \cdot G'' \cup \tau^2 \cdot G''$ and has 24 components. In view of § 5.5.7, we wish to show $\Gamma \subset G''$; it will then follow that Γ centralizes G. As τ has order 3, it suffices to show that no element of Γ has order 3. This will follow from:

5.5.9. Lemma. *Let γ be a self-homeomorphism of $M = \mathbf{SO}(8)/\mathbf{SO}(4) \times \mathbf{SO}(4)$ of period 3. Then γ has a fixed point on M.*

Remark. The idea of the proof is to calculate the LEFSCHETZ number $L(\gamma) = \Sigma(-1)^j \operatorname{trace}. \gamma_j$, where γ_j is the linear automorphism of $\mathbf{H}^j(M; \mathbf{R})$ induced by γ, and use the LEFSCHETZ Fixed Point Theorem, which says that γ has a fixed point if $L(\gamma) \neq 0$. By means of the HIRSCH formula it is easy to see that ($b_j = \dim. \mathbf{H}^j(M; R)$) $b_0 = b_{16} = 1$, $b_4 = b_{12} = 3$, $b_8 = 4$, $b_j = 0$ if $j > 16$, and $b_j = 0$ if $j \not\equiv 0 \pmod 4$. Now $\gamma^3 = 1$ implies that γ_j has order 3 or 1, whence, if we define $R(\vartheta) = \begin{pmatrix} \cos(2\pi\vartheta) & \sin(2\pi\vartheta) \\ -\sin(2\pi\vartheta) & \cos(2\pi\vartheta) \end{pmatrix}$, each γ_j is a direct sum of matrices (1) or $R(1/3)$. Thus $\operatorname{trace}. \gamma_0 = \operatorname{trace}. \gamma_{16} = 1$, $\operatorname{trace}. \gamma_8 = 4, +1$ or -2, and $\operatorname{trace}. \gamma_4 = \operatorname{trace}. \gamma_{12} = 0$ or 3. It follows that $L(\gamma) = 0$ if and only if $\operatorname{trace}. \gamma_4 = 0$ and $\operatorname{trace}. \gamma_8 = -2$, i.e., if and only if γ_4 is similar to $\begin{pmatrix} R(1/3) & \\ & 1 \end{pmatrix}$ and γ_8 is similar to $\begin{pmatrix} R(1/3) & \\ & R(1/3) \end{pmatrix}$. C.T.C. WALL has kindly shown me that γ_8 has an eigenvalue $+1$; it follows that $L(\gamma) \neq 0$, so γ has a fixed point on M. This is a consequence of:

5.5.10. Lemma (C. T. C. Wall). *Let ξ be a self-homeomorphism of $M = \mathbf{SO}(8)/\mathbf{SO}(4) \times \mathbf{SO}(4)$ such that some power of ξ is homotopic to* 1, *and let ξ_8 be the induced linear transformation of $\mathbf{H}^8(M; \mathbf{R})$. Then ξ_8 has an eigenvalue $+1$.*

Proof. Let $L = \mathbf{SO}(8)$ and $U = \mathbf{SO}(4) \times \mathbf{SO}(4)$, and choose a maximal torus T of L such that $T \subset U \subset L$. The Weyl groups W_L and W_U act on the classifying space B_T, hence on $\mathbf{H}^*(B_T; \mathbf{R})$. $\mathbf{H}^*(B_T; \mathbf{R})$ is a polynominal ring on 4 generators x_j of degree 2. It is well-known to follow from [1, Prop. 30.1] that $\mathbf{H}^*(L/U; \mathbf{R})$ is isomorphic to the quotient of the ring I_U of W_U-invariant elements of $\mathbf{H}^*(B_T; \mathbf{R})$ by the ideal I'_L generated by the W_L-invariant elements of degree > 0. I_U is generated by

$$x_1^2 + x_2^2,\ x_1 x_2,\ x_3^2 + x_4^2 \text{ and } x_3 x_4;$$

I'_L is generated by $\Sigma_i\, x_i^2$, $\Sigma_{i<j}\, x_i^2 x_j^2$, $\Sigma_{i<j<k}\, x_i^2 x_j^2 x_k^2$ and $x_1 x_2 x_3 x_4$; it follows that $\mathbf{H}^*(M = L/U; \mathbf{R})$ is the graded algebra on generators $a = x_1 x_2 + x_3 x_4$, $b = x_1 x_2 - x_3 x_4$ and $c = x_1^2 + x_2^2$ of degree 4 with relations $a^2 = b^2 = c^2$ and $abc = 0$. Thus $\mathbf{H}^j(M; \mathbf{R})$ has an additive basis $\{1\}$ if $j = 0$, $\{a, b, c\}$ if $j = 4$, $\{a^2, ab, bc, ca\}$ if $j = 8$, $\{a^3, b^3, c^3\}$ if $j = 12$, $\{a^4\}$ if $j = 16$, and zero otherwise.

