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Autor(en): **Thomas, Emery**

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On the Mod 2 Cohomology of Certain H-spaces

by EMERY THOMAS[1]), University of California (Berkeley)

1. Introduction. Let X be a topological space and G an abelian group. We denote by $H^i(X; G)$ the i-th (singular) cohomology group of X with coefficients in G. Define an integer-valued function ν_G by

$$\nu_G(X) = \text{least positive integer } n \text{ such that } H^n(X; G) \neq 0 .$$

In case X is G-acyclic we set $\nu_G(X) = 0$. For simplicity we write the function as ν_r if $G = Z_r$, the integers mod r, $r \geq 2$.

Suppose now that X is an H-space – that is, X has a continuous multiplication with unit – and suppose (for the remainder of the paper) that X satisfies the following conditions.

(1) The integral cohomology groups of X are finitely generated in each dimension.

(2) $H^*(X; Z_2)$ is finitely generated as a vector space, and is primitively generated as a HOPF algebra.

In [9] we showed that for such H-spaces, $\nu_2(X) = 2^r - 1$ for some $r \geq 0$. The purpose of this note is to prove

Theorem 1. *Let X be an H-space satisfying the above two conditions. Then*

$$\nu_2(X) = 0, 1, 3, 7 \text{ or } 15 .$$

Moreover if X has no 2-torsion, then

$$\nu_2(X) = \nu_Q(X) = 0, 1, 3 \text{ or } 7,$$

where Q denotes the rational numbers.

Examples of H-spaces having these values of ν_2 are the various LIE groups (for $\nu_2 = 0, 1, 3$) and the sphere of dimension seven (for $\nu_2 = 7$). At present no H-space is known for which $\nu_2 = 15$, and it seems unlikely that this value can occur.

The method of proof for Theorem 1 is based on the work of ADAMS [1], [2]. For related results in case X is a topological group, see CLARK [7].

I would like to thank W. BROWDER for arousing my interest in this problem and for his helpful suggestions about its solution.

2. Truncated polynomial algebras. We work in the category of commutative, associative, graded and connected algebras of finite type over Z_2. For

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such algebras A we identify A_0 with Z_2 and denote the complement by $\bar{A}$. Following § 1 we define $\nu(A)$ to be the least positive integer n such that $A_n \neq 0$.

Define

$$D^1 A = \bar{A}, \; D^n A = D^{n-1} A \cdot \bar{A} \quad (n \geq 2).$$

We call $D^2 A$ the ideal of decomposable elements, and if an element $a \notin D^2 A$ we call it indecomposable.

We will say that A is a *truncated polynomial algebra of height* n (for $n \geq 2$) if $A \approx A'/D^n A'$, where A' is a polynominal algebra. We call A finite-dimensional if it is so as a mod 2 vector space.

We now assume that A is a finite-dimensional truncated polynomial algebra of height three, and furthermore that A is an algebra over the mod 2 STEENROD algebra, $\mathscr{A}$. Such algebras were studied in [9], and we record here the results needed from that paper.

Let $a \in A$ be an indecomposable element and suppose that deg $a = 2^{r+1}k + 2^r$, where $r, k \geq 1$. Then (see (2.1) and (3.3) of [9]):

(2.1) *There is an indecomposable element* a' *of degree* $2^{r+1}k$ *such that*

$$a \equiv \mathrm{Sq}^{2^r}(a') \bmod D^2 A.$$

(2.2) $\mathrm{Sq}^{2^r}(a) = \Sigma_{i=0}^{r} \mathrm{Sq}^{2^i}(d_i)$, *where* $d_i \in D^2 A$. *In particular, if* $(D^2 A)_l = 0$, *for* $l = 2^{r+1}(k+1) - 2^i (0 \leq i \leq r)$, *then* $\mathrm{Sq}^{2^r}(a) = 0$.

