...-degenerate singular integral equations and holomorphic affine bundles over compact RIEMANN surfaces. I.

Ω-degenerate singular integral equations and holomorphic affine bundles over compact RIEMANN surfaces. I.[0])

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It is well known [2] that the classical theory of systems of FREDHOLM equations can be developed by carrying out the following two steps:

(i) proving FREDHOLM's theorems for degenerate equations, i.e. equations whose kernel is of the form $\sum_{\nu=1}^{n} B_\nu(x_1)C_\nu(x_2)$, B_ν and C_ν being square matrices,

(ii) approximating arbitrary kernels by degenerate kernels.

Since a degenerate FREDHOLM equation is equivalent to a system of linear equations, this approach leads quickly to the desired results.

In this paper we shall deal with (systems of) singular integral equations in one variable, which are then defined on a system S of (not necessarily disjoint) curves on a compact RIEMANN surface X. The integrals involved are therefore of the form $\int_S K(x_1, x_2)\, \Omega(x_2, x_1)$ where $K(x_1, x_2)$ is the kernel of the singular integral equation and $\Omega(x_2, x_1)$ is a CAUCHY-kernel on X. Again, a singular integral equation whose kernel can be written in the form (i) shall be called degenerate. However, since for RIEMANN surfaces X of higher genus the CAUCHY-kernel Ω is not unique but depends on the choice of certain divisors, one will have to speak of Ω-degenerate (instead of simply degenerate) singular integral equations. Clearly, the dominant equation of a singular integral equation and the adjoint of the dominant equation have Ω-degenerate kernels. Since the discussion of the dominant equation and its adjoint represents the main difficulty (see [7]) in the theory of singular integral equations, a theory of-Ω-degenerate integral equations is far from being trivial, contrary to the classical case of FREDHOLM equations.

Yet it is possible (§ 3) to associate with a given Ω-degenerate singular integral equation an equivalent transmission problem with values in a trivial vector bundle over X (terminology as in [11]), provided the singular integral equation is regularizable and satisfies certain conditions (which are trivially fullfilled if S is the disjoint union of simple closed curves). Since one can associate with a transmission problem of the described type a holomorphic affine bundle over X (see [11]) such that the holomorphic solutions of the transmission problem correspond bijectively to the holomorphic sections in this affine bundle, one gets finally (§ 4) a bijective correspondence between the set of integrable

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solutions of the original singular integral equation and the set of holomorphic sections in the affine bundle over X. This is somewhat surprising since in the case of the RIEMANNian sphere there are only denumerably many isomorphy classes of holomorphic vector bundles (see [4]) while the set of Ω-degenerate singular integral equations has obviously cardinality $\aleph$.

One of the advantages of this approach consists in the following result. Introducing suitable families $\mathfrak{B} \to M$ of compact RIEMANN surfaces (see [5]), families of CAUCHY-kernels $(\Omega_t : t \in M)$, families of systems $(S_t : t \in M)$ of curves, and families of matrices $(B_{\nu t} : t \in M)$ and $(C_{\nu t} : t \in M)$, one can investigate the corresponding family of Ω_t-degenerate singular integral equation and ask for solutions which depend in the same way on the parameter $t \in M$ as the above families do. There it turns out that a family $\mathfrak{W} \to \mathfrak{B} \to M$ of holomorphic affine bundles can be constructed so that the set of those solutions of the integral equation which depend on t as desired corresponds bijectively to the set of those families of holomorphic sections in $W_t \to \mathfrak{B}_t$ (terminology as in [5]) which depend on t in the same way. Therefore known results ([5], [10]) on the existence of families of holomorphic sections in certain fiber bundles lead immediately to similar results about solutions of families of Ω_t-degenerate singular integral equations.

In § 5 we determine the vector bundle associated with the adjoint of an Ω-degenerate singular integral equation in terms of the vector bundle associated with the integral equation itself. I turns out to be the dual of the previous vector bundle. Therefore the RIEMANN-ROCH theorem for vector bundles gives immediately two of F. NOETHER's theorems on singular integral equations. The remaining one, stating necessary and sufficient conditions for the solubility of inhomogeneous Ω-degenerate equations, is a consequence of CAUCHY's integral formula resp. CAUCHY's integral representation.

In § 6 we discuss briefly how the proof of F. NOETHER's theorems for general singular integral equations can be reduced to the case of Ω-degenerate singular integral equations. This furnishes then a new foundation of the theory of singular integral equations based on PLEMELJ's theorems and the theory of affine bundles over compact RIEMANN surfaces.

Finally the previously developed method is applied to certain non-linear singular integral equations for which then an existence and uniqueness theorem is derived.

§ 1. Auxiliary results

I. Given a compact RIEMANN surface X of genus g, the positive divisor $D = \sum_{\kappa=1}^{k} n_\kappa x_\kappa^0$ of degree $\sum_{\kappa=1}^{k} n_\kappa = g$ shall be called a normalization divisor ([6])

if $\dim_C(-D) = 1$, i.e. if every meromorphic function on X which is holomorphic in $X - \{x_1^0, \ldots, x_k^0\}$ and has pole order $\leq n_\varkappa$ at $x_\varkappa^0$, $\varkappa = 1, \ldots, k$, is constant. In case the support $\operatorname{supp} D = \{x_1^0, \ldots, x_k^0\}$ of D consists of a single point, D is a normalization divisor if and only if this point is not a WEIERSTRASS point.

In generalization of the CAUCHY-kernel $\dfrac{dz}{z - w}$ on the RIEMANNian sphere it can be shown ([8], [12]) that, given the normalization divisor D on X and a point $* \in X - \operatorname{supp} D$, there exists a meromorphic differential form $\Omega(x_1, x_2)$ of degree one on $X \times X$ along the first factor (i.e. in complex coordinates t_1 for the first factor and t_2 for the second factor of $X \times X$ the differential form $\Omega(x_1, x_2)$ can be written in the form $f(t_1, t_2) dt_1$ where $f(t_1, t_2)$ is a suitable meromorphic function of t_1 and t_2) having the following properties:

(i) $\Omega(x_1, x_2)$ has divisor

$$\geq X \times \{*\} - \{*\} \times X + D \times X - X \times D - \Delta$$

where Δ is the diagonal of $X \times X$,

(ii) for every $x_2 \notin \{*\} \cup \operatorname{supp} D$ the restriction of $\Omega(x_1, x_2)$ to $X \times \{x_2\}$ is a differential form which has divisor

$$\geq \{x_2\} \times \{x_2\} - \{*\} \times \{x_2\} + D \times \{x_2\}$$

and residue $+1$ at $\{x_2\} \times \{x_2\}$ (and hence residue -1 at $\{*\} \times \{x_2\}$). $\Omega(x_1, x_2)$ is called the *CAUCHY-kernel of X associated with D and $*$.*

Clearly, $\Omega(x_1, x_2)$ is unique. The existence of $\Omega(x_1, x_2)$ can be established (see [8], [12]) by constructing for each $x_2 \notin \{*\} \cup \operatorname{supp} D$ the unique meromorphic differential form on $X \times \{x_2\}$ which satisfies (ii) and then showing that this gives rise to a meromorphic differential form on $X \times (X - \{*\} \cup \operatorname{supp} D)$ along the first factor which can be extended to a meromorphic differential form on $X \times X$. The divisor can then be calculated explicitly.

(i) and (ii) lead to the following

Proposition 1.1: *Let $\mathfrak{C}$ be a piecewise smooth, compact arc in X which does not meet $\{*\} \cup \operatorname{supp} D$. Then for every continuous complex valued function f on $\mathfrak{C}$ the integral*

$$(\Omega f)(x_2) = \frac{1}{2\pi i} \int_{\mathfrak{C}} f(x_1)\, \Omega(x_1, x_2)$$

is a meromorphic function on $X - \mathfrak{C}$ whose divisor is $\geq * - D$.