Now observe that, for $y = a, b$, or c, $\mathbf{H}^4(M; \mathbf{R})$ is spanned by elements z such that $z^2 = y^2$. We will prove: *If $y \in \mathbf{H}^4(M; \mathbf{R})$ such that $\mathbf{H}^4(M; \mathbf{R})$ is spanned by elements z with the property that $z^2 = y^2$, then y is a nonzero multiple of a, b or c. To see this, let $y = \lambda a + \mu b + \nu c$*, and suppose that $z = \alpha a + \beta b + \gamma c$ with $y^2 = z^2$. Then $\lambda^2 + \mu^2 + \nu^2 = \alpha^2 + \beta^2 + \gamma^2$, $\lambda\mu = \alpha\beta$, $\lambda\nu = \alpha\gamma$ and $\mu\nu = \beta\gamma$. It is now easily checked that the elements z span $\mathbf{H}^4(M; \mathbf{R})$ if and only if precisely one of λ, μ, ν is nonzero.

Suppose that η is an automorphism of $\mathbf{H}^*(M; \mathbf{R})$. The preceding paragraph shows that $\eta(a)$ is a nonzero multiple of a, b or c. Thus $\eta(a)^2 = \alpha a^2$ with $\alpha > 0$. If η is of finite order, this implies $\eta(a)^2 = \eta(a^2) = a^2$, whence η has an eigenvalue $+1$ on $\mathbf{H}^8(M; \mathbf{R})$. The Lemma is now proved by taking η to be the automorphism of $\mathbf{H}^*(M; \mathbf{R})$ induced by ξ. *Q.E.D.*

5.5.11. $M = \mathbf{SO}(p+q)/\mathbf{SO}(p) \times \mathbf{SO}(q)$ with p odd, p odd, $p > 2$ and $q > 2$. M is the Grassmann manifold of oriented q-planes in an oriented $\mathbf{R}^m$, where $m = 2n = p + q$. $\mathbf{O}(m)$ acts effectively on M by isometries, so $G = \mathbf{SO}(m)$, $G' = \mathbf{O}(m)$ and $K = \mathbf{SO}(p) \times \mathbf{SO}(q)$. As p and q are odd, the symmetry $s = \begin{pmatrix} -I_p & \\ & I_q \end{pmatrix} \in G', s \notin G$, whence $G' = G \cup s \cdot G$. If $p \neq q$, then K has no outer automorphism and $\mathbf{I}(M) = G'$. If $p = q$, then the only outer automorphism of K is the exchange of the two factors $\mathbf{SO}(p)$; it then follows that $\mathbf{I}(M) = G' \cup \beta \cdot G'$ where $\beta : M \to M$ by $x \to x^\perp$. $x^\perp$ is oriented such that $x \wedge x^{\perp}$ gives the original orientation of $\mathbf{R}^m$. β centralizes G'. As $p = q$ is odd, we have $x \wedge x^{\perp} = - x^{\perp} \wedge x$; thus $\beta^2 = - I \in G'$.

The LIE algebra $\mathfrak{G}$ has basis $\{X_{ij}\}_{1 \leqq i < j \leqq m}$ where $X_{ij} = E_{ij} - E_{ji}$, and the subalgebra $\mathfrak{K}$ corresponding to K has basis

$$\{X_{ij}\}_{1 \leqq i < j \leqq p} \cup \{X_{ij}\}_{p < i < j \leqq m}.$$

The KILLING form on $\mathfrak{G}$ is proportional to the positive definite form $B(X, Y) = - \frac{1}{2}\,\mathrm{trace}(XY)$, for both are $Ad(G)$-invariant forms on $\mathfrak{G}$. Using B for the metric on M, $\{X_{ij}\}_{1 \leqq i \leqq p < j \leqq m}$ is an orthonormal basis of $\mathfrak{P}$. It follows that $\varrho(e_{j_1} \wedge \ldots \wedge e_{j_q},\ (\cos\vartheta_1 \cdot e_{j_1} + \sin\vartheta_1 \cdot e_{k_1}) \wedge \ldots \wedge (\cos\vartheta_q \cdot e_{j_q} + \sin\vartheta_q \cdot e_{k_q}))^2 = = \Sigma\, \vartheta_j^2$ if $|\vartheta_j| \leqq \pi$ and if the i_q, together with those k_q for which $\sin\vartheta_q \neq 0$, form a set of distinct indices.