Here Sq^i denotes the (mod 2) STEENROD operator. It follows from (2.1) that

$$\nu(A) = 2^r, \text{ for some } r \geq 0. \tag{2.3}$$

Let $M = 2^s$, for some $s \geq 2$, and let $a \in A_{M+2^r}$, where $0 \leq r \leq s-2$. Since A is an $\mathscr{A}$-algebra, $a^2 = \mathrm{Sq}^{M+2^r}(a)$. By the ADEM relations [3]

$$\mathrm{Sq}^{M+2^r} = \mathrm{Sq}^{2^{r+1}} \mathrm{Sq}^{M-2^r} + \mathrm{Sq}^M \mathrm{Sq}^{2^r},$$

and since $r \leq s-2$, $M - 2^r = 2^{r+1} e + 2^r$ for some $e \geq 1$. Therefore by (4.2) of [9],

$$\mathrm{Sq}^{M-2^r} = \Sigma_{i=0}^{r} \mathrm{Sq}^{2^i} \alpha_i,$$

where $\alpha_i \in \mathscr{A}$, and consequently

$$a^2 = \mathrm{Sq}^{2^{r+1}} [\Sigma_{i=0}^{r} \mathrm{Sq}^{2^i} \alpha_i(a)] + \mathrm{Sq}^M \mathrm{Sq}^{2^r}(a). \tag{2.4}$$

For the remainder of the section *we assume that* $A_i = 0$ *if* i *is odd*. We use (2.4) to prove

(2.5) Lemma. *Suppose that* $\nu(A) = 2^q$, *where* $q \geq 3$. *Then* $A_s = 0$, *for* $2^q < s < 2^q + 2^{q-1}$. *Moreover if* $\mathrm{Sq}^{2^{q-1}} A_{2^q} = 0$, *then* $A_t = 0$, *for* $2^q < t < 2^{q+1}$.

Notice that the second assertion follows at once from 2.1 and the first assertion. By 2.1 (and the above hypothesis) it suffices to prove the first assertion just for the case $s = 2^q + 2^r$, where $1 \leq r \leq q-2$. By the definition of $\nu(A)$ we have $(D^2A)_j = 0$ for $j < 2^{q+1}$. Let $a \in A$ be on element of degree $2^q + 2^r$. Thus a is indecomposable and hence if $a^2 = 0$, then $a = 0$, since A is a truncated polynomial algebra of height three. Set $M = 2^q$ in 2.4. Then

$$\deg \alpha_i(a) = 2^{q+1} - 2^i \quad (0 \leq i \leq r \leq q-2)$$

and hence each element $\alpha_i(a)$ is also indecomposable. Therefore by 2.2,

$$\mathrm{Sq}^{2^r}(a) = 0, \ \mathrm{Sq}^{2^i}\alpha_i(a) = 0 \quad (0 \leq i \leq r)$$

and hence by (2.4) $a^2 = 0$, showing that $A_{2^q+2^r} = 0$ as asserted.

In § 4 we use (2.5) to show that $\nu_2(X) \leq 15$, if X is an H-space satisfying the two conditions in § 1.

For the remainder of the section let A denote a fixed finite-dimensional truncated polynomial algebra of height 3, which is an algebra over $\mathcal{A}$, such that $\nu(A) = 16$. Then by (2.5), $A_{16+2i} = 0$ for $1 \leq i \leq 3$. We show

(2.6) Lemma. (i) $\mathrm{Sq}^{2^i} A_{24} = 0$, *for* $i = 1, 3$.
(ii) $\mathrm{Sq}^{2^j} A_{28} = 0$, *for* $j = 2, 3$;
(iii) $\mathrm{Sq}^{2^k} A_{30} = 0$, *for* $k = 1, 2, 3$.

Set $D = D^2A$. Notice that (2.2) implies at once that

$$\mathrm{Sq}^8 A_{24} = \mathrm{Sq}^4 A_{28} = \mathrm{Sq}^2 A_{30} = 0,$$

since $D_j = 0$ for $j < 32$. Thus to complete the proof of (2.6) (i) we must show that $\mathrm{Sq}^2(a) = 0$, for $a \in A_{24}$. Setting $b = \mathrm{Sq}^2(a)$, we have $b^2 = \mathrm{Sq}^{26}(b)$. By the ADEM relations [3],

$$\mathrm{Sq}^{26} = \mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^{20} + \alpha_1\,\mathrm{Sq}^1 + \alpha_2\,\mathrm{Sq}^2,$$

where $\alpha_1, \alpha_2 \in \mathcal{A}$. Moreover $\mathrm{Sq}^2\,\mathrm{Sq}^2 = \mathrm{Sq}^3\,\mathrm{Sq}^1$ and $\mathrm{Sq}^1 \equiv 0$ in A since the odd-dimensional elements of A are zero. Therefore

$$b^2 = \mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^{20}(b).$$

Since deg $\mathrm{Sq}^{20}(b) = 46$, it follows from (2.1) that

$$\mathrm{Sq}^{20}(b) = \mathrm{Sq}^2(c) + d,$$

where $c \in A_{44}$ and $d \in D_{46}$. Moreover by 2.5 $D_{46} = A_{16} \cdot A_{30}$, and by the CARTAN product formula, $\mathrm{Sq}^2(A_{16} \cdot A_{30}) = 0$. Hence

$$b^2 = \mathrm{Sq}^4 \mathrm{Sq}^2[\mathrm{Sq}^2(c) + d] = 0\,,$$

which shows that $b = 0$, thus proving (2.6) (i). The remaining cases in (2.6) are proved similarly (using (2.1) and (2.5)) and are left to the reader.