Denoting the canonical bundle ([3]) over X by $K_X \to X$, a continuous differential form on the piecewise smooth, compact arc $\mathfrak{C}$ in X is meant to be a continuous section in $K_X \to X$ over $\mathfrak{C}$. With this notion we get as a dual of Proposition 1.1.

Proposition 1.2: *Let $\mathfrak{C}$ be a piecewise smooth, compact arc in X which does not meet $\{*\} \cup \operatorname{supp} D$. Then for every continuous differential form η on $\mathfrak{C}$ the integral*

$$(\Omega\eta)(x_2) = \frac{1}{2\pi i} \int_{\mathfrak{C}} \Omega(x_2, x_1)\, \eta(x_1)$$

is a meromorphic differential form on $X - \mathfrak{C}$ whose divisor is $\geq D - *$.

Now let $\mathfrak{B} \xrightarrow{\pi} M$ be a differentiable (holomorphic) family of compact RIEMANN surfaces of genus g in the sense of KODAIRE-SPENCER ([5]). The restriction of the product family $\mathfrak{B} \times \mathfrak{B} \xrightarrow{\pi\times\pi} M \times M$ to the diagonal Δ_M of $M \times M$ is then in a canonical way a differentiable (holomorphic) family over M which shall be denoted by $\mathfrak{B} \times_M \mathfrak{B} \xrightarrow{\pi\times_M\pi} M$. Obviously, $(\pi \times_M \pi)^{-1}(t) = \pi^{-1}(t) \times \times \pi^{-1}(t)$ holds. Now suppose that $(D_t : t \in M)$ is a differentiable (holomorphic) family of normalization divisors in $\mathfrak{B} \to M$, i.e. $D_t = \sum\limits_{\kappa=1}^{n} n_\kappa x^0_{\kappa,t}$ is a normalization divisor in $\pi^{-1}(t)$ for every $t \in M$ and $(x^0_{\kappa,t} : t \in M)$ is a differentiable (holomorphic) submanifold of $\mathfrak{B}$ for each $\varkappa = 1, \ldots, k$. Assume furthermore that $(*_t : t \in M)$ is a differentiable (holomorphic) family of points in $\mathfrak{B} \to M$ which is disjoint from $\operatorname{supp}(D_t : t \in M)$. Then the previously indicated construction of $\Omega(x_1, x_2)$ together with Theorem 2.1 (Theorem 18.1 of [5]) shows that for every $t_0 \in M$ there is a neighborhood U of t_0 such that the CAUCHY-kernel Ω_t of $\pi^{-1}(t)$ associated with D_t and $*_t$ depends differentiably (holomorphically) upon $t \in U$. This and the uniqueness of the CAUCHY-kernel imply that Ω_t varies differentiably (holomorphically) with $t \in M$. Summing up we get

Proposition 1.3: *Let $\mathfrak{B} \to M$ be a differentiable (holomorphic) family of compact RIEMANN surfaces of genus g, $(D_t : t \in M)$ a differentiable (holomorphic) family of normalization divisors on $\mathfrak{B} \to M$, and $(*_t : t\ M)$ a differentiable (holomorphic) family of points on $\mathfrak{B} \to M$ which is disjoint from $(D_t : t \in M)$. Then $(\Omega_t : t \in M)$ is a differentiable (holomorphic) family of meromorphic differential forms on $\mathfrak{B} \times_M \mathfrak{B} \to M$.*

Given the differentiable (holomorphic) family $\mathfrak{B} \to M$, *a differentiable (holomorphic) family of piecewise smooth, compact arcs* $(\mathfrak{C}_t : t \in M)$ is a continuous mapping $\mathfrak{C} : M \times I \to \mathfrak{B}$[1]) such that

(i) $\pi \circ \mathfrak{C} = pr_1$ (= canonical projection of $M \times I$ onto M)

(ii) I is the union of finitely many closed subintervals I_ϱ, $\varrho = 1, \ldots, r$, such that $\mathfrak{C} | M \times I_\varrho$ is differentiable and both, $\mathfrak{C} | M \times I_\varrho$ and $\frac{\partial \mathfrak{C}}{\partial \tau} | M \times I_\varrho$, depend differentiably (holomorphically) on $t \in M$.

[1]) I denotes the closed unit interval $\{\tau : 0 \leq \tau \leq 1\}$.

Given the differentiable (holomorphic) family of piecewise smooth, compact arcs $(\mathfrak{C}_t : t \in M)$, the *function* $f : \bigcup_{t\in M} \mathfrak{C}_t \to \mathbf{C}$ is said *to depend continuously (differentiably, holomorphically)* on $t \in M$ if for every $\varrho = 1, \ldots, r$ the composition $f \circ \mathfrak{C} | M \times I_\varrho$ has this property. A corresponding definition can be given for a differential form η on $\bigcup_{t\in M} \mathfrak{C}_t$, i.e. a section over $\bigcup_{t\in M} \mathfrak{C}_t$ in the family $\mathfrak{K} \to \mathfrak{B} \to M$ of canonical bundles associated with $\mathfrak{B} \to M$.

With these notions, Proposition 1.1 and Proposition 1.2 generalize in the obvious way to families. The precise formulation is left to the reader.

II. Let S be a closed subset of the RIEMANN surface X and $x \in S$. Then S is said to be a *star of smooth arcs at* x if there exists a neighborhood U of x and finitely many simple smooth arcs $\mathfrak{C}_\nu : I \to U$ such that

(i) $\mathfrak{C}_\nu(0) = x$ for every ν

(ii) $\bigcup \mathfrak{C}_\nu(I) = S \cap U$

(iii) $\mathfrak{C}_{\nu_1}(I) \cap \mathfrak{C}_{\nu_2}(I) = \{x\}$ whenever $\nu_1 \neq \nu_2$.

The number of the arcs $\mathfrak{C}_\nu$ is called the *order of the star at* x.

S is said to be *locally a star of smooth arcs* if for every $x \in S$ the set S is a star of smooth arcs at x. Clearly, a piecewise smooth, simple arc can be interpreted as a set with the above properties, regardless whether the arc contains cusps or not

By replacing X by a differentiable (holomorphic) family $\mathfrak{B} \to M$ we can talk about a *differentiable (holomorphic) family of smooth arcs at* $v_0 \in S$ where S is a closed subset of $\mathfrak{B}$: in this case the arcs $\mathfrak{C}_\nu$ are required to be differentiable (holomorphic) families of smooth, compact arcs such that

(i) $\mathfrak{C}_\nu(\pi(v_0), 0) = v_0$ for every ν

(ii) $\bigcup \mathfrak{C}_\nu(\pi(U) \times I) = S \cap U$

(iii) $\mathfrak{C}_{\nu_1}(\{t\} \times I) \cap \mathfrak{C}_{\nu_2}(\{t\} \times I) = \{\mathfrak{C}_{\nu_1}(t, 0)\}$ whenever $\nu_1 \neq \nu_2$.