Let $g \in G' = \mathbf{O}(m)$ be a CLIFFORD translation of M. Then $g \in \mathbf{SO}(m)$ by Lemma 5.3.1. Thus g can be conjugated in G' and we can assume $g = \mathrm{diag.}\,\{R(\vartheta_1), \ldots, R(\vartheta_n)\}$ where $0 \leqq \vartheta_1 \leqq \ldots \leqq \vartheta_n \leqq \pi$ and where $R(\vartheta) = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}$. Let $p = 2u - 1$ and let $x = e_{p+1} \wedge \ldots \wedge e_m$. Then $g(x) = (-\sin\vartheta_u \cdot e_{2u-1} + \cos\vartheta_u \cdot e_{2u}) \wedge e_{2u+1} \wedge \ldots \wedge e_m$, giving $\varrho(x, g(x))^2 = \vartheta_u^2$, so g is of constant displacement ϑ_u. Thus, permuting basis vectors, we see $\vartheta_1 = \vartheta_2 = \ldots = \vartheta_n$; let ϑ denote the common value of the ϑ_j; $0 \leqq \vartheta \leqq \pi$ and

$$g = \mathrm{diag.}\,\{R(\vartheta), \ldots, R(\vartheta)\}.$$

If q were 1, i.e., if M were an odd dimensional sphere, we could say nothing more about g; that is why the odd dimensional spheres are the most complicated GRASSMANN manifolds from the viewpoint of CLIFFORD translations. As $q > 1$, we have $y = e_1 \wedge e_3 \wedge e_5 \wedge e_6 \wedge \ldots \wedge e_{q+2} \in M$. $g(y) = (\cos\vartheta \cdot e_1 + \sin\vartheta \cdot e_2) \wedge (\cos\vartheta \cdot e_3 + \sin\vartheta \cdot e_4) \wedge e_5 \wedge e_6 \wedge \ldots \wedge e_{q+1} \wedge (\cos\vartheta \cdot e_{q+2} + \sin\vartheta \cdot e_{q+3})$. If $\sin\vartheta \neq 0$, this would give $\varrho(y, g(y))^2 = 3\,\vartheta^2 \neq \vartheta^2$, which is impossible because g is of constant displacement ϑ. Thus $\sin\vartheta = 0$, so $g = \pm I$. It follows that $\Gamma \cap G' \subset \{\pm I\}$. Thus Γ is central in G if $p \neq q$.

If $p = q$, suppose that $\gamma \in \Gamma \cap \beta \cdot G'$, say $\gamma = \beta g$ with $g \in G'$. $\gamma^2 = = \beta g \beta g = \beta^2 g^2 = - g^2 \in \Gamma \cap G$, so $g^2 = \pm I$. $g^2 = - I$ would imply that we could conjugate γ and assume $g = \begin{pmatrix} 0 & -I_p \\ I_p & 0 \end{pmatrix}$, whence γ would have a fixed point $e_{p+1} \wedge \ldots \wedge e_{2p=m}$ on M. Thus $g^2 = I$ and we may assume $g = \begin{pmatrix} I_r & \\ & -I_t \end{pmatrix}$. We define $x = e_1 \wedge \ldots \wedge e_p$ and $y = (e_1 + e_{p+1}) \wedge \ldots \wedge (e_p + e_{2p}) \in M$. If $r - p = u \geqq 0$, then $\gamma(x) = e_{p+1} \wedge \ldots \wedge e_{2p}$ and $\gamma(y) = \pm (e_1 - e_{p+1}) \wedge \ldots \wedge (e_u - e_{p+u}) \wedge (e_{u+1} + e_{p+u+1}) \wedge \ldots \wedge (e_p + e_{2p})$, whence $\varrho(x, \gamma(x))^2 = p(\pi/2)^2$ and $\varrho(y, \gamma(y))^2 = u(\pi/2)^2$. Thus $p = u$ and $g = I$. Similarly, if

$t - p = v \geqq 0$, then $p = v$ and $g = -I$. Thus $\gamma = \pm \beta$. It follows that $\Gamma \subset \{\pm I, \pm \beta\}$, and consequently centralizes G.