We set

$$D^+ = \sum_{1=1}^{\infty} A_i \cdot A_i,\ D_n^- = \sum_{\substack{o<i<j\\ i+j=n}} A_i \cdot A_j,\ D^- = \sum_{n=3}^{\infty} D_n^-\,,$$

which are well-defined subspaces of A since A is a truncated polynomial algebra of height three. Moreover, as a mod 2 vector space

$$D = D^+ \oplus D^-\,,$$

a splitting we use to show

(2.7) Lemma. $A_{32+2i} = 0$, *for* $1 \leq i \leq 3$.

Since $D_{32+2j} = 0$ for $1 \leq j \leq 3$, it suffices (by (2.1)) to show that $A_{34} = A_{36} = 0$, and to do this it suffices to show that if $a \in A_{32+2^r}$ $(r = 1, 2)$, then $a^2 = 0$. Now by (2.2), $\mathrm{Sq}^{2^r}(a) = 0$, and therefore by (2.4) (taking $M = 32$ and $r = 1$ or 2),

$$a^2 = \mathrm{Sq}^{2^{r+1}}[\Sigma_{i=1}^{r} \mathrm{Sq}^{2^i} \alpha_i(a)]\,.$$

Now deg $\alpha_i(a) = 64 - 2^i$ and hence by (2.2),

$$\mathrm{Sq}^{2^i} \alpha_i(a) = \sum_{j=1}^{i} \mathrm{Sq}^{2^j}(d_j)\,,$$

where $d_j \in D_{64-2j}$ $(1 \leq j \leq i \leq r \leq 2)$. But

$$D_{64-2j} = A_{16} \cdot A_{48-2j} + A_{24} \cdot A_{40-2j} + A_{28} \cdot A_{36-2j} + A_{30} \cdot A_{34-2j}\,,$$

and hence by the CARTAN product formula and (2.6),

$$\mathrm{Sq}^{2^{r+1}} \mathrm{Sq}^{2^j}(D_{64-2j}) \subset D^-\,.$$

Since $a^2 \in D^+$ and since $D^+ \cap D^- = 0$, this shows that $a^2 = 0$, completing the proof of the lemma.

3. The operations of ADAMS. The proof of Theorem 1 will require the secondary cohomology operations of J.F. ADAMS, defined in [1]. These are functions Φ_{ij} defined for each pair of integers, i, j such that $0 \leq i \leq j$ and $i \neq j - 1$. We will use the following properties of these operations (see [1; §§ 3, 4.2]).

(3.1) *Let X be a space. The operation Φ_{ij} is defined on cohomology classes*

$u \in H^*(X)$ (mod 2 coefficients) such that $\mathrm{Sq}^{2^r}(u) = 0$ for $0 \le r \le j$. If $u \in H^m(X)$ $(m > 0)$, then $\Phi_{ij}(u)$ is an element of the group $H^{m+2^i+2^j-1}(X)$, modulo an indeterminancy subgroup $Q_{ij}(X)$. Moreover, if $i < j$, then

$$Q_{ij}(X) = \mathrm{Sq}^{2^i} H^{m+2^j-1}(X) + \sum_{0 \le l \le j} b_l H^{m+2^l-1}(X),$$

where $b_l \in \mathscr{A}$ and $\deg b_l = 2^i + 2^j - 2^l$.

Take an integer $k \ge 3$ and suppose that $u \in H^m(X)$ $(m > 0)$ is a class such that $Sq^{2^r}(u) = 0$, for $0 \le r \le k$. ADAMS' main result is ([1; 4.6.1]):

(3.2) Theorem (ADAMS). *There is a relation*

$$\Sigma a_{ij} \Phi_{ij}(u) = [\mathrm{Sq}^{2^{k+1}}(u)],$$

(independent of X) which holds modulo the total indeterminancy of the left-hand side.

