On the Lattice Space Forms

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1. Introduction

In a recent paper [1] we gave a global classification for homogeneous pseudo-RIEMANNian manifolds of constant nonzero curvature. We described certain families of such manifolds, and proved [1, Theorem 12] that a complete connected homogeneous pseudo-RIEMANNian manifold of constant positive curvature is isometric to an element of one of those families; for negative curvature, one replaces the metric by its negative. The most complicated of those families are the families $\mathcal{L}$, $\mathcal{L}_3$, $\mathcal{L}_4$, $\mathcal{L}_6$ and $\mathcal{L}_8$, the families of so-called lattice space forms. The purpose of this note is to improve their description, examine their interrelations, and correct a minor error in their enumeration.

We first give a criterion (Theorem 3.1) for two manifolds, each isometric to an element of $\mathcal{L}$ or to an element of an $\mathcal{L}_m$, to be isometric to each other, i.e., to be equal as elements of $\mathcal{L}$ or $\mathcal{L}_m$. We then decompose $\mathcal{L}$ into a disjoint union of subsets $\mathcal{L}(+, h, n)$, $\mathcal{L}(-, h, n)$ and $\mathcal{L}(\pm, h, n)$, and decompose $\mathcal{L}_m$ into a disjoint union of subsets $\mathcal{L}_m(h, n)$; here (h, n) refers to the dimension n and the signature of metric $ds^2 = -\sum_1^h dx_i^2 + \sum_{h+1}^n dx_i^2$ of the manifolds under consideration. Our first main result (Theorem 4) is that $\mathcal{L}(+, h, n)$, $\mathcal{L}(\pm, h, n)$ and $\mathcal{L}_4(h, n)$ are each in one to one correspondence in a natural fashion with the quotient of the upper half plane by the modular group, and that $\mathcal{L}(-, h, n)$ is in one to one correspondence with the quotient of the upper half plane by a maximal parabolic (= unipotent) (mod 2) subgroup of the modular group. The other main result (Theorem 5.2) is that $\mathcal{L}_3(h, n)$, $\mathcal{L}_6(h, n)$ and $\mathcal{L}_8(h, n)$ each has just one element.

2. Preliminaries

2.1. Let $\mathbf{R}_h^{n+1}$ be an $(n+1)$-dimensional real vectorspace with a symmetric bilinear form Q of signature $(h - 's,\ (n-h+1) + 's)$, and let $\mathbf{O}^h(n+1)$ denote the orthogonal group of Q. An *orthonormal basis* of $\mathbf{R}_h^{n+1}$ is an ordered basis $\{v_1, \ldots, v_{n+1}\}$ such that $Q(v_i, v_j) = \delta_{ij}$ if $i > h$, $= -\delta_{ij}$ if $i \leqq h$. A *skew basis* of $\mathbf{R}_h^{n+1}$ is, if $2h \geqq n+1$, an ordered basis

$$\{f_1, \ldots, f_k;\ v_{k+1}, \ldots, v_h;\ e_1, \ldots, e_k\}$$

[1]) The author was supported by a National Science Foundation fellowship while preparing this paper.

where $k = n - h + 1$, $Q(f_i, e_j) = 2\delta_{ij} = -2Q(v_i, v_j)$ and $Q(f_i, f_j) = Q(e_i, e_j) = Q(f_i, v_p) = Q(v_p, e_j) = 0$. A similar definition holds if $2h \leqq n + 1$. There is a one to one correspondence between orthonormal bases and skew bases which sends an orthonormal basis $\{v_i\}$ to

$$\{v_{h+1} - v_1, \ldots, v_{h+k} - v_k;\ v_{k+1}, \ldots, v_h;\ v_{h+1} + v_1, \ldots, v_{h+k} + v_k\}$$

when $2h \geqq n + 1$.