Let $\mathfrak{C} : I \to X$ be an arc. Then $\mathfrak{C}(0)$ and $\mathfrak{C}(1)$ are usually called the endpoints of the arc. To assign to $\mathfrak{C}$ an orientation means to distinguish one of the endpoints of $\mathfrak{C}$ as the initial point of the oriented arc. Now let the closed subset S of X be a star of smooth arcs at $x \in S$, the star being described as above. Then an *orientation of the star at* x consists in assigning an orientation to each of the arcs $\mathfrak{C}_\nu$. An equivalent description – which shall be used from now on – of an orientation can be gotten by assigning to each index ν the integer $\sigma_x(\nu) = +1$ if x is the initial point of $\mathfrak{C}_\nu$ and the integer $\sigma_x(\nu) = -1$ is x is not the initial point of $\mathfrak{C}_\nu$. With this notation, $\sum_\nu \sigma_x(\nu)$ is called the *total orientation of the star* at x and is denoted by $|\sigma_x|$. These definitions carry over immediately to families, in which case we require that for every ν the orientation of $\mathfrak{C}_\nu | \{t\} \times I$ is independent of $t \in M$.

Given the star S of smooth arcs at x which is described as above, we can find to every point $x' \in S \cap U$ a neighborhood U' such that $S \cap U'$ is again a star of smooth arcs at x' which then is of order 2 at x'. An orientation of the star at x induces in an obvious way an orientation of the star at x' which then has total orientation 0. Suppose now that the closed subset S of X is locally a star of smooth arcs and that for every point $x \in S$ an orientation σ_x is given. Then the set $\Sigma = \{\sigma_x : x \in S\}$ of orientation is said to be *coherent* if for any two points x' and x'' of S and the neighborhoods U' of x' and U'' of x'' which where used in defining $\sigma_{x'}$ and $\sigma_{x''}$, both $\sigma_{x'}$ and $\sigma_{x''}$ induce the same orientation on S in $U' \cap U''$.

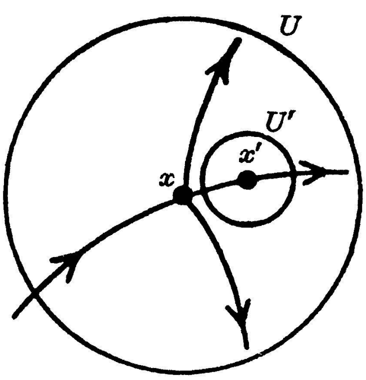

Fig. 1

As an immediate consequence of the various definitions of II. we get

Proposition 1.4: *Let X be a compact* Riemann *surface, S a closed subset of X which is locally a star of smooth arcs, and Σ a coherent set of orientations on S. Then there are finitely many simple, smooth, oriented arcs $\mathfrak{C}^{(1)}, \ldots, \mathfrak{C}^{(l)}$ on X such that*

(i) $S = \bigcup_{\lambda=1}^{l} \mathfrak{C}^{(\lambda)}(I)$

(ii) $\mathfrak{C}^{(\lambda_1)}(I) \cap \mathfrak{C}^{(\lambda_2)}(I)$ *is either empty or else a common endpoint of $\mathfrak{C}^{(\lambda_1)}$ as well as $\mathfrak{C}^{(\lambda_2)}$ provided $\lambda_1 \neq \lambda_2$*

(iii) *the orientation of $\mathfrak{C}^{(\lambda)}$ agrees with the one induced by Σ in each point of $\mathfrak{C}^{(\lambda)}$.*

If l is chosen minimal, then the arcs $\mathfrak{C}^{(\lambda)}$ are unique (up to order).

Clearly, Proposition 1.4 remains valid for families.

III. Finally we need a result concerning the determinant of a certain matrix.

Lemma 1.5: *Let $B_1, \ldots, B_n$, $C_1, \ldots, C_n$ be $q \times q$ matrices over the field of complex numbers. Denoting the $q \times q$ unit matrix by 1 we have*

$$\det \begin{pmatrix} 1 - C_1 B_1, & -C_1 B_2, \ldots, & -C_1 B_n \\ -C_2 B_1, & 1 - C_2 B_2, \ldots, & -C_2 B_n \\ \ldots & & \\ -C_n B_1, & -C_n B_2, \ldots, & 1 - C_n B_n \end{pmatrix} = \det\Big(1 - \sum_{\nu=1}^{n} B_\nu C_\nu\Big)$$

Proof: In order to prove this identity, we may assume that $\det C_\nu \neq 0$ for $\nu = 1, \ldots, n$. Then we get

$$\det \begin{pmatrix} 1 - C_1 B_1, \ldots, & -C_1 B_n \\ \ldots & \\ -C_n B_1, \ldots, & 1 - C_n B_n \end{pmatrix} =$$

$$= \det \begin{pmatrix} C_1 & & 0 \\ & \ddots & \\ 0 & & C_n \end{pmatrix} \det \begin{pmatrix} C_1^{-1} - B_1, & - B_2, \ldots, & - B_n \\ \ldots & & \\ - B_1, \ldots & , & C_n^{-1} - B_n \end{pmatrix}$$

$$= \det \begin{pmatrix} C_1^{-1} & , - C_2^{-1}, 0 & , \ldots\ldots, 0 \\ 0 & , C_2^{-1} \quad , - C_3^{-1}, & \ldots\ldots, 0 \\ \ldots\ldots & & \\ -B_1, - B_2 & , \ldots\ldots & , C_n^{-1} - B_n \end{pmatrix} \det \begin{pmatrix} C_1 & & 0 \\ & \ddots & \\ 0 & & C_n \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & , -1 & , \quad 0, \ldots, 0 \\ 0 & , 1 & , -1, \quad \ldots, 0 \\ \ldots & & \\ - B_1 C_1, & - B_2 C_2, \ldots & , 1 - B_n C_n \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & , & 0 & , \ldots, 0 \\ 0 & , & 1 & , \ldots, 0 \\ \ldots & & & \\ - B_1 C_1, & - B_1 C_1 - B_2 C_2, & \ldots, & 1 - \sum_{\nu=1}^{n} B_\nu C_\nu \end{pmatrix}$$

$$= \det \left(1 - \sum_{\nu=1}^{n} B_\nu C_\nu\right) .$$

§ 2. PLEMELJ's theorems

Let X be a RIEMANN surface and S a closed subset of X which is locally a star of smooth arcs. Let f be a complex valued function on S. Then f is called *HÖLDER continuous on* S if for every point $x \in S$ there is a complex coordinate t at x whose coordinate neigborhood is contained in the neighborhood U used in § 1, II., (i) — (iii), such that for each ν the restriction of $f \circ t^{-1}$ to $t(\mathfrak{C}_\nu(I))$ is HÖLDER continuous in the usual sense in some neighborhood of $t(x)$. Obviously, this definition is independent of the choice of the complex coordinate t. Accordingly a complex valued function k on $S \times S$ is said to be *HÖLDER continuous on* $S \times S$ if for every ν the restriction of $k \circ (t^{-1} \times t^{-1})$ to $t(\mathfrak{C}_\nu(I)) \times t(\mathfrak{C}_\nu(I))$ is HÖLDER continuous in the usual sense in some neighborhood of $(t(x), t(x))$.

Now let X be a compact RIEMANN surface with normalization divisor D and Ω the CAUCHY-kernel of X associated with D and $*$. Let furthermore S be a closed subset of X which is locally a star of smooth arcs and Σ a coherent set of orientations on S. In order to define the *singular integral*

$$\frac{1}{2\pi i} \int_{(S,\Sigma)} f(x_1)\, \Omega(x_1, x_2) \quad x_2 \in S \tag{1}$$

where f is a complex valued function on S, we need the following well known ([7], Appendix 1) result. Given $x_2 \in S$ and the complex coordinate t at x_2

whose coordinate neighborhood is contained in the neighborhood U used in § 1, II., (i) – (iii), there exists a positive real number ε_0 such that for every ν and for every positive real number $\varepsilon \leq \varepsilon_0$ the set $\{x \in U: |t(x)| = \varepsilon\} \cap \mathfrak{C}_\nu(I)$ consists of exactly one point. Since $\mathfrak{C}_\nu$ is simple, this implies that for every arc $\mathfrak{C}^{(\lambda)}$ in the splitting of S described in Proposition 1.4 and for all ε which are sufficiently small the difference $\mathfrak{C}^{(\lambda)} - \mathfrak{C}^{(\lambda)} \cap \{x \in U: |t(x)| \leq \varepsilon\}$ is a connected subarc of $\mathfrak{C}^{(\lambda)}$. This subarc equipped with the induced orientation is denoted by $\mathfrak{C}_\varepsilon^{(\lambda)}$. With these notations we define the singular integral (1) to be

$$\lim_{\varepsilon \to 0} \sum_{\lambda=1}^{l} \frac{1}{2\pi i} \int_{\mathfrak{C}_\varepsilon^{(\lambda)}} f(x_1)\, \Omega(x_1, x_2) \tag{1'}$$

provided that this limit exists.