5.5.12. $M = E_6/D_5 \times T^1$. Here we need only know [11, § 90] that $\mathbf{I}(M)$ has exactly two components. The EULER characteristic $\chi(M)$ is the quotient of the order $2^7.3^4.5$ of the WEYL group of $\mathbf{E}_6$ by the order $2^7.3.5$ of the WEYL group of $D_5 \times T^1$. Thus $\chi(M) = 27$, so $\chi(M/\Gamma) = 27/(\text{order of } \Gamma)$ because $M \to M/\Gamma$ is a covering. Thus the order of Γ divides 27. But $\Gamma \cap G = \{1\}$ because rank. $G =$ rank. K, whence the order of Γ divides the order 2 of $\mathbf{I}(M)/G$. It follows that $\Gamma = \{1\}$, which is central in $\mathbf{I}(M)$.

5.5.13. $M = E_6/A_5 \times A_1$. Here again we must know [11, §90] that $\mathbf{I}(M) = G \cup \alpha \cdot G$. $G = ad(\mathbf{E}_6)$, $s \in K$, and we may assume that $K \cup \alpha \cdot K$ is the isotropy subgroup of $\mathbf{I}(M)$.

Conjugation by α induces an outer automorphism of K; we must check that it induces an outer automorphism of G. If α induces an inner automorphism of G, then we have $a \in G$ which induces the same automorphism of G, and thus commutes with s. As the universal covering $\mathbf{E}_6 \to G$ has multiplicity 3, we can take $s' \in \mathbf{E}_6$ over $s \in G$, and cube it if necessary, to ensure that $s'^2 = 1$. We define $B' = \{g \in \mathbf{E}_6 : [s', g] = s'gs'g^{-1}$ is central in $\mathbf{E}_6\}$. Now suppose $g \in B', h = [s', g], h \neq 1$. Then $s'g = hgs'$, whence $g = s's'g = s'hgs' = hs'gs' = hhgs's' = h^2g$, so $h = 1$ because $\mathbf{E}_6$ has center cyclic order 3. Thus B' is the centralizer of s' in $\mathbf{E}_6$. As $\mathbf{E}_6$ is simply connected and s' has order 2, it follows [11, § 101] that B' is connected (also, see [6, Prop. 3.11]). B' is the full inverse image of the centralizer B of s in G; B is thus connected. This implies $a \in K$, which is impossible because α induces an outer automorphism of K. It follows that α induces an outer automorphism of $G = ad(\mathbf{E}_6)$; this proves that the centralizer of α in G is of rank 4.

As A_1 has no outer automorphism, α must centralize a torus $T^1 \subset A_1$, and α must induce an outer automorphism of A_5. Thus α centralizes a torus $T^3 \subset A_5$. It follows that K contains a maximal torus $T^3 \cdot T^1 = T^4$ of the centralizer of α in G. By Lemma 5.3.2, every element of $\alpha \cdot G$ has a fixed point on M, and $\Gamma \subset G$. But every element of G has a fixed point because rank. $G =$ rank. K. Thus $\Gamma = \{1\}$, which is central in $\mathbf{I}(M)$.

5.5.14. $M = \mathbf{E}_6/(\mathbf{F}_4$ or $ad(\mathbf{C}_4))$. $G = \mathbf{E}_6$ and K is either $\mathbf{F}_4$ or $ad(\mathbf{C}_4) = \mathbf{Sp}(4)/\{\pm I\}$; $\mathbf{I}(M) = G \cup s \cdot G$ because $s \notin G$ and K has no outer automorphism. Thus $\Gamma \subset G$ by Lemma 5.3.1.