If A is a linear transformation of $\mathbf{R}_h^{n+1}$ and β is a basis of $\mathbf{R}_h^{n+1}$, then A_β will denote the matrix of A relative to β. J_2 will denote the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and, if p is even, J_p will denote the matrix $\begin{pmatrix} J_2 & & \\ & \ddots & \\ & & J_2 \end{pmatrix}$ of order p. I will denote the identity transformation, and I_q will denote the identity matrix of order q. Finally, if p is even, $R(\theta)_p$ will denote the rotation matrix

$$\cos(2\pi\theta)\, I_p + \sin(2\pi\theta)\, J_p.$$

2.2. Now suppose that $2h > n$ and $k = n + 1 - h \equiv 0 \pmod 4$, and choose a skew basis β of $\mathbf{R}_h^{n+1}$. Given an antisymmetric nonsingular $k \times k$ matrix d, we have $t(d) \in \mathbf{O}^h(n+1)$ defined by $t(d)_\beta = \begin{pmatrix} I_k & 0 & 2d \\ 0 & I_{h-k} & 0 \\ 0 & 0 & I_k \end{pmatrix}$. Now suppose that $dJ_k + J_k d = 0$; this requires $k \equiv 0 \pmod 4$. We identify the complex number field $\mathbf{C}$ with the matrices $(aI_k + bJ_k)$ by $a + \sqrt{-1}\,b \to (aI_k + bJ_k)$, and observe that ud is an antisymmetric nonsingular matrix for every $0 \neq u \in \mathbf{C}$. Let ε be one of the signs $+$ or $-$ or $\pm$. Given $u \in \mathbf{C}$, $u \notin \mathbf{R}$, we define $\mathfrak{L}(u, \varepsilon)$ to be the subgroup of $\mathbf{O}^h(n+1)$ generated by $t(d)$ and $\varepsilon t(ud)$.

Now suppose, in addition, that h is even, i.e., that $h - k$ is even. For every nonzero integer m, we have $r(m) \in \mathbf{O}^h(n+1)$ defined by $r(m)_\beta = R(1/m)_{n+1}$. Given $u \in \mathbf{C}$, $u \notin \mathbf{R}$, let $L(u)$ be the additive lattice $\{1, u\}$ in $\mathbf{C}$. If $m = 3, 4, 6$ or 8, and if $\exp(4\pi\sqrt{-1}/m)\, L(u) = L(u)$, then we define $\mathfrak{L}_m(u)$ to be the subgroup of $\mathbf{O}^h(n+1)$ generated by $r(m)$, $t(d)$ and $t(ud)$.

In [1, §§ 9.3–9.4], it was stated but not proved that $\mathfrak{L}(u, \varepsilon)$ and $\mathfrak{L}_m(u)$ are well defined, up to conjugacy in $\mathbf{O}^h(n+1)$. This will follow from § 2.4.

2.3. $\mathbf{S}_h^n$ is the quadric $Q(x, x) = 1$ in $\mathbf{R}_h^{n+1}$, together with the induced structure as a pseudo-RIEMANNian manifold with metric of signature

$$ds^2 = -\sum_1^h dx_i^2 + \sum_{h+1}^n dx_i^2.$$

$\mathbf{O}^h(n+1)$ is the full group of isometries of $\mathbf{S}_h^n$, and acts transitively on the points of $\mathbf{S}_h^n$. $\mathbf{S}_h^n$ is thus homogeneous, and it has constant sectional curvature $+1$. We refer to [1, § 4] for details.

We have proved [1, §§ 9.3–9.4] that $\mathbf{S}_h^n/\mathfrak{L}(u,\varepsilon)$ and $\mathbf{S}_h^n/\mathfrak{L}_m(u)$ inherit from $\mathbf{S}_h^n$ the structure of homogeneous pseudo-RIEMANNian manifold of constant curvature $+1$. The manifolds $\mathbf{S}_h^n/\mathfrak{L}(u,\varepsilon)$ form a family $\mathcal{L}$, where we identify isometric manifolds. The manifolds $\mathbf{S}_h^n/\mathfrak{L}_m(u)$ form a family $\mathcal{L}_m$, again identifying isometric manifolds. $\mathcal{L}$ and $\mathcal{L}_m$ are the families of lattice space forms.