According to definition, the existence of (1′) is a purely local problem, whence one has to deal only with the geometric situation as described in § 1, II., (i) – (iii). Using [7], § 12, (12.3) one sees from the argument in [7], § 12, following formula (12.3) that the sufficient conditions for the existence of (1) are given by

Proposition 2.1: *Let X be a compact* Riemann *surface, Ω the* Cauchy-*kernel of X associated with D and $*$, S a closed subset of X which is locally a star of smooth arcs and which satisfies $S \cap (\{*\} \cup \operatorname{supp} D) = \varnothing$, and $\Sigma = \{\sigma_x : x \in S\}$ a coherent set of orientations on S. Let f be a* Hölder *continuous function on S such that*

$$|\sigma_x|\, f(x) = 0 \quad \textit{for every} \quad x \in S\,. \tag{2}$$

Then the singular integral (1) *exists for every point $x_2 \in S$.*

Corresponding definitions and results hold for singular integrals

$$\frac{1}{2\pi i} \int_{(S,\Sigma)} \Omega(x_1, x_2)\, \eta(x_2) \qquad x_1 \in S$$

where η is a Hölder continuous differential form on S.

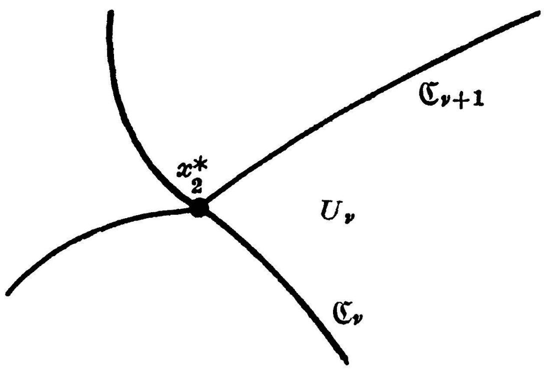

Fig. 2

Let us consider the geometric situation as described in § 1, II., (i) – (iii). For a given point $x_2^* \in S$ of order n we choose ε_0 as before. Then $\{x \in U: |t(x)| < \varepsilon_0\} - \bigcup_{\nu=1}^{n} \mathfrak{C}_\nu(I) \cap U$ consists of exactly n connected components $U_1, \ldots, U_n$. These connected components shall be indexed in such a way that the boundary of U consists of a part of the set $\{x \in U: |t(x)| = \varepsilon_0\}$

together with $\mathfrak{C}_\nu(I) \cap U$ and $\mathfrak{C}_{\nu+1}(I) \cap U$ where we set $\mathfrak{C}_{n+1} = \mathfrak{C}_1$ (note that this may require a reindexing of the $\mathfrak{C}_\nu$). Furthermore we assume that, when going through $\{x \in U : |t(x)| = \varepsilon_0\}$ in the positive sense starting out at $\mathfrak{C}_1(I)$, we meet $U_1, U_2, \ldots, U_n$ in this order.

Using the orientation $\sigma_{x_2^*}$ at the point x_2^* of order n we define now integer valued functions σ_ν, $\nu = 1, \ldots, n$, as follows (U_{n+1} is identified with U_1)

(i) if $n \neq 2$, then for $\nu = 1, \ldots, n$ and $x \in \overline{U}_\nu \cap \overline{U}_{\nu+1} \cap S$ the function $\sigma_\nu(x)$ is defined to be $\sigma_{x_2^*}(\nu)$

(ii) if $n = 2$, then we set

$$\sigma_1(x) = \begin{cases} -\sigma_{x_2^*}(1) & \text{for } x \in \mathfrak{C}_1(I) - \{x_2^*\} \\ \sigma_{x_2^*}(2) & \text{for } x \in \mathfrak{C}_2(I) \end{cases}$$

$$\sigma_2(x) = \begin{cases} \sigma_{x_2^*}(1) & \text{for } x \in \mathfrak{C}_1(I) \\ -\sigma_{x_2^*}(2) & \text{for } x \in \mathfrak{C}_2(I) - \{x_2^*) \end{cases}$$

Now let f be a HÖLDER continuous function on S. Then the function

$$(\Omega f)(x_2) = \frac{1}{2\pi i} \int_{(S,\Sigma)} f(x_1)\Omega(x_1, x_2) \quad x_2 \in X - S$$

is defined, provided $S \cap (\{*\} \cup \operatorname{supp} D) = \Phi$. Moreover, it is holomorphic in a sufficiently small neighborhood of S. We shall have to investigate the behavior of this function as we approach S.

Assume for the time being that x_2 is a point of order 2 of S with total orientation 0 and that the angle between the oriented tangents at $\mathfrak{C}_1$ and $\mathfrak{C}_2$ in x_2^* is different from π. Then, given the complex coordinate t at x_2^*, we get for x_1 and x_2 in a sufficiently small neighborhood of x_2^*

$$\Omega(x_1, x_2) = \frac{dt(x_1)}{t(x_1) - t(x_2)} + h(t(x_1), t(x_2))\, dt(x_1)$$

where $h(t_1, t_2)$ is a suitable holomorphic function of t_1 and t_2. Therefore well known theorems ([7], § 16) imply immediately that there is a neighborhood V of x_2^* in X such that Ωf can be extended continuously into $\overline{U}_\mu \cap V$, $\mu = 1, 2$. This extension restricted to $S \cap V$ is HÖLDER continuous ([7], § 19). Moreover, denoting the extension of Ωf to $\overline{U}_\mu \cap V$ by $\Omega_\mu f$, [7], § 17, implies the validity of PLEMELJ's formulas

$$\left.\begin{aligned} (\Omega_{\mu+1} f)(x_2) - (\Omega_\mu f)(x_2) &= \sigma_\mu(x_2) f(x_2) \\ (\Omega_{\mu+1} f)(x_2) + (\Omega_\mu f)(x_2) &= \frac{1}{\pi i} \int_{(S,\Sigma)} f(x_1)\, \Omega(x_1, x_2) \end{aligned}\right\} \quad x_2 \in S \cap V \qquad (3)$$

In order to get the formulas (3) in the general case too, we adjoin to the star of smooth arcs consisting of $\mathfrak{C}_1, \ldots, \mathfrak{C}_n$ a line segment $\mathfrak{C}_0$ with one end point in x_2^* such that

(i) the orientation of $\mathfrak{C}_0$ at x_2^* is given by -1, i.e. $\sigma_{x_2^*}(0) = 1$

(ii) the angle between the oriented tangents at $\mathfrak{C}_0$ and $\mathfrak{C}_\nu$, $\nu = 1, \ldots, n$, in x_2^* is different from π

(iii) $\mathfrak{C}_0$ does not enter U_μ.