Here the summation is taken over all i, j such that $0 \le i \le j \le k$ and $i \ne j-1$. We apply this in the next section to prove Theorem 1, using in fact the following simple consequence of (3.2).

(3.3) *Let X be a space and let m, k be positive integers with $k \ge 3$. Suppose that $H^{2j-1}(X) = 0$ for $m < j \le m + 2^k$ and that $\mathrm{Sq}^{2^i} H^{2m}(X) = 0$ for $0 \le i \le k$. Then for any element $u \in H^{2m}(X)$,*

$$\mathrm{Sq}^{2^{k+1}}(u) = \Sigma_{2 \le j \le k}\, a_j \Phi_{0j}(u),$$

with zero indeterminancy, where the elements $a_j \in \mathscr{A}$.

This follows at once from 3.1 and 3.2.

4. Proof of Theorem 1. The relation between the work of § 2 and the proof of Theorem 1 is given by the projective plane, P_2X, for an H-space X. This was defined by STASHEFF [8] and the cohomology of P_2X is studied in [6], where a group homomorphism ι is defined with $\iota: H^{q+1}(P_2X) \to H^q(X)$. (For the remainder of the paper we use mod 2 coefficients.) ι is an $\mathscr{A}$-homomorphism and the image of ι is the subspace of $H^*(X)$ spanned by the primitive classes. Moreover if $\nu_2(X) = 2d - 1$, then (see [6; 3.1]).

(4.1) *ι is a monomorphism for $q \le 4d - 2$ and is an epimorphism for $q \le 4d - 3$.*

Denote by P^- the subspace of $H^*(X)$ spanned by the odd-dimensional primitive classes. Let $\{u_i\}$ be a basis for P^- and choose classes $\{a_i\}$ in $H^*(P_2X)$ so that $\iota a_i = u_i$. Define $\tilde{A}$ to be the subalgebra of $H^*(P_2X)$ generated by $\{a_i\}$. The following result is proved in [6].

(4.2) *The algebra* $\tilde{A}$ *is a truncated polynomial algebra of height three. Moreover there is an ideal* N *in* $H^*(P_2X)$ *such that*

$$H^*(P_2X) = \tilde{A} \oplus N\,, \quad \textit{(group direct sum)}$$

and $\mathcal{A}_1(N) \subset N$, *where* $\mathcal{A}_1$ *denotes the subalgebra of* $\mathcal{A}$ *generated by the squares of even degree. Furthermore if* $\nu_2(X) = 2d - 1$, *then*

$$H^{2i}(P_2X) = \tilde{A}_{2i}, \quad \textit{for} \ \ 1 \leq i \leq 2d\,.$$

Thus if we set

$$A' = H^*(P_2X)/N\,,$$

we obtain a finite-dimensional truncated polynomial algebra of height three, which is an $\mathcal{A}_1$-algebra. Moreover

$$\text{(4.3)} \qquad A'_{2i} \approx H^{2i}(P_2X), \quad \textit{for} \ \ 1 \leq i \leq 2d\,,$$

and the isomorphism is an $\mathcal{A}_1$-map. Let $\mathcal{B}$ denote the ideal of $\mathcal{A}$ generated by Sq^1. Then $\mathcal{A} = \mathcal{B} \oplus \mathcal{A}_1$, as a Z_2-module. Since the elements of A' all have even degree we can make A' into an $\mathcal{A}$-algebra by setting $\mathcal{B}(A') = 0$. Furthermore, $\nu(A') = 2d$.

We use these facts to show

(4. 4) Lemma. *Suppose that* $\nu_2(X) = 2d - 1$, *where* d *is an even integer* ≥ 4. *Then*

$$H^{2d+s}(P_2X) = 0, \quad \textit{for} \ \ 0 < s < d\,.$$

Furthermore if $\mathrm{Sq}^d H^{2d}(P_2X) = 0$, *then*

$$H^{2d+t}(P_2X) = 0, \quad \textit{for} \ \ 0 < t < 2d\,.$$

The proof uses the following result of W. Browder [5].

(4. 5) Theorem (Browder). *Let* X *be an* H*-space as in* § 1 *and let* $v \in H^{2q}(X)$ $(q \geq 1)$ *be a primitive class. Then there is a class* $u \in H^{2q-1}(X)$ *such that* $v = \mathrm{Sq}^1(u)$.

Clearly if $q \leq \nu_2(X)$, then the class u is itself primitive.