To study the lattice space forms, we must know when elements of the families $\mathcal{L}$ and $\mathcal{L}_m$ are isometric. An element of one family cannot be isometric to an element of another family, so we may study the families separately. As $\mathbf{O}^h(n+1)$ is the full group of isometries of $\mathbf{S}_h^n$, it is easily seen that a manifold $\mathbf{S}_h^n/\mathfrak{L}(u,\varepsilon)$ is isometric to $\mathbf{S}_{h'}^{n'}/\mathfrak{L}(u',\varepsilon')$ if and only if $n=n'$, $h=h'$ and $\mathfrak{L}(u,\varepsilon)$ is conjugate to $\mathfrak{L}(u',\varepsilon')$ in $\mathbf{O}^h(n+1)$, and that $\mathbf{S}_h^n/\mathfrak{L}_m(u)$ is isometric to $\mathbf{S}_{h'}^{n'}/\mathfrak{L}_m(u')$ if and only if $n=n'$, $h=h'$ and $\mathfrak{L}_m(u)$ is conjugate to $\mathfrak{L}_m(u')$ in $\mathbf{O}^h(n+1)$.

2.4. In order to study conjugacy of groups $\mathfrak{L}(u,\varepsilon)$ and $\mathfrak{L}_m(u)$, we will need

Lemma 2.4. *Let d and d' be antisymmetric nonsingular real $k\times k$ matrices which anticommute with J_k. Then there is a real nonsingular $k\times k$ matrix g_1 which commutes with J_k, and such that $g_1 d\,{}^t g_1 = d'$.*

Proof. Let $\gamma=\{s_1,\ldots,s_k\}$ be an orthonormal basis of $V=\mathbf{R}_0^k$ and let J be given by $J_\gamma = J_k$. V carries the structure of a complex vectorspace of dimension $k/2$ where J is scalar multiplication by $\sqrt{-1}$; now

$$\gamma' = \{s_1, s_3, \ldots, s_{k-1}\}$$

is a $\mathbf{C}$-basis for V. We define a real-linear transformation η of V by $\eta(s_i) = (-1)^{i+1}s_i$. Now η, d and d' each anticommutes with J; thus $d\eta$ and $d'\eta$ commute with J, i.e., $d\eta$ and $d'\eta$ are $\mathbf{C}$-linear. Let Q' be the symmetric $\mathbf{C}$-bilinear form on V defined by $Q'(s_{2i+1}, s_{2j+1}) = \delta_{ij}$; it is easily checked that $Q'(x, d\eta y) + Q'(d\eta x, y) = 0 = Q'(x, d'\eta y) + Q'(d'\eta x, y)$ for every $x, y \in V$; thus $(d\eta)_{\gamma'}$ and $(d'\eta)_{\gamma'}$ are nonsingular antisymmetric complex matrices. If follows that there is a nonsingular complex matrix h_1 of order $k/2$ such that $h_1 (d\eta)_{\gamma'} {}^t h_1 = (d'\eta)_{\gamma'}$. If h is defined by $h_{\gamma'} = h_1$, then $(\eta h \eta^{-1})_{\gamma'} = \bar{h}_1$. It follows that $d' = h_\gamma d\,{}^t h_\gamma$. Define $g_1 = h_\gamma$. Q.E.D.

It is clear that the choice of the original skew basis β of $\mathbf{R}_h^{n+1}$ does not effect the conjugacy class of $\mathfrak{L}(u,\varepsilon)$ or $\mathfrak{L}_m(u)$ in $\mathbf{O}^h(n+1)$. The above lemma shows that the choice of d does not effect the conjugacy class. Thus the groups $\mathfrak{L}(u, \varepsilon)$ and $\mathfrak{L}_m(u)$ are well defined up to conjugacy in $\mathbf{O}^h(n+1)$.