Let A be a connected subgroup of type $A_1 \times A_5$ in $\mathbf{E}_6$ [4, p. 219]; replacing A by a conjugate we have $sAs^{-1} = A$. Here s is the symmetry at $x \in M$;

it follows that $N = A(x)$ is a connected totally geodesic submanifold of M (and thus RIEMANNian symmetric) and that the restriction $a \to a|_N$ defines a homomorphism $f: A \to \mathbf{I}(N)$. $f(A) = \mathbf{I}_0(N)$ because A is connected and semisimple, and because $f(A)$ induces every transvection from x along N. Let D be the center of $\mathbf{E}_6$; D is cyclic of order 3, acts freely on M, and is contained in A. It follows that $f(D)$ is a central cyclic subgroup of order 3 in $\mathbf{I}_0(N)$.

Let B be the kernel of f. As $B \subset K$, rank. $B \leqq 4$; thus B is discrete or of type A_1, and $\mathbf{I}_0(N)$ is either of type $A_1 \times A_5$ or of type A_5. This implies [9] that the universal RIEMANNian covering of N is a product $N_1 \times N_2$ where N_1 is a single point or a sphere $\mathbf{S}^2$, and where N_2 is a complex GRASSMANN manifold, $\mathbf{SU}(6)/\mathbf{SO}(6)$ or $\mathbf{SU}(6)/\mathbf{Sp}(3)$. $f(A \cap K)$ is the isotropy subgroup of $\mathbf{I}_0(N)$ at x, and has rank $\leqq 4$; thus N_2 is not a complex GRASSMANN manifold. Looking at the various possibilities, now, we see that $f(D)$ is the center of $\mathbf{I}_0(N)$.

Let $g \in A$ be a CLIFFORD translation of M. As N is totally geodesic in M, $f(g)$ is a CLIFFORD translation of N. If $N_2 = \mathbf{SU}(6)/\mathbf{SO}(6)$, then it follows that $f(g)$ is central in $\mathbf{I}_0(N)$, and we conclude that $f(g) \in f(D)$. This implies that $g = dk$ with $d \in D$ and $k \in K$; now $g \in D$ by Theorem 5.2.2.

Suppose that $N_2 = \mathbf{SU}(6)/\mathbf{Sp}(3)$. The projection of N_2 into N is one to one because $\mathbf{I}_0(N)$ has a central element of order 3. Thus $N_2 = A_5(x)$ is a totally geodesic submanifold of M; we have a projection $f': A \to \mathbf{I}(N_2)$ given by restriction; $f'(A) = \mathbf{I}_0(N_2)$, $f'(D)$ is the center of $\mathbf{I}_0(N_2)$, and $f'(g)$ is a CLIFFORD translation of N_2. $\mathbf{I}_0(N_2)$ is isomorphic to $\mathbf{SU}(6)/\{\pm I\}$; making the identification and viewing $\mathbf{SU}(6)$ as acting on $\mathbf{C}^6$, § 5.5.4 provides an orthonormal basis of $\mathbf{C}^6$ for which $f'(g) = \pm$ diag. $\{a'; a, \ldots, a\}$. This choice of orthonormal basis of $\mathbf{C}^6$ amounts to a choice of maximal torus in $\mathbf{I}_0(N_2)$, which in turn is a choice of maximal torus T^5 in the A_5 factor of A such that $T^5 \subset T^6$ where T^6 is a maximal torus of A which we may assume to contain g. If $b \in G$ normalizes T^6, then it follows that $f'(bgb^{-1}) = \pm$ $\pm$ diag. $\{a, \ldots, a;\ a';\ a, \ldots, a\}$.

Let $\mathfrak{T}$ be the LIE algebra of the torus $T = T^6$ chosen above. $\mathbf{E}_6$ has a subgroup of type $D_5 \times T^1$ in which T is a maximal torus. In particular, $\mathfrak{T}$ is a sum $\mathfrak{U} + \mathfrak{V} + \mathfrak{W}$ of 2-dimensional subspaces, and G has elements u and v which normalize T, such that $ad(u)|_{\mathfrak{U}}$ and $ad(v)|_{\mathfrak{V}}$ are the identity transformations, and $ad(u)|_{\mathfrak{V}+\mathfrak{W}}$ and $ad(v)|_{\mathfrak{U}+\mathfrak{W}}$ are $-I$. We can make a choice $\mathfrak{X}$ of $\mathfrak{U}, \mathfrak{V}$ or $\mathfrak{U} + \mathfrak{V}$ such that $\mathfrak{Y} = \mathfrak{X} \cap \mathfrak{T}^5$ has the property: $2 \leqq \dim. \mathfrak{Y} \leqq 3$; if y is the corresponding element, u, v or uv of G, then $f'(ygy^{-1}) = \pm$ diag. $\{a, \ldots, a;\ a';\ a, \ldots, a\}$ implies $a = a'$. Thus $f'(g) \in f'(D)$, and, as with the other possibility of N_2, $g \in D$.

