The equivalence of two definitions of quasiconformal mappings.

The equivalence of two definitions of quasiconformal mappings[1])

by Lipman Bers

We give here a new proof to the known fact (cf. Mori [7], Bers [3], Yûjôbô [12], Pfluger [11]) that the so-called analytic and geometric definitions of quasiconformality are equivalent. The proof uses a minimum of real variable techniques; no mention is made of absolute continuity in the sense of Tonelli. We rely instead on the theory of Beltrami's equations as exposed in Ahlfors-Bers [2] and on a theorem of Beurling-Ahlfors [5].

Let $z \to w(z) = u(x, y) + iv(x, y)$ be an orientation preserving homeomorphism of a plane domain D onto another. If the partial derivatives of w, in the sense of distribution theory, are locally square integrable functions, we denote by $K_a(D, w)$ the smallest constant $K \geq 1$ such that the inequality

$$\left|\frac{\partial w}{\partial x}\right|^2 + \left|\frac{\partial w}{\partial y}\right|^2 \leq \left(K + \frac{1}{K}\right)\frac{\partial(u, v)}{\partial(x, y)} \tag{1}$$

holds a.e. in D. If there is no such number, or if w does not have locally square integrable generalized derivatives, we set $K_a(D, w) = \infty$. If

$$K_a(D, w) < \infty\,,$$

w is said to be K-quasiconformal according to the *analytic definition* [4, 6, 9].

A *topological rectangle* R is a conformal image of a closed rectangle

$$0 \leq \xi \leq m,\; 0 \leq \eta \leq 1\,,$$

the images of the vertices being distinguished. We write $m = \operatorname{mod} R$. For $R \subset D$, $w(R)$ is also a topological rectangle, in view of Riemann's mapping theorem. We set

$$K_g(D, w) = \sup\left(\frac{\operatorname{mod} w(R)}{\operatorname{mod} R}\right), \quad R \subset D\,. \tag{2}$$

It is immediate that $K_g(D, w) = K_g(w(D), w^{-1})$. Mappings with

$$K_g(D, w) < \infty$$

are called K-quasiconformal according to the *geometric definition* [1, 8, 10].

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Theorem. $K_a(D, w) = K_g(D, w)$.

The proof requires several lemmas. The crux of the argument is contained in Lemma 8 below.

If $K_a(D, w) \leq K < \infty$, then $w(z)$ satisfies a BELTRAMI *equation*

$$\frac{\partial w}{\partial \bar{z}} = \mu(z) \frac{\partial w}{\partial z} \tag{3}$$

with a measurable coefficient $\mu(z)$ satisfying

$$|\mu(z)| \leq \frac{K-1}{K+1} . \tag{4}$$

Indeed, inequality (1) may be written as

$$\left| \frac{\partial w}{\partial \bar{z}} \right| \leq \frac{K-1}{K+1} \left| \frac{\partial w}{\partial z} \right| .$$

Thus the theory exposed in [2] is applicable. In particular

$$K_a(D, \varphi \circ w \circ \psi) = K_a(\psi(D), w)$$

if φ and ψ are conformal.

Now let D and $w(D)$ be JORDAN domains, z_0 a point in D and φ and ψ conformal mappings of D and $w(D)$ onto the unit disc with

$$\varphi(z_0) = \psi(w(z_0)) = 0 .$$

Then $W = \psi \circ w \circ \varphi^{-1}$ is a self-mapping of the unit disc with $W(0) = 0$. If $K(D, w) \leq K < \infty$, W is a solution of a BELTRAMI equation with a coefficient satisfying (4). Using [2] we obtain

Lemma 1. *If* $K_a(D, w) < \infty$, D *and* $w(D)$ *are* JORDAN *domains and* $z_0 \in D$, *then* w *has a uniform modulus of continuity depending only on* $K_a(D, w)$, D, $w(D)$, z_0 *and* $w(z_0)$.

Let R be a topological rectangle, $\mu(z)$, $z \in R$, a measurable function satisfying (4) for some $K \geq 1$, w and w_1 two homeomorphisms of D satisfying (3). Then $w \circ w_1^{-1}$ is a conformal mapping so that $\operatorname{mod} w(R) = \operatorname{mod} w_1(R)$. Hence we may define: $\operatorname{mod}(R, \mu) = \operatorname{mod} w(R)$.

Lemma 2. *Let* R *be a topological rectangle and* $\{\mu_j(z)\}$ *a sequence of measurable functions in* R *such that* $|\mu_j(z)| \leq k < 1$ *and* $\mu_j(z) \to \mu(z)$ *a.e. Then* $\operatorname{mod}(R, \mu_j) \to \operatorname{mod}(R, \mu)$.