3. The Equivalence Theorem

3.1. Let $\mathbf{H}$ denote the upper half plane, consisting of all complex numbers with positive imaginary part. The *modular group* is the group Γ consisting of all transformations $\gamma : u \to \gamma(u) = \dfrac{au+b}{cu+d}$ of $\mathbf{H}$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an integral matrix of determinant $+1$. We write $\gamma = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The *mod* 2 *parabolic modular group* Γ' is the subgroup of Γ consisting of all $\gamma = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ (mod 2), i.e., such that c is even but a and d are odd. Γ' is maximal among the subgroups of Γ with all elements congruent mod 2 to parabolic ($=$ unipotent) transformations of $\mathbf{H}$. Γ' has a subgroup of index 2, the "modular group of level two" given by $b \equiv 0$ (mod 2), which is well known.

Convention. Observe that $\mathfrak{L}(u, \varepsilon) = \mathfrak{L}(-u, \varepsilon)$ and $\mathfrak{L}_m(u) = \mathfrak{L}_m(-u)$. As u has nonzero imaginary part, we may replace it by $-u$ if necessary. We will now always assume $u \in \mathbf{H}$ when we speak of $\mathfrak{L}(u, \varepsilon)$ or $\mathfrak{L}_m(u)$.

The Equivalence Theorem is:

Theorem 3.1. *Suppose that* $2h > n$ *and* $k = n + 1 - h \equiv 0$ (*mod* 4), *and let* $u, u' \in \mathbf{H}$. *Then* $\mathfrak{L}(u, \varepsilon)$ *and* $\mathfrak{L}(u', \varepsilon')$ *are conjugate subgroups of* $\mathbf{O}^h(n+1)$, *if and only if* (1) $\varepsilon = \varepsilon'$, (2) *there is an element* $\gamma \in \Gamma$ *such that* $\gamma(u) = u'$, *and* (3) $\gamma \in \Gamma'$ *if* $\varepsilon = -$. *Suppose further that* $h \equiv 0$ (*mod* 2), *that* $m = 3, 4, 6$ *or* 8, *and that* $\exp(4\pi\sqrt{-1}/m)\, L(u) = L(u)$ *and* $\exp(4\pi\sqrt{-1}/m)\, L(u') = L(u')$ *where* $L(v)$ *denotes the additive lattice* $\{1, v\}$ *in* $\mathbf{C}$. *Then* $\mathfrak{L}_m(u)$ *and* $\mathfrak{L}_m(u')$ *are conjugate subgroups of* $\mathbf{O}^h(n+1)$, *if and only if* $\gamma(u) = u'$ *for some* $\gamma \in \Gamma$.

The rest of § 3 is devoted to the proof of this theorem.

3.2. Let $u, u' \in \mathbf{H}$, let $L(u) = \{1, u\}$ and $L(u') = \{1, u'\}$ be the corresponding additive lattices in $\mathbf{C}$, and suppose that $\gamma(u) = u'$ where $\gamma = \pm \begin{pmatrix} a & b \\ c & p \end{pmatrix} \in \Gamma$. Let $\alpha = (cu+p)^{-1}$, nonzero element of $\mathbf{C}$. Then

$$L(u') = \{1, u'\} = \left\{1, \frac{au+b}{cu+p}\right\} = \alpha\,\{cu+p, au+b\} = \alpha L(u)\,.$$

Now α is represented by $sI_k + tJ_k$ under $1 \to I_k$ and $\sqrt{-1} \to J_k$, and, as d is an antisymmetric nonsingular $k \times k$ real matrix which anticommutes with J_k, the same is true for $d' = \alpha d$. By Lemma 2.4, there is a nonsingular real matrix g_1 of degree k, which commutes with J_k, such that $g_1 d {}^t g_1 = \alpha d$. Let $g \in \mathbf{O}^h(n+1)$, $g_\beta = \begin{pmatrix} g_1 & 0 & 0 \\ 0 & I_{h-k} & 0 \\ 0 & 0 & {}^t g_1^{-1} \end{pmatrix}$. Then $gt(wd)g^{-1} = t(\alpha wd) = t(w\alpha d)$ for every $w \in \mathbf{C}$. This proves that $g\mathfrak{L}(u, +)g^{-1} = \mathfrak{L}(u', +)$, $g\mathfrak{L}(u, \pm)g^{-1} = \mathfrak{L}(u', \pm)$, and, if $h \equiv 0 \pmod 2$, $g\mathfrak{L}_m(u)g^{-1} = \mathfrak{L}_m(u')$.