Let $\gamma \in \Gamma$. Then $\gamma \in G$, so $\gamma = hgh^{-1}$ for some $h \in G$ and some $g \in A$. g is a CLIFFORD translation of M, and we have just seen that this implies $g \in D$. Now $\gamma = g$ is central in G. Thus Γ is central in $\mathbf{I}_0(M)$.

5.5.15. The statement of Theorem 5.5.1 has now been verified for every compact connected simply connected RIEMANNian symmetric space with simple group of isometries. *Q.E.D.*

6. The main theorems

6.1. CLIFFORD translations and homogeneity

Our main result, an immediate consequence of Theorem 3.3, Corollary 4.5.3 and Theorem 5.5.1, is:

Theorem: *Let Γ be the group of deck transformations of the universal RIEMANNian covering $\pi: M \to N$ of a complete connected locally symmetric RIEMANNian manifold N. Then N is a RIEMANNian homogeneous manifold if and only if Γ is a group of CLIFFORD translations of M.*

6.2. Symmetry of locally symmetric homogeneous spaces

From Theorems 3.3 and 5.5.1 we have:

Theorem: *Let N be a connected locally symmetric RIEMANNian homogeneous manifold such that, in CARTAN's symmetric space decomposition of the universal RIEMANNian covering manifold of N, none of the compact irreducible factors is a group manifold, on odd dimensional sphere, a complex projective space of odd complex dimension > 1, $\mathbf{SU}(2m)/\mathbf{Sp}(m)$ with $m > 1$, or $\mathbf{SO}(4n+2)/\mathbf{U}(2n+1)$ with $n > 0$. Then N is a RIEMANNian symmetric manifold.*

As we have seen, this theorem does not remain true if we drop any of the restrictions on the factors of the universal covering manifold of N.

6.3. CLIFFORD translations and symmetry

From Theorems 3.3 and 5.5.1, or from Theorems 6.1 and 6.2, we have:

Theorem: *Let Γ be the group of deck transformations of a universal RIEMANNian covering $\pi: M \to N$ of a complete connected locally symmetric RIEMANNian manifold N, and suppose that none of the compact irreducible factors in CARTAN's symmetric space decomposition of M is a group manifold, an odd dimensional sphere, a complex projective space of odd complex dimension > 1,*

$\mathbf{SU}(2m)/\mathbf{Sp}(m)$ *with* $m > 1$, *or* $\mathbf{SO}(4n+2)/\mathbf{U}(2n+1)$ *with* $n > 0$. *Then* N *is a* RIEMANN*ian symmetric manifold if and only if* Γ *is a group of* CLIFFORD *translations of* M.

6.4. The fundamental group of a symmetric space

Theorem: *Let* N *be a* RIEMANN*ian symmetric manifold. Given* $x \in N$, *the fundamental group* $\pi_1(N, x)$ *is abelian.*

Remark. My original proof depended on a reduction to the case where N is locally irreducibile and then on the results of § 3–5. The proof given below is due to H. SAMELSON; it is a considerable improvement.

Proof. It suffices to prove that every element $\zeta \in \pi_1(N, x)$ can be represented by an arc of a closed geodesic γ_ζ through x. For then the symmetry s_x reverses orientation of γ_ζ and consequently induces an automorphism $\zeta \to \zeta^{-1}$ of $\pi_1(N, x)$. If the transformation $a \to a^{-1}$ of a group A is automorphism, then A is abelian. Thus $\pi_1(N, x)$ will be abelian.

In order to represent $\zeta \in \pi_1(N, x)$ by a closed geodesic, we first represent it by a geodesic arc δ_ζ of minimal length L_ζ based at x. If η is the free homotopy class of ζ, then homogeneity of N shows that L_ζ is minimal for the length of a closed curve representing η. If δ_ζ had a corner at x, then rounding that corner would give a representative of η of length less than L_ζ, which is impossible. Thus δ_ζ is an arc of a closed geodesic γ_ζ through x, and the Theorem is proved. Observe that this paragraph is equivalent to [20, Th. 2].

Q.E.D.

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