Proof. We may assume that R is the unit disc made into a topological rectangle by choosing four "vertices" $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ on the boundary. Let W^{μ_j} be the homeomorphism of $|z| \leq 1$ onto itself with $W^{\mu_j}(0) = 0$, $W^{\mu_j}(1) = 1$

and $\partial W^{\mu_j}/\partial \bar{z} = \mu_j(z) \partial W^{\mu_j}/\partial z$ and let W^μ be defined similarly. By [2], p. 399, $W^{\mu_j} \to W^\mu$ uniformly in the closed unit disc. Since $\operatorname{mod}(R, \mu_j) = \operatorname{mod} W^{\mu_j}(R)$ is a continuous function of the cross-ratio of the points $W^{\mu_j}(\zeta_i)$, $i = 1, 2, 3, 4$, and similarly for $\operatorname{mod}(R, \mu)$, the conclusion follows.

Lemma 3. *Let R be a topological rectangle, $\mu(z)$, $z \in R$, a measurable function satisfying* (4). *Then* $\operatorname{mod}(R, \mu) \leq K \operatorname{mod} R$.

Proof. If $\mu(z)$ is smooth, every homeomorphic solution of (3) is smooth and has a positive jacobian (cf. [2], p. 391). In this case the desired inequality follows by GRÖTZSCH's classical argument [7]. The general case is reduced to this special one by Lemma 2, since it is easy to find a sequence of smooth μ_j satisfying (4) and converging a.e. to μ.

Lemma 4. $K_g(D, w) \leq K_a(D, w)$.
This is an immediate corollary of Lemma 3.

Lemma 5. *If* $K_a(D, w) < \infty$, *then* $K_g(D, w) = K_a(D, w)$.

Proof. Set $K_a(D, w) = K$ and assume that $1 < K < \infty$. (Otherwise there is nothing to prove.) In view of Lemma 4 it suffices to show that for every δ, $0 < \delta < K - 1$, there exists a sequence of squares $Q_j \subset D$ with

$$\lim \operatorname{mod} w(Q_j) \geq K - \delta . \tag{5}$$

Set $\mu(z) = (\partial w/\partial \bar{z})/(\partial w/\partial z)$ for $\partial w/\partial z \neq 0$, $\mu(z) = 0$ for $\partial w/\partial z = 0$; then w satisfies (4) and $\operatorname{ess\,sup} |\mu(z)| = (K - 1)/(K + 1)$. Let Δ denote the annulus

$$\frac{K - \delta - 1}{K + \delta + 1} \leq |\mu| \leq \frac{K - 1}{K + 1}$$

in the μ-plane; then $\mu^{-1}(\Delta) \subset D$ has positive measure. Let $\epsilon_j \downarrow 0$ be a given sequence. We can find a sequence of measurable sets Δ_j such that

$$\Delta_{j+1} \subset \Delta_j \subset \Delta\ , \quad \operatorname{diam} \Delta_j \leq \epsilon_j\ , \quad \operatorname{mes} \mu^{-1}(\Delta_j) > 0\ .$$

Indeed, if Δ is subdivided into finitely many measurable sets of diameter not exceeding ϵ_1, at least one of them, say Δ_1 must be such that $\operatorname{mes} \mu^{-1}(\Delta) > 0$. If Δ_1 is subdivided into finitely many measurable sets of diameter not exceeding ϵ_2, at least one of them, say Δ_2, is such that $\operatorname{mes} \mu^{-1}(\Delta_2) > 0$, etc. Let $\mu_0 = |\mu_0| e^{i\alpha}$ be the intersection of the closures of the Δ_j. For each j let $z_j \in D$ be a point at which the set $\mu^{-1}(\Delta_j)$ has metric density one; such points exist by LEBESGUE's theorem. Each z_j is the center of a square Q_j with one side parallel to the ray $z = re^{i\alpha/2}$, $0 < r < \infty$, and such that

$$Q_j \subset D, \quad \operatorname{mes} Q_j = m_j^2 < \epsilon_j, \quad \operatorname{mes}\,[Q_j \cap \mu^{-1}(\Delta_j)] \geq (1 - \epsilon_j)\, m_j^2\ .$$

Hence

$$\text{mes}\,\{z \mid z \in Q_j,\ \mid \mu(z) - \mu_0 \mid > \epsilon_j\} \leq \epsilon_j m_j^2 .$$

Let Q be the square obtained from Q_j by the mapping $z \to (z - z_j)/m_j$. For $z \in Q$ set $\mu_j(z) = \mu(z_j + m_j z)$. Then $\mu_j(z) \to \mu_0$ in measure. Selecting if need be a subsequence we may assume that $\mu_j(z) \to \mu_0$ a.e. in Q. By Lemma 2

$$\text{mod}\,(Q, \mu_j) \to \text{mod}\,(Q, \mu_0) = \frac{1 + \mid \mu_0 \mid}{1 - \mid \mu_0 \mid} \geq K - \delta .$$

Noting that $\text{mod}\, w(Q_j) = \text{mod}\,(Q, \mu_j)$ we obtain (5).