Now suppose that $\gamma \in \Gamma'$. As a and p are odd, and c is even, it follows that, given integers s and v, *v is odd if and only if $vp - sc$ is odd.* Now $u' = \alpha(au + b)$ and $1 = \alpha(cu + p)$; thus $u\alpha = pu' - b$ and $\alpha = -cu' + a$. Given integers s and v, we have

$$\begin{aligned} g \cdot t(d)^s(-t(ud))^v \cdot g^{-1} &= (-1)^v \cdot gt((s + uv)d)\, g^{-1} = (-1)^v \cdot t((s + uv)\alpha d) = \\ &= (-1)^v \cdot t((as - scu')d + (vpu' - vb)d) = \\ &= (-1)^v \cdot t((as - vb)d) \cdot t((vp - sc)u'd) = t(d)^{as-vb}(-t(u'd))^{vp-sc} \end{aligned}$$

because $v \equiv vp - sc \pmod 2$. Thus $g\mathfrak{L}(u, -)g^{-1} = \mathfrak{L}(u', -)$.

This proves the sufficiency of our conjugacy conditions.

3.3. Let $\mathfrak{L}(u)$ be a subgroup $\mathfrak{L}(u, \varepsilon)$ or $\mathfrak{L}_m(u)$ of $\mathbf{O}^h(n+1)$, and let $\mathfrak{L}(u')$ be another subgroup $\mathfrak{L}(u', \varepsilon')$ or $\mathfrak{L}_m(u')$ of the same type. Suppose that there exists an element $g \in \mathbf{O}^h(n+1)$ such that $g\mathfrak{L}(u)g^{-1} = \mathfrak{L}(u')$.

The various possibilities for $\mathfrak{L}(u, \varepsilon)$ are characterized as follows. Every eigenvalue of every element of $\mathfrak{L}(u, +)$ is equal to $+1$. $\mathfrak{L}(u, \pm)$ contains $-I$. $\mathfrak{L}(u, -)$ has elements with eigenvalues equal to -1, but it does not contain $-I$. Thus $g\mathfrak{L}(u, \varepsilon)g^{-1} = \mathfrak{L}(u', \varepsilon')$ implies $\varepsilon = \varepsilon'$. Again looking at eigenvalues, we see that $g\mathfrak{L}(u, +)g^{-1} = \mathfrak{L}(u', +)$, whether $\mathfrak{L}(u)$ is $\mathfrak{L}(u, \varepsilon)$ or $\mathfrak{L}_m(u)$.

We will prove that $u' = \gamma(u)$ for some $\gamma \in \Gamma$, as a consequence of

$$g\mathfrak{L}(u, +)g^{-1} = \mathfrak{L}(u', +).$$

For g must preserve the nullspace of $t(d) - I$. In our skew basis β of $\mathbf{R}_h^{n+1}$, it follows that $g_\beta = \begin{pmatrix} g_1 & g_2 & g_3 \\ 0 & g_4 & g_5 \\ 0 & 0 & g_6 \end{pmatrix}$, where $g \in \mathbf{O}^h(n+1)$ implies $g_6 = {}^t g_1^{-1}$. Thus $w \in \mathbf{C}$ gives $gt(wd)g^{-1} = t(g_1 wd {}^t g_1)$, where $\mathbf{C}$ is identified with the $\mathbf{R}$-linear combinations of I_k and J_k. As u and I_k span $\mathbf{C}$ over $\mathbf{R}$, it follows that $g_1 J = \pm J g_1$ and $g_1 d {}^t g_1 = \alpha d$ for $0 \neq \alpha \in \mathbf{C}$. We may replace g_1 by $g_1 \cdot \operatorname{diag}\{1, -1; \ldots; 1, -1\}$ if necessary, and assume $g_1 J = J g_1$. Thus $g \cdot t(d)^s t(ud)^v g^{-1} = t(\alpha d)^s t(\alpha u d)^v$ for any integers s and v. It follows that the lattices $L(u) = \{1, u\}$ and $L(u') = \{1, u'\}$ in $\mathbf{C}$ are related by $L(u') = \alpha L(u)$. i.e., that $\{1, u'\} = \{\alpha, \alpha u\}$. As $u, u' \in \mathbf{H}$, $L(u)$ and