Assuming that f satisfies the additional hypothesis (2) of Proposition 2.1 we get for $x_2 \in U_\mu$

$$\Omega(x_1, x_2) = \sum_{\lambda=1}^{l} \frac{1}{2\pi i} \int_{\mathfrak{C}_{e_0}^{(\lambda)}} f(x_1)\, \Omega(x_1, x_2) + \sum_{\nu=1}^{n} \frac{1}{2\pi i} \int_{\mathfrak{C}_\nu} f(x_1)\, \Omega(x_1, x_2) =$$

$$= \sum_{\lambda=1}^{l} \frac{1}{2\pi i} \int_{\mathfrak{C}_{e_0}^{(\lambda)}} f(x_1)\, \Omega(x_1, x_2) + \sum_{\nu=1}^{n} \left(\frac{1}{2\pi i} \int_{\mathfrak{C}_\nu} f(x_1)\, \Omega(x_1, x_2) + \frac{\sigma_{x_2^*}(\nu)}{2\pi i} f(x_2^*) \int_{\mathfrak{C}_0} \Omega(x_1, x_2) \right)$$

Denoting by $\sigma_{x_2^*}(\nu)\, \mathfrak{C}_0$ the arc $\mathfrak{C}_0$ equipped with the orientation $\sigma_{x_2^*}(0)\, \sigma_{x_2^*}(\nu)$ and letting $\mathfrak{C}'_\nu$ be the oriented arc gotten from the oriented arcs $\mathfrak{C}_\nu$ and $\sigma_{x_2^*}(\nu)\, \mathfrak{C}_0$, we have

$$\frac{1}{2\pi i} \int_{\mathfrak{C}_\nu} f(x_1)\, \Omega(x_1, x_2) + \frac{\sigma_{x_2^*}(\nu)}{2\pi i} f(x_2^*) \int_{\mathfrak{C}_0} \Omega(x_1, x_2) = \frac{1}{2\pi i} \int_{\mathfrak{C}'_\nu} f'_\nu(x_1)\, \Omega(x_1, x_2)$$

where f'_ν coincides with f on $\mathfrak{C}_\nu$ and equals to the constant function $f(x_2^*)$ on $\mathfrak{C}_0$. To the latter integral, however, the previously discussed special case applies, whence we get the formulas (3) in a rather obvious way for $n \geq 2$. For the sake of maintaining the above notation we reduce the case $n = 1$ to the case $n = 2$ by adding to the arc $\mathfrak{C}_1$ the previously described arc $\mathfrak{C}_0$ and extending the definition of f to $\mathfrak{C}_0$ as above.

The local situation we dealt with so far can be globalized as follows (see also [11]). For every point $x \in X$ we denote by $\mathfrak{U}(x)$ the filter of neighborhoods of x. The trace of $\mathfrak{U}(x)$ on $X - S$ (see [1]) shall be denoted by $\mathfrak{U}_S(x)$. Now we form the set $\overline{\overline{X}}$ of all pairs $(x, \mathfrak{F})$ where x is a point of X and $\mathfrak{F}$ is a filter that refines $\mathfrak{U}_S(x)$ and has a basis such that each element of this basis is a connected component of some element of $\mathfrak{U}_S(x)$. The mapping of $\overline{\overline{X}}$ onto X which sends $(x, \mathfrak{F})$ into x is denoted by pr. Given $(x, \mathfrak{F}) \in \overline{\overline{X}}$ we denote for every $F \in \mathfrak{F}$ by $\overline{\overline{F}}$ the set of all elements $(x', \mathfrak{F}')$ in $\overline{\overline{X}}$ such that $F \in \mathfrak{F}'$. Defining $\mathfrak{V}(x, \mathfrak{F})$ to be the filter generated by $\{\overline{\overline{F}} : F \in \mathfrak{F}\}$ we associate with every element of $\overline{\overline{X}}$ a filter of subsets of $\overline{\overline{X}}$. As one can check easily there is a topology on $\overline{\overline{X}}$ such that the filter of neighborhoods of any element $(x, \mathfrak{F})$ coincides with $\mathfrak{V}(x, \mathfrak{F})$. Obviously, $pr: \overline{\overline{X}} \to X$ is a continuous mapping. Now let $(X_j : j \in J)$ be the family of connected components of $X - S$. Then the restriction of pr to $pr^{-1}(X_j)$ is a homeomorphism onto X_j. The closure $\overline{\overline{X}}_j$ of $pr^{-1}(X_j)$ in $\overline{\overline{X}}$ is both closed and open in $\overline{\overline{X}}$. Furthermore, $pr(\overline{\overline{X}}_j)$ equals the closure $\overline{X}_j$ of X_j in X. Denoting the filter of subsets of X generated by the set $\{\overline{F}:F \in \mathfrak{F}\}$ by $\overline{\mathfrak{F}}$, we call two distinct elements $(x, \mathfrak{F})$ and $(x', \mathfrak{F}')$ neighbors if $x = x' \in S$ and $\sup(\mathfrak{F}, \mathfrak{F}')$ is strictly coarser than the filter of all sets containing x. This means – using the previous notation – that for a suitable choice of the index ν the set U_ν belongs to one of the two filters $\mathfrak{F}$ and $\mathfrak{F}'$, while $U_{\nu+1}$ belongs to the other one. If $U_\nu \in \mathfrak{F}$, then we say that $(x, \mathfrak{F})$ preceeds $(x, \mathfrak{F}')$. Clearly, every element $(x, \mathfrak{F})$, where $x \in S$, has at most one predecessor and at most one successor; it has exactly one if x is a point of order > 1. Denoting by $\overline{\overline{X}}_\beta$ the set of all elements $\overline{\overline{x}}$ in $\overline{\overline{X}}$ which either have a successor or else lie above a point of order 1 on S, we get a canonical mapping $\beta: \overline{\overline{X}}_\beta \to \overline{\overline{X}}$ which sends every point $\overline{\overline{x}} \in \overline{\overline{X}}_\beta$ for which $pr(\overline{\overline{x}})$ has order > 1 into its successor and leaves every point $\overline{\overline{x}} \in \overline{\overline{X}}_\beta$ for which $pr(\overline{\overline{x}})$ has order 1 fixed. Clearly, β is a continuous mapping which maps $\overline{\overline{X}}_\beta$ onto itself.

The previously defined functions $\sigma_\nu(x)$ are now transplanted onto $\overline{\overline{X}}$ by setting $\sigma(\overline{\overline{x}}) = \sigma_\nu(pr\,\overline{\overline{x}})$, provided $\overline{\overline{x}} = (pr\,\overline{\overline{x}}, \mathfrak{F}) \in \overline{\overline{X}}_\beta$ and the set U_ν is contained in $\mathfrak{F}$. Finally we define the function $\overline{\overline{\Omega}}f$ on $\bigcup_{j \in J} pr^{-1}(X_j)$ by $(\overline{\overline{\Omega}}f)(\overline{\overline{x}}) = (\overline{\Omega}f)(pr\,\overline{\overline{x}})$.

With these notations the previously treated local situation gives immediately rise to the following generalization of PLEMELJ's theorem and formulas ([7]).

Theorem 2.2: *Assuming that the hypotheses of Proposition 2.1 are fulfilled, the function $\overline{\overline{\Omega}}f$ can be extended continuously to $\overline{\overline{X}}$. Furthermore, for every point $\overline{\overline{x}} \in \overline{\overline{X}}_\beta$*

$$(\overline{\overline{\Omega}}f)(\beta\overline{\overline{x}}) - (\overline{\overline{\Omega}}f)(\overline{\overline{x}}) = \sigma(\overline{\overline{x}})\, f(pr\,\overline{\overline{x}})$$

$$(\overline{\overline{\Omega}}f)(\beta\overline{\overline{x}}) + (\overline{\overline{\Omega}}f)(\overline{\overline{x}}) = \frac{1}{\pi i} \int_{(S,\Sigma)} f(x_1)\, \Omega(x_1, pr\,\overline{\overline{x}})$$

holds.