Lemma 6. *If* $w_j \to w$ *uniformly in* D *and* $K_a(D, w_j) \leq K < \infty$, *then* $K_a(D, w) \leq K$.

Proof. Set $w = u + iv$, $w_j = u_j + iv_j$. Let D_0 be a relatively compact subdomain of D. It suffices to show that $K_a(D_0, w) \leq K$. By [2] Theorem 5, and the hypothesis

$$\iint_{D_0} \frac{\partial(u_j, v_j)}{\partial(x, y)}\, dx\, dy = \text{mes}\, w_j(D_0) = O(1) .$$

Since (1) holds for each w_j,

$$\iint_{D_0} \left(\left| \frac{\partial w_j}{\partial x} \right|^2 + \left| \frac{\partial w_j}{\partial y} \right|^2 \right) dx\, dy = O(1) .$$

This shows that the partial derivatives of w_j are square-integrable functions in D_0 and that we may assume, selecting if need be a subsequence, that

$$\frac{\partial w_j}{\partial x} \to \frac{\partial w}{\partial x}, \quad \frac{\partial w_j}{\partial y} \to \frac{\partial w}{\partial y} \quad \text{weakly in } L_2(D_0) . \tag{6}$$

Next, let ω be a smooth function with compact support in D_0. Then

$$\begin{aligned} \iint_{D_0} \omega \frac{\partial(u_j, v_j)}{\partial(x, y)}\, dx\, dy &= \iint_{D_0} v_j \frac{\partial(\omega, u_j)}{\partial(x, y)}\, dx\, dy, \\ \iint_{D_0} \omega \frac{\partial(u, v)}{\partial(x, y)}\, dx\, dy &= \iint_{D_0} v \frac{\partial(\omega, u)}{\partial(x, y)}\, dx\, dy . \end{aligned} \tag{7}$$

If w and w_j are smooth, this follows by integration by parts. In the general case one approximates w (or w_j) together with its first derivatives, in the mean, by smooth functions. If ω is also non-negative, then

$$\iint_{D_0} \omega \left(\left| \frac{\partial w_j}{\partial x} \right|^2 + \left| \frac{\partial w_j}{\partial y} \right|^2 \right) dx\, dy \leq \left(K + \frac{1}{K} \right) \iint_{D_0} \omega \frac{\partial(u_j, v_j)}{\partial(x, y)}\, dx\, dy . \tag{8}$$

But by (6)

$$\iint\limits_{D_0} \omega\left(\left|\frac{\partial w}{\partial x}\right|^2+\left|\frac{\partial w}{\partial y}\right|^2\right)dx\,dy \le \liminf \iint\limits_{D_0} \omega\left(\left|\frac{\partial w_j}{\partial x}\right|^2+\left|\frac{\partial w_j}{\partial y}\right|^2\right)dx\,dy$$

and by (6) and (7)

$$\lim \iint\limits_{D_0} \omega \frac{\partial(u_j, v_j)}{\partial(x, y)}\,dx\,dy = \iint\limits_{D_0} \omega \frac{\partial(u, v)}{\partial(x, y)}\,dx\,dy\,,$$

so that by (8)

$$\iint\limits_{D_0} \omega\left(\left|\frac{\partial w}{\partial x}\right|^2+\left|\frac{\partial w}{\partial y}\right|^2\right)dx\,dy \le \left(K+\frac{1}{K}\right)\iint\limits_{D_0} \omega \frac{\partial(u, v)}{\partial(x, y)}\,dx\,dy\,.$$

A simple limiting argument shows that this holds also if ω is the characteristic function of a rectangle in D_0. This implies that (1) holds a.e.

We note now a corollary of the BEURLING-AHLFORS theorem [5].