$L(u')$ carry the same orientation; thus $u' = a\alpha u + b\alpha$ and $1 = c\alpha u + p\alpha$ where $\begin{pmatrix} a & b \\ c & p \end{pmatrix}$ is an integral matrix of determinant $+1$. Now define $\gamma = \pm \begin{pmatrix} a & b \\ c & p \end{pmatrix} \in \Gamma$, and observe that $u' = \frac{u'}{1} = \frac{a\alpha u + b\alpha}{c\alpha u + p\alpha} = \frac{au+b}{cu+p} = \gamma(u)$. Except for the groups $\mathfrak{L}(u, -)$ and $\mathfrak{L}(u', -)$, this proves the necessity of our conjugacy conditions.

Suppose that $g\mathfrak{L}(u, -)g^{-1} = \mathfrak{L}(u', -)$. We must prove $\gamma \in \Gamma'$.

$$g \cdot t(d)^s(-t(ud))^v g^{-1} = (-1)^v t(\alpha(s + vu)d) = \\ = (-1)^v \cdot t((as - vb)d) \cdot t((vp - sc)u'd)$$

as at the end of § 3.2. Thus $(-1)^v = (-1)^{vp-sc}$ for any integers s and v. It follows that p is odd and c is even. Thus $1 = ap - bc \equiv a \pmod 2$ proves that a is odd. Now $\begin{pmatrix} a & b \\ c & p \end{pmatrix} \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \pmod 2$, proving $\gamma \in \Gamma'$. This completes the proof of the necessity of the conjugacy conditions.

Theorem 3.1 is now proved.

3.4. In [1, § 9.3], I made the error of stating that $\mathfrak{L}(u, \varepsilon)$ is conjugate to $\mathfrak{L}(u', \varepsilon)$, if and only if the lattices $L(u)$ and $L(u')$ are equal. From the proof of Theorem 3.1, it is clear that this condition is too strong. If $\varepsilon = +$ or $\varepsilon = \pm$, then the correct condition is that $L(u') = \alpha L(u)$ where $0 \neq \alpha \in \mathbf{C}$.

A similar error [1, § 9.4] was made with respect to the conjugacy of $\mathfrak{L}_m(u)$ and $\mathfrak{L}_m(u')$. Here the correct condition on the lattices is $L(u') = \alpha L(u)$ where $0 \neq \alpha \in \mathbf{C}$.

Remark. The condition of Theorem 3.1 is not surprising, for the condition that two lattices L_1 and L_2 in $\mathbf{C}$ are related by $L_1 = \alpha L_2$, $0 \neq \alpha \in \mathbf{C}$, is clearly the same as the condition that an integral unimodular fractional linear transformation sends the ratio of a pair of generators of L_1 to the ratio of a pair of generators of L_2.

4. Parameterization of $\mathcal{L}$ and $\mathcal{L}_4$

$\mathcal{L}$ is a disjoint union $\mathcal{L}(+) \cup \mathcal{L}(-) \cup \mathcal{L}(\pm)$ where

$$\mathcal{L}(\varepsilon) = \bigcup_{h=4t}^{\infty} \bigcup_{t=1}^{\infty} \mathcal{L}(\varepsilon, h, 4t + h - 1)$$

and $\mathcal{L}(\varepsilon, h, n)$ consists of all manifolds $\mathbf{S}_h^n/\mathfrak{L}(u, \varepsilon)$ in $\mathcal{L}$. Similarly, $\mathcal{L}_m = \bigcup_{s=2t}^{\infty} \bigcup_{t=1}^{\infty} \mathcal{L}_m(2s, 4t + 2s - 1)$ where $\mathcal{L}_m(h, n)$ consists of all manifolds $\mathbf{S}_h^n/\mathfrak{L}_m(u)$ in $\mathcal{L}_m$. We will study the structure of $\mathcal{L}(\varepsilon, h, n)$ and $\mathcal{L}_m(h, n)$.