Remark 2.3: The generalization of the corresponding results of PLEMELJ's is not complete without phrasing HÖLDER continuity of the extension $\overline{\overline{\Omega}}f$ on $\overline{\overline{X}}_j - pr^{-1}(X_j)$. This, however, can be done in a rather obvious way using the fact that apart from points above points of order 1 on S the projection pr is a locally topological mapping from $\overline{\overline{X}}_j - pr^{-1}(X_j)$ onto its image, which is contained in S.

Remark 2.4: Since the proof of Proposition 2.1 and Theorem 2.2 is purely local, they can be carried over word by word to HÖLDER continuous differential forms η on S which satisfy condition (2). Then we have as in Proposition 2.1 that the singular integral

$$\frac{1}{2\pi i} \int_{(S,\Sigma)} \Omega(x_2, x_1)\,\eta(x_1)$$

exists for every point $x_2 \in S$. Furthermore, denoting

$$\frac{1}{2\pi i} \int_{(S,\Sigma)} \Omega(x_2, x_1)\,\eta(x_1), \quad x_2 \in X - S$$

by $(\Omega\eta)(x_2)$ we get the extension property stated in Theorem 2.2 also for $\overline{\overline{\Omega}}\eta$. However, as far as the formulas contained in Theorem 2.2 are concerned, we shall have to replace $\sigma(\overline{\overline{x}})$ by $-\sigma(\overline{\overline{x}})$ due to the fact the roles of x_1 and x_2 in $\Omega(x_1, x_2)$ are now interchanged.

In case we deal with a family $\mathfrak{V} \xrightarrow{\pi} M$ of compact RIEMANN surfaces and a closed subset S of $\mathfrak{V}$ which is locally a family of stars of smooth arcs, we define in exactly the same way as before the spaces $\overline{\overline{\mathfrak{V}}}$ and $\overline{\overline{\mathfrak{V}}}_j$ and the mapping $pr: \overline{\overline{\mathfrak{V}}} \to \mathfrak{V}$. Then $\overline{\overline{\mathfrak{V}}} \xrightarrow{\overline{\overline{\pi}}} M$, where $\overline{\overline{\pi}} = \pi \circ pr$, is a family of spaces $(\overline{\overline{V}}_t : t \in M)$ and we define for every $t \in M$ the sets $\overline{\overline{V}}_{t\beta}$, the mapping β_t, and the functions σ_t. Then the statements of Theorem 2.2 still hold for every $t \in M$ as we replace $\overline{\overline{\Omega}}f$ by the family $(\overline{\overline{\Omega}}_t f : t \in M)$ of functions. In addition we have

Corollary 2.5: *The family $(\overline{\overline{\Omega}}_t f : t \in M)$ of continuous functions on $(\overline{\overline{V}}_t : t \in M)$ constitutes a continuous function $\overline{\overline{\Omega}}f$ on $\overline{\overline{\mathfrak{V}}}$.*

A proof of this corollary which of course is local in character can be found in [11].

§ 3. Ω-degenerate singular integral equations

Let X be a compact RIEMANN surface, Ω the CAUCHY-kernel of X associated with the normalization divisor D and $*$, S a closed subset of X which is locally a star of smooth arcs, and Σ a coherent set of orientations on S. Given a $q \times q$ matrix A on S, a $q \times 1$ matrix g on S, and a $q \times q$ matrix K on $S \times S$ such that every entry of A, g, and K is HÖLDER continuous[2]), we ask for HÖLDER continuous $q \times 1$ matrices f on S fulfilling

$$A(x_1)\,f(x_1) + \frac{1}{\pi i} \int_{(S,\Sigma)} K(x_1, x_2)\,f(x_2)\,\Omega(x_2, x_1) = g(x_1) \quad {}^{3)} \qquad (4)$$

[2]) In this case we simply speak of HÖLDER continuous matrices on S.

[3]) It can be shown (see [7], § 51) that every integrable solution of (4) is HÖLDER continuous.

The integer q shall be called the *rank* of (4).

The *kernel* $K(x_1, x_2)$ of the singular integral equation is called *degenerate* – and (4) itself Ω-degenerate - if there are HÖLDER continuous matrices $B_1, C_1, \ldots, B_n, C_n$ on S such that

$$K(x_1, x_2) = \sum_{\nu=1}^{n} B_\nu(x_1)\, C_\nu(x_2) \quad \text{for} \quad x_1, x_2 \in S \tag{5}$$

In the general case we subject (4) to the following requirements:

I 1) *neither* $\det(A(x_1) + K(x_1, x_1))$ *nor* $\det(A(x_1) - K(x_1, x_1))$ *vanishes on* S, *i.e. the singular integral equation is regularizable*

I 2′) $|\sigma_{x_2}|\, K(x_1, x_2) = 0$ *holds on* $S \times S$.

As soon as we talk about Ω-degenerate singular integral equations whose kernel is given by (5), we replace I 2′) by

I 2) $|\sigma_{x_2}|\, C_\nu(x_2) = 0$ *for every* ν *and every* $x_2 \in S$

We call n the *height of the kernel* K in the representation (5).

Before investigating Ω-degenerate singular integral equations we make few general remarks about such integral equations.

Calling a complex valued function $k(x_1, x_2)$ on $S \times S$ *degenerate* if it admits a representation (5) with B_ν and C_ν being complex valued functions, we see immediately that the kernel $K(x_1, x_2)$ of (4) is degenerate if and only if every entry of this kernel is degenerate. Therefore we have

Proposition 3.1: *If* $K(x_1, x_2)$ *is degenerate, then* $K(x_2, x_1)$ *as well as* $K^t(x_1, x_2)$[4]) *are degenerate.*

Denoting the classical CAUCHY-kernel $\dfrac{dx_1}{x_1 - x_2}$ by Ω_{P1} we have

Proposition 3.2: *The* Ω_{P1}*-degenerate integral operators form a subalgebra of the algebra of all singular integral operators.*

This follows immediately from [7], (45.9) and (45.11).

In order to associate with the singular integral equation (4) with degenerate kernel a holomorphic affine bundle, we first transform the integral equation into a transmission problem[5]). For that purpose assume that f is a solution of the integral equation. Then we form for every $\nu = 1, \ldots, n$

$$(\Omega F_\nu)(x_2) = \frac{1}{2\pi i} \int_{(S,\Sigma)} C_\nu(x_1)\, f(x_1)\, \Omega(x_1, x_2), \quad x_2 \in X - S.$$

ΩF_ν is meromorphic in $X - S$ and has divisor (i.e. each entry of ΩF_ν has divisor) $\geq * - D$. Furthermore, Hypothesis I 2) implies that Theorem 2.2 is applicable, whence we get for every $\nu = 1, \ldots, n$ and every $\bar{\bar{x}} \in \bar{\bar{X}}_\beta$

[4]) K^t denotes the transpose of the matrix K.