Proof of (4. 4). It follows from (4.1) and (4.5) that

$$H^{2d+1}(P_2X) = \mathrm{Sq}^1 H^{2d}(P_2X)\,.$$

Let $u \in H^{2d}(P_2X)$ and set $v = \mathrm{Sq}^1(u)$. Then,

$$v^2 = \mathrm{Sq}^{2d+1}(v) = \mathrm{Sq}^{2d+1}\,\mathrm{Sq}^1(u)\,.$$

By the ADEM relations [3], since d is even,

$$\mathrm{Sq}^{2d+1} = \mathrm{Sq}^2\,\mathrm{Sq}^{2d-1} + \mathrm{Sq}^{2d}\,\mathrm{Sq}^1$$

and therefore,

$$v^2 = \mathrm{Sq}^2\,\mathrm{Sq}^{2d-1}(v) = \mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^{2d-2}(v)\,,$$

using the fact that $\mathrm{Sq}^1\,\mathrm{Sq}^1 = 0$ and that $\mathrm{Sq}^1\,\mathrm{Sq}^{2d-2} = \mathrm{Sq}^{2d-1}$. Let $w = \mathrm{Sq}^{2d-2}(v)$. Then $\iota(w)$ is a primitive class of degree $4d-2$. Hence by (4.5) there is a primitive class $y \in H^{4d-3}(X)$ such that $\mathrm{Sq}^1(y) = \iota(w)$, and therefore by (4.1) there is a class $x \in H^{4d-2}(P_2X)$ such that $\mathrm{Sq}^1(x) = w$. Hence

$$v^2 = \mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^1(x) = 0.$$

But by Theorem 1.1 of [6], $v^2 = 0$ implies that $v = 0$, and hence $H^{2d+1}(P_2X) = 0$.

As remarked above we have already proved that $\nu_2(X) = 2^q - 1$ for some $q \geq 0$ [9]. Thus in the present situation we can assume that $d = 2^{q-1}$ with $q \geq 3$. Therefore by 4.3 and 2.5 applied to the algebra A',

$$H^{2d+2i}(P_2X) = A'_{2d+2i} = 0\,,$$

for $0 < i < d/2$, and also for $d/2 \leq i < d$ if $\mathrm{Sq}^d\, H^{2d}(P_2X) = 0$. Therefore by 4.5 and 4.1,

$$H^{2d+t}(P_2X) = 0$$

for $0 < t < d$, and also for $d \leq t < 2d$ if $\mathrm{Sq}^d\, H^{2d}(P_2X) = 0$, which completes the proof of the lemma.

We now can prove the first part of Theorem 1. Suppose that X is an H-space such that $\nu_2(X) = 2^q - 1$, where $q \geq 5$. We show that this leads to a contradiction.

Let k denote the least positive integer such that $\mathrm{Sq}^{2^{k+1}}\, H^{2^q}(P_2X) \neq 0$. By (4.4), $k \geq q-2$ and hence $k \geq 3$. Let $u \in H^{2^q}(P_2X)$ be a class such that $\mathrm{Sq}^{2^{k+1}}(u) \neq 0$. Since

$$\mathrm{Sq}^{2^q}(u) = u^2 \neq 0\,,$$

we see that $k = q-1$ or $q-2$. From (4.4) and (3.3) (taking $2m = 2^q$ in (3.3)) it follows that

$$\mathrm{Sq}^{2^{k+1}}(u) = \Sigma_{2\leq j\leq k}\, a_j\, \Phi_{0j}(u)\,.$$

But deg $\Phi_{0j}(u) = 2^q + 2^j$, where $2 \leq j \leq k$, and therefore by (4.4) and the definition of the integer k, $\Phi_{0j}(u) = 0$. Hence $\mathrm{Sq}^{2^{k+1}}(u) = 0$, which is a contradiction. Therefore $q \leq 4$, which proves the first assertion in Theorem 1.

Suppose now that X is an H-space with no 2-torsion. We complete the proof of Theorem 1 by showing that $q \leq 3$ (i.e., that $\nu_2(X) = 0, 1, 3$ or 7). Since X has no 2-torsion it is immediate that $\nu_2(X) = \nu_Q(X)$.

By Borel [4; Prop. 7.2] $H^*(X)$ is generated by odd-dimensional primitive generators, since X is without 2-torsion. From this it follows (see [9]) that the ideal N given in 4.2 is an $\mathscr{A}$-submodule of $H^*(P_2X)$; and moreover, that if $\nu_2(X) = 2d - 1$, then

(4.6) $H^{2j-1}(P_2X) = 0,\ 1 \leq j \leq 3d - 1;\ H^k(P_2X) = \tilde{A}_k,\ 0 \leq k \leq 6d - 2,$

where $\tilde{A}$ is the algebra given in 4.2.