We first look at the easy cases. Let $u \in \mathbf{H} =$ upper half plane. For appropriate h and n, and any ε, $\mathfrak{L}(u, \varepsilon) \subset \mathbf{O}^h(n+1)$ is defined. The lattice $L(u)$ is preserved by multiplication by $\exp(4\pi\sqrt{-1}/4) = -1$, so, for appropriate h and n, $\mathfrak{L}_4(u) \subset \mathbf{O}^h(n+1)$ is defined.

Theorem 3.1 now yields:

Theorem 4. *Let* $\mathbf{M} = \mathbf{H}/\Gamma$, *identification space of the upper half plane under the modular group; let* $\mathbf{M}' = \mathbf{H}/\Gamma'$, *identification space under the mod 2 parabolic modular group. Then each* $\mathcal{L}(+, h, n)$, *each* $\mathcal{L}(\pm, h, n)$ *and each* $\mathcal{L}_4(h, n)$ *is parameterized by* $\mathbf{M}$, *and each* $\mathcal{L}(-, h, n)$ *is parameterized by* $\mathbf{M}'$. *More precisely, for appropriate* h *and* n, *the maps*

$$u \to \mathbf{S}^n_h/\mathfrak{L}(u, +), u \to \mathbf{S}^n_h/\mathfrak{L}(u, \pm), u \to \mathbf{S}^n_h/\mathfrak{L}_4(u)$$

and $u \to \mathbf{S}^n_h/\mathfrak{L}(u, -)$ *induce one to one maps of* $\mathbf{M}$ *onto* $\mathcal{L}(+, h, n)$, $\mathbf{M}$ *onto* $\mathcal{L}(\pm, h, n)$, $\mathbf{M}$ *onto* $\mathcal{L}_4(h, n)$, *and* $\mathbf{M}'$ *onto* $\mathcal{L}(-, h, n)$, *respectively.*

Given $u, u' \in \mathbf{H}$, we can find an analytic arc (such as the straight line segment) in $\mathbf{H}$ from u *to* u'. This results in a real analytic deformation of $\mathfrak{L}(\varepsilon, u)$ (or $\mathfrak{L}_4(u)$) over to $\mathfrak{L}(\epsilon, u')$ (or $\mathfrak{L}_4(u')$). It follows that any two elements of $\mathcal{L}(\varepsilon, h, n)$ (or of $\mathcal{L}_4(h, n)$) are real-analytically homeomorphic. This is notable because, by Theorem 4, $\mathcal{L}(\varepsilon, h, n)$ and $\mathcal{L}_4(h, n)$ carry the structure of a real analytic V-manifold in the sense of BAILEY. On the other hand, we cannot assert that two distinct elements of $\mathcal{L}(\varepsilon, h, n)$ or of $\mathcal{L}_4(h, n)$ be affinely equivalent, for, utilizing irreducibility of the holonomy group and the fact that the manifolds have the same constant curvature, such an affine equivalence would be seen to be an isometry.

5. Uniqueness in $\mathcal{L}_3$, $\mathcal{L}_6$ and $\mathcal{L}_8$

5.1. We will prove that each $\mathcal{L}_3(h, n)$, each $\mathcal{L}_6(h, n)$, and each $\mathcal{L}_8(h, n)$, has just one element. This will require a lemma on lattices in $\mathbf{C}$. The lemma also gives an explanation as to why m is not arbitrary in the definitions of the groups $\mathfrak{L}_m(u)$.

Let L be a discrete additive lattice in $\mathbf{C}$, and suppose that $L = xL$ for some non-real unimodular complex number x. As L is discrete, we may choose $a \in L$ with the property that $|b| \geqq |a| > 0$ for every nonzero $b \in L$; we then define L' to be the sublattice of L generated by a and xa. Suppose that $x = \cos(t) + \sqrt{-1}\sin(t)$, t real. Then

$$|ra + sxa|^2 = |a|^2\,|r^2 + 2rs\cos(t) + s^2|$$

for any $r, s \in \mathbf{R}$.