[5]) What follows is well known (see [7]) for the dominant equation and its adjoint equation.

$$\begin{aligned}(\bar{\Omega}F_\nu)(\beta\bar{\bar{x}}) - (\bar{\Omega}F_\nu)(\bar{\bar{x}}) &= \sigma(\bar{\bar{x}})\, C_\nu(pr\,\bar{\bar{x}})\, f(pr\,\bar{\bar{x}}) \\ (\bar{\Omega}F_\nu)(\beta\bar{\bar{x}}) + (\bar{\Omega}F_\nu)(\bar{\bar{x}}) &= \frac{1}{\pi i}\int_{(S,\Sigma)} C_\nu(x_2)\, f(x_2)\, \Omega(x_2, pr\,\bar{\bar{x}})\end{aligned} \tag{6}$$

On the other hand, the integral equation can be rewritten as

$$\{A(pr\,\bar{\bar{x}}) + \sum_{\nu=1}^{n} B_\nu(pr\,\bar{\bar{x}})\, C_\nu(pr\,\bar{\bar{x}})\}\, f(pr\,\bar{\bar{x}}) +$$

$$+ \sum_{\nu=1}^{n} B_\nu(pr\bar{\bar{x}})\,\{-C_\nu(pr\bar{\bar{x}})\, f(pr\bar{\bar{x}}) + \frac{1}{\pi i}\int_{(S,\Sigma)} C_\nu(x_2)\, f(x_2)\, \Omega(x_2, pr\bar{\bar{x}})\} = g(pr\bar{\bar{x}})$$

or equivalently, putting $K_+(x) = A(x) + \sum_{\nu=1}^{n} B_\nu(x)\, C_\nu(x)$

$$\begin{aligned}&K_+(pr\,\bar{\bar{x}})\, f(pr\,\bar{\bar{x}}) + \sum_{\nu=1}^{n} B_\nu(pr\,\bar{\bar{x}})\,\{(1-\sigma(\bar{\bar{x}}))\,(\bar{\Omega}F_\nu)(\beta\bar{\bar{x}}) + \\ &+ (1+\sigma(\bar{\bar{x}}))\,(\bar{\Omega}F_\nu)(\bar{\bar{x}})\} = g(pr\,\bar{\bar{x}})\,.\end{aligned} \tag{7}$$

Since $\det K_+(x)$ does not vanish on S, we can substitute (7) into first set of equations (6) which then can be rewritten as

$$\begin{aligned}&(\bar{\Omega}F_\nu)(\beta\bar{\bar{x}}) + (\sigma(\bar{\bar{x}})-1)\, C_\nu(pr\,\bar{\bar{x}})\, K_+^{-1}(pr\,\bar{\bar{x}}) \sum_{\mu=1}^{n} B_\mu(pr\,\bar{\bar{x}})\,(\bar{\Omega}F_\mu)\,\beta\bar{\bar{x}}) = \\ &= (\bar{\Omega}F_\nu)(\bar{\bar{x}}) - (\sigma(\bar{\bar{x}})+1)\, C_\nu(pr\,\bar{\bar{x}})\, K_+^{-1}(pr\,\bar{\bar{x}}) \sum_{\mu=1}^{n} B_\mu(pr\,\bar{\bar{x}})\,(\bar{\Omega}F_\mu)(\bar{\bar{x}}) + \\ &+ \sigma(\bar{\bar{x}})\, C_\nu(pr\,\bar{\bar{x}})\, K_+^{-1}(pr\,\bar{\bar{x}})\, g(pr\,\bar{\bar{x}})\end{aligned} \tag{8'}$$

Forming the $nq \times 1$ matrix $\bar{\Omega}F$ given by $(\bar{\Omega}F)^t = ((\bar{\Omega}F_1)^t, \ldots, (\bar{\Omega}F_n)^t)$ and abbreviating $\sigma(\bar{\bar{x}})$ by σ, $B_\nu(pr\,\bar{\bar{x}})$ by B_ν, $C_\nu(pr\,\bar{\bar{x}})$ by C_ν, $K_+(pr\,\bar{\bar{x}})$ by K_+, and $g(pr\,\bar{\bar{x}})$ by g we can put (8′) in the form

$$\begin{pmatrix} 1+(\sigma-1)\, C_1K_+^{-1}B_1, (\sigma-1)\, C_1K_+^{-1}B_2, \ldots, (\sigma-1)\, C_1K_+^{-1}B_n \\ \vdots \\ (\sigma-1)\, C_nK_+^{-1}B_1, (\sigma-1)\, C_nK_+^{-1}B_2, \ldots, 1+(\sigma-1)\, C_nK_+^{-1}B_n \end{pmatrix} (\bar{\Omega}F)(\beta\bar{\bar{x}}) = \tag{8''}$$

$$= \begin{pmatrix} 1-(\sigma+1)\, C_1K_+^{-1}B_1, & -(\sigma+1)\, C_1K_+^{-1}B_2, \ldots, & -(\sigma+1)\, C_1K_+^{-1}B_n \\ \vdots & & \\ -(\sigma+1)\, C_nK_+^{-}\, B_1, & -(\sigma+1)\, C_nK_+^{-1}B_2, \ldots, & 1-(\sigma+1)\, C_nK_+^{-1}B_n \end{pmatrix} .$$

$$\cdot\, (\bar{\Omega}F)(\bar{\bar{x}}) + \sigma \begin{pmatrix} C_1K_+^{-1}g \\ \vdots \\ C_nK_+^{-1}g \end{pmatrix}$$

In order to see that the factors of $\bar{\Omega}F(\beta\bar{\bar{x}})$ and $\bar{\Omega}F(\bar{\bar{x}})$ are matrices with determinant nowhere zero, we remark that both are either the unit matrix or else equal to

$$\begin{pmatrix} 1 - 2\,C_1 K_+^{-1} B_1, & - 2\,C_1\, K_+^{-1} B_2, \dots, & - 2C_1\, K_+^{-1} B_n \\ \vdots & & \\ - 2\,C_n K_+^{-1} B_1, & \dots, & 1 - 2C_n K_+^{-1} B_n \end{pmatrix}$$

According to Lemma 1.5 the determinant of this matrix equals

$$\det(1 - 2 \sum_{\nu=1}^{n} K_+^{-1} B_\nu C_\nu) = \det K_+^{-1} \cdot \det K_-$$

where $K_-(x) = A(x) - \sum_{\nu=1}^{n} B_\nu(x)\, C_\nu(x)$.

Now we put (8′) in its final form and denote for that purpose the affine transformation of $\mathbf{C}^{nq}$ which sends $F^t = (F_1^t, \dots, F_n^t)$, $F_1, \dots, F_n \in \mathbf{C}^q$, into

$$\begin{pmatrix} 1 - 2\,C_1(pr\,\bar{\bar{x}})\, K_+^{-1}(pr\,\bar{\bar{x}})\, B_1(pr\,\bar{\bar{x}}), \dots, & - 2\,C_1(pr\,\bar{\bar{x}})\, K_+^{-1}(pr\,\bar{\bar{x}})\, B_n\,(pr\,\bar{\bar{x}}) \\ \vdots & \\ - 2\,C_n(pr\,\bar{\bar{x}})\, K_+^{-1}(pr\,\bar{\bar{x}})\, B_1(pr\,\bar{\bar{x}}), \dots, & 1 - 2\,C_n(pr\,\bar{\bar{x}})\, K_+^{-1}(pr\,\bar{\bar{x}})\, B_n\,(pr\,\bar{\bar{x}}) \end{pmatrix} F$$

$$+ \begin{pmatrix} C_1(pr\,\bar{\bar{x}})\; K_+^{-1}(pr\,\bar{\bar{x}})\, g\,(pr\,\bar{\bar{x}}) \\ \vdots \\ C_n(pr\,\bar{\bar{x}})\; K_+^{-1}(pr\,\bar{\bar{x}})\, g\,(pr\,\bar{\bar{x}}) \end{pmatrix} \tag{9}$$

by $T(pr\,\bar{\bar{x}})F$. Then (8″) becomes simply

$$(\bar{\bar{\Omega}} F)\,(\beta \bar{\bar{x}}) = T(pr\,\bar{\bar{x}})^{\sigma(\bar{\bar{x}})} (\bar{\bar{\Omega}} F)\;(\bar{\bar{x}}) \tag{8}$$

where M^{-1} means the inverse of M in the affine group of $\mathbf{C}^{nq}$.