Lemma 7. *For every* $K \ge 1$ *there exists a number* K^* *with the following property. Let* $K_g(D, w) \le K < \infty$ *and let* D_0 *be a relatively compact* JORDAN *subdomain of* D. *Then there exists a homeomorphism* Ω *of the closure of* D_0 *onto that of* $w(D_0)$ *with* $K_a(D, \Omega) \le K^*$ *and* $w(z) = \Omega(z)$ *on the boundary* $\dot{D}_0$ *of* D_0.

Proof. Choose a point $\hat{z}$ on $\dot{D}_0$ and set $\hat{Z} = w(\hat{z})$. Let $z \to \varphi(z)$ and $Z \to \psi(Z)$ be conformal homeomorphisms, of D_0 and $w(D_0)$, respectively, onto the half-plane $U = \{\zeta \mid \operatorname{Im} \zeta > 0\}$ with $\varphi(\hat{z}) = \psi(\hat{Z}) = \infty$. Set

$$\gamma(\xi) = \psi \circ w \circ \varphi^{-1}(\xi), \quad -\infty < \xi < +\infty\,.$$

For a real ξ and an $h > 0$ make D_0 into a topological rectangle $R_{\xi,h}$ by choosing as "vertices" the points $\varphi^{-1}(\xi - h)$, $\varphi^{-1}(\xi)$, $\varphi^{-1}(\xi + h)$ and $\hat{z}$. Then $\operatorname{mod} R_{\xi,h} = 1$. The "vertices" of $w(R_{\xi,h})$ are the points

$$\psi^{-1}(\gamma(\xi - h)),\ \psi^{-1}(\gamma(\xi)),\ \psi^{-1}(\gamma(\xi + h))$$

and $\hat{Z}$, and $\operatorname{mod} w(R_{\xi,h})$ is a continuous function of the ratio

$$(\gamma(\xi + h) - \gamma(\xi))\,/\,(\gamma(\xi) - \gamma(\xi - h))\,.$$

Since $K^{-1} \le \operatorname{mod} w(R_{\xi,h}) \le K$, there exists a $\varrho > 0$ depending only on K such that

$$0 < \frac{1}{\varrho} \le \frac{\gamma(\xi + h) - \gamma(\xi)}{\gamma(\xi) - \gamma(\xi - h)} \le \varrho\,. \tag{9}$$

According to [5], condition (9) implies that

$$\zeta = \xi + i\eta \to F(\zeta) = \tfrac{1}{2}\int_0^1 [(1+i)\,\gamma(\xi + \tau\eta) + (1-i)\,\gamma(\xi - \tau\eta)]\,d\tau$$

is a homeomorphism of the closed upper half-plane onto itself with $F(\xi) = g(\xi)$ and $K_a(U, F) = K^* < \infty$, where K^* depends only on K. Set $\Omega = \psi^{-1} \circ F \circ \varphi$; this mapping has the required properties.

Lemma 8. *If* $K_g(D, w) < \infty$, *then* $K_a(D, w) < \infty$.

Proof. Set $K_g(D, w) = K$. We show that for every square $Q \subset D$,

$$K_a(Q, w) \leq K^*,$$

the number in Lemma 7. For every integer $j > 0$ subdivide Q into 4^j congruent squares. Lemma 7 implies that there exists a homeomorphism w_j of Q such that $K_a(q, w_j) \leq K^*$ for each of the 4^j small squares q and $w_j = w$ on the boundary of each small square. Hence w_j is a homeomorphism of Q onto $w(Q)$ and $K_a(Q, w_j) \leq K^*$. By Lemma 1 the w_j are equicontinuous and, by construction, $w_j \to w$ on a dense set. Hence $w_j \to w$ uniformly and, by Lemma 6, $K_a(Q, w) \leq K^*$.

Combining Lemmas 5 and 8 we obtain the theorem.

Now set

$$K_1(D, w) = \inf \operatorname{mod} w(R) \quad \text{for all } R \subset D \text{ and } \operatorname{mod} R = 1$$

where R is a topological rectangle.

The argument used in proving Lemma 5 shows that $K_1(D, w) = K_a(D, w)$ whenever $K_a(D, w) < \infty$. The argument used in proving Lemmas 7 and 8 shows that $K_a(D, w)$ is finite whenever $K_1(D, w)$ is. Thus

$$K_1(D, w) = K_a(D, w)\,.$$

The geometric definition can be given a local form by setting (cf. PFLUGER [9]).

$$K^*(D, w) = \sup_{z_0 \in D} \lim_{r \to 0} K_g(S_r(z_0), w)$$

$S_r(z_0)$ being the disc $|z - z_0| < r$. We have that

$$K^*(D, w) = K_g(D, w)\,;$$

the proof is immediate via the equivalence theorem.

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