Suppose that we have $b \in L$, $b \notin L'$. $b = ra + sxa$ for some real r and s

because a and xa are linearly independent over $\mathbf{R}$, consequence of $x \notin \mathbf{R}$. Adding an element of L' to b, we may assume that $|r| \leqq \frac{1}{2}$, $|s| \leqq \frac{1}{2}$, and $b \neq 0$. Thus $0 < |b|^2 \leqq |a|^2 \, |\frac{1}{2} + \frac{1}{2}\cos(t)| < |a|^2$.

Now $|a| \leqq |a + xa|$ implies $\cos(t) \geqq -\frac{1}{2}$. As $x^n L = x^{n-1} L = \ldots = L$, the same must be true of every power of x. Thus $\cos(nt) \geqq -\frac{1}{2}$ for every integer $n > 0$. It follows that

$$x = \exp(2\pi \sqrt{-1}/3) \quad \text{or} \quad x = \exp(\pm 2\pi \sqrt{-1}/4),$$

whence $\cos(t) = -\frac{1}{2}$ or $\cos(t) = 0$. In either case, we have $0 < |b|^2 < |a|^2$, contradicting $b \notin L'$. We have proved:

Lemma 5.1. *Let* L *be a discrete additive lattice in* $\mathbf{C}$; *suppose that*

$$x \in \mathbf{C}, \; x \notin \mathbf{R}, \; |x| = 1,$$

and $xL = L$. *Then* $L = \{a, xa\}$, *where* $|a| \leqq |b|$ *for every nonzero* $b \in L$, *and* $x = \exp(\pm 2\pi \sqrt{-1}/3)$ *or* $x = \pm \sqrt{-1}$. *In other words, there is a nonzero complex number* a *such that either*

$$L = a\{1, \exp(2\pi \sqrt{-1}/3)\} \text{ or } L = a\{1, \sqrt{-1}\}.$$

Remark. Lemma 5.1 provides an alternative treatment of the last part of [1, § 10.4].

5.2. Theorem 3.1, the remark at the end of § 3.4, and Lemma 5.1, combine to give a proof of:

Theorem 5.2. *If* $m = 3$, 6 or 8, *then each* $\mathcal{L}_m(h, n)$ *has just one element. In other words, for appropriate* h *and* n, *we have:*

1. *Every* $S^n_h/\mathfrak{L}_3(u)$ *is isometric to* $S^n_h/\mathfrak{L}_3(\exp(2\pi\sqrt{-1}/3))$.
2. *Every* $S^n_h/\mathfrak{L}_6(u)$ *is isometric to* $S^n_h/\mathfrak{L}_6(\exp(2\pi\sqrt{-1}/3))$.
3. *Every* $S^n_h/\mathfrak{L}_8(u)$ *is isometric to* $S^n_h/\mathfrak{L}_8(\sqrt{-1})$.

5.3. Let M^n_h and N^n_h be connected homogeneous pseudo-Riemannian manifolds of constant curvature $+1$ with isomorphic fundamental groups. $\pi_1(M^n_h) \cong G \cong \pi_1(N^n_h)$ for some abstract group G. Let $\mathbf{Z}$ denote the infinite cyclic group.

If G is finite, then [1, Theorem 12] M^n_h is isometric to N^n_h. If G is an extension of $\mathbf{Z}$ by an element of order 4, then [1, Theorem 12] M^n_h is isometric to N^n_h. If G is an extension of $\mathbf{Z} \times \mathbf{Z}$ by an element of order $m \neq 1$, 2 or 4, then ([1, Theorem 12] and Theorem 5.2) $m = 3, 6$ or 8 and M^n_h is isometric to N^n_h. If M^n_h is not isometric to N^n_h, then ([1, Theorem 12] and

Theorems 4 and 5.2) either G is an extension of $\mathbf{Z}$ by an element of order 1 or 2, or G is an extension of $\mathbf{Z} \times \mathbf{Z}$ by an element of order 1, 2 or 4. This covers all possible G.

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(Received March 2, 1962)