Suppose now that X is an H-space without 2-torsion such that $\nu_2(X) = 15$. We show that this leads to a contradiction. For this X, (4.6) implies

(4.7) $H^{2j-1}(P_2X) = 0,\ 1 \leq j \leq 23;\ A'_k \approx H^k(P_2X),\ 0 \leq k \leq 46,$

where, as before, $A' = H^*(P_2X)/N$. Since N is an $\mathscr{A}$-submodule, the above isomorphism is an $\mathscr{A}$-map. Applying (2.6) and (2.7) to A' we obtain the following facts.

$$\begin{aligned}
&(4.8) \qquad && \mathrm{Sq}^{2^i} H^{24}(P_2X) = 0 \text{ for } i = 0, 1, 3 . \\
& && \mathrm{Sq}^{2^j} H^{28}(P_2X) = 0 \text{ for } j = 0, 2, 3 . \\
& && \mathrm{Sq}^{2^k} H^{30}(P_2X) = 0 \text{ for } k = 0, 1, 2, 3. \\
&(4.9) && H^{32+2j}(P_2X) = 0 \text{ for } 1 \leq j \leq 3 .
\end{aligned}$$

We now obtain the contradiction, caused by assuming that $\nu_2(X) = 15$. First of all it follows from (3.3), (4.7), (4.8) and (4.9), that $\mathrm{Sq}^{16} H^{30}(P_2X) = 0$. But by the Adem relations and (4.8) this shows that $\mathrm{Sq}^{30} H^{30}(P_2X) = 0$. Thus if $u \in H^{30}(P_2X)$,

$$u^2 = \mathrm{Sq}^{30}(u) = 0 ,$$

and therefore $u = 0$, showing that $H^{30}(P_2X) = 0$. Now

$$H^{30}(P_2X) = \mathrm{Sq}^2 H^{28}(P_2X)$$

(see (2.1) and (4.7)) and hence by (4.8), $\mathrm{Sq}^{2^j} H^{28}(P_2X) = 0$ for $0 \leq j \leq 3$. Let $v \in H^{28}(P_2X)$. By the Adem relations

$$\mathrm{Sq}^{28} = \mathrm{Sq}^{12}\,\mathrm{Sq}^{16} + \sum_{i=0}^{3} \beta_i\, \mathrm{Sq}^{2^i},$$

where $\beta_i \in \mathscr{A}$, and therefore

$$v^2 = \mathrm{Sq}^{28}(v) = \mathrm{Sq}^{12}\,\mathrm{Sq}^{16}(v) .$$

Hence by 3.3,

$$v^2 = \mathrm{Sq}^{12}\, a_2(\Phi_{02}(v)) ,$$

since $H^{36}(P_2X) = 0$. Now deg $a_2 = 12$ and therefore by the Adem relations

$$\mathrm{Sq}^{12}\, a_2 = \sum_{j=0}^{2} \alpha_j\, \mathrm{Sq}^{2^j} \quad (\alpha_j \in \mathscr{A}) .$$

Since $\Phi_{02}(v) \in H^{32}(P_2X)$, this shows by (4.9) that $v^2 = 0$. Thus $v = 0$ and hence $H^{28}(P_2X) = 0$.

Since $H^{28}(P_2X) = \mathrm{Sq}^4 H^{24}(P_2X)$ we obtain by (4.8) that $\mathrm{Sq}^{2^i} H^{24}(P_2X) = 0$ for $0 \leq i \leq 3$. The fact that $H^{24}(P_2X) = 0$ now follows by a similar argument to that used above. One must use (4.33) of [1], setting $k = 3$ in that lemma. We leave the details to the reader.

Finally, since $H^{24}(P_2X) = 0$ we see by (2.5) (taking $q = 4$) and (4.7) that $H^{16+j}(P_2X) = 0$ for $0 < j < 16$. Hence by (3.3), $\mathrm{Sq}^{16} H^{16}(P_2X) = 0$, which implies that $H^{16}(P_2X) = 0$. But this is a contradiction, since $\nu_2(X) = 15$. Hence, assuming that $q = 4$ has led to a contradiction, and therefore $q \leq 3$ – that is, $\nu_2(X) = 0, 1, 3,$ or 7, completing the proof of the theorem.

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