We note immediately that the condition 4) of [11], § 4, for the problem (32) of [11] is satisfied in our case whence the methods of [11] are applicable to our situation (note that the holomorphic fiber bundle which is among the data for the problem (32) of [11] reduces in our situation to the trivial vector bundle of rank nq over X).

Before we continue dealing with the relation between the singular integral equation (4) and the transmission problem (8) we should like to derive from (7) a necessary condition for the existence of solutions of (4). For that purpose let $x \in S$ be a point of order > 2 for which $|\sigma_x| = 0$ holds (in this case x has necessarily an even order). Then (7) implies that

$$\sum_{\nu=1}^{n} B_\nu(pr\,\bar{\bar{x}})\, \{(1 - \sigma(\bar{\bar{x}}))\;\, \Omega F_\nu(\beta \bar{\bar{x}}) + (1 + \sigma(\bar{\bar{x}}))\;\, \Omega F_\nu(\bar{\bar{x}})\} \tag{7'}$$

does not depend on the choice of $\bar{\bar{x}} \in pr^{-1}(x)$. If for every $\bar{\bar{x}} \in pr^{-1}(x)$ the relation $\sigma(\bar{\bar{x}}) + \sigma(\beta \bar{\bar{x}}) = 0$ holds, then the first set of formulas (6) shows easily that (7) does not depend on the choice of $\bar{\bar{x}} \in pr^{-1}(x)$. However, if there is an $\bar{\bar{x}} \in pr^{-1}(x)$ such that $\sigma(\bar{\bar{x}}) = \sigma(\beta\bar{\bar{x}})$, then a straightforward computation again involving the first set of formulas (6) shows that

$$\sum_{\nu=1}^{n} B_\nu(pr\,\bar{\bar{x}})\, C_\nu(pr\,\bar{\bar{x}})\, f(pr\,\bar{\bar{x}}) = 0\,.$$

Since we do not wish to impose conditions on the solutions of (4) we arrive at the following necessary conditions for the existence of solutions of (4)

I 3) *for every point* $x \in S$ *of order* > 2 *for which* $\sigma_x = 0$ *holds and for which there exists a point*

$$x \in pr^{-1}(\bar{\bar{x}}) \quad \textit{satisfying} \quad \sigma(\bar{\bar{x}}) = \sigma(\beta\bar{\bar{x}})\,,$$

$$K(x, x) = \sum_{\nu=1}^{n} B_\nu(x)\, C_\nu(x) = 0$$

holds.

We proved that every solution of (4) gives rise to a solution of the transmission problem (8) which has divisor $\geq * - D$. Now assume that a meromorphic solution of the transmission problem (8) is given which has divisor $\geq * - D$ on $X - S$, i.e. a meromorphic $nq \times 1$ matrix G on $X - S$ for which the matrix $\bar{\bar{G}}$ on $pr^{-1}(X - S)$ defined by $\bar{\bar{G}}(\bar{\bar{x}}) = G(pr\,\bar{\bar{x}})$ can be extended continuously to $\bar{\bar{X}}$ and whose extension satisfies the equation

$$\bar{\bar{G}}(\beta\bar{\bar{x}}) = T(pr\,\bar{\bar{x}})^{\sigma(\bar{\bar{x}})}\,\bar{\bar{G}}(\bar{\bar{x}}) \qquad \bar{\bar{x}} \in \bar{\bar{X}}_\beta\,.$$

Keeping (7) in mind, we consider for $x \in S$ and $\bar{\bar{x}} \in pr^{-1}(x)$ the expression

$$K_+^{-1}(x)\, g(x) - K_+^{-1}(x) \sum_{\nu=1}^{n} B_\nu(x)\, \{(1 - \sigma(\bar{\bar{x}}))\, \bar{\bar{G}}_\nu(\beta\bar{\bar{x}}) + (1 + \sigma(\bar{\bar{x}}))\, \bar{\bar{G}}_\nu(\bar{\bar{x}})\} \tag{7''}$$

and claim that this matrix depends only on $pr\,\bar{\bar{x}} = x$. This is obvious for every point $x \in S$ of odd order since in this case I 2) implies $\bar{\bar{G}}(\beta\bar{\bar{x}}) = \bar{\bar{G}}(\bar{\bar{x}})$ for all $\bar{\bar{x}} \in pr^{-1}(x)$. The same argument holds for every point $x \in S$ of even order satisfying $|\sigma_x| \neq 0$. If $x \in S$ is of order 2 and $|\sigma_x| = 0$, then a direct computation shows that (7″) is independent of the choice of $\bar{\bar{x}} \in pr^{-1}(x)$. As remarked previously, (7′) and hence (7″) has the required property if x is of even order > 2, $|\sigma_x| = 0$, and if for every $\bar{\bar{x}} \in pr^{-1}(x)$ the relation $\sigma(\bar{\bar{x}}) + \sigma(\beta\bar{\bar{x}}) = 0$ holds. In the remaining case, where $\sigma(\bar{\bar{x}}) = \sigma(\beta\bar{\bar{x}})$ holds we get from (8) by a straightforward calculation

$$\sum_{\nu=1}^{n} B_\nu(x)\, \bar{\bar{G}}_\nu(\beta\bar{\bar{x}}) = (K_-(x)\, K_+^{-1}(x))^{\sigma(\bar{\bar{x}})} \sum_{\nu=1}^{n} B_\nu(x)\, \bar{\bar{G}}_\nu(\bar{\bar{x}}) +$$
$$+\ \sigma(\bar{\bar{x}})\, (K_-(x)\, K_+^{-1}(x))^{\frac{1}{2}(\sigma(\bar{x})-1)} \sum_{\nu=1}^{n} B_\nu(x)\, C_\nu(x)\, K_+^{-1}(x)\, g(x)$$

whence I 3) implies that also in this last case (7″) does not depend on the choice of $\overline{\overline{x}}$.

As shown in [11] there is a local representation of $\overline{\overline{G}}$ in the form $H(x)\,h(x)$ where h is a $nq \times 1$ matrix which is holomorphic in a suitable neighborhood of the point in discussion, while H is a holomorphic mapping (notation as in § 1, II., and § 2) of $\cup\ U_\mu$ into $GA(nq, \mathrm{C})$ which restricted to each U_μ can be extended to a HÖLDER continuous mapping of $\overline{U_\mu}$ into $GA(nq, \mathrm{C})$. Therefore (7″) is a HÖLDER continuous matrix on S which now shall be denoted by f. Since (8) and (8′) are equivalent, $\overline{\overline{G}}$ satisfies the equation gotten from (8′) by replacing $\overline{\overline{\Omega F_\nu}}$ by $\overline{\overline{G}}_\nu$. Consequently, using the above definition of f by (7″), we get for $\nu = 1, \ldots, n$

$$\overline{\overline{G}}_\nu(\beta \overline{\overline{x}}) - \overline{\overline{G}}_\nu(\overline{\overline{x}}) = \sigma(\overline{\overline{x}})\, C_\nu(pr\, \overline{\overline{x}})\, f(pr\, \overline{\overline{x}})\,.$$

Since f is HÖLDER continuous, we can define ΩF_ν as before and get then, using (6)

$$\overline{\overline{G}}_\nu(\beta \overline{\overline{x}}) - \overline{\overline{G}}_\nu(\overline{\overline{x}}) = \overline{\overline{\Omega F}}_\nu(\beta \overline{\overline{x}}) - \overline{\overline{\Omega F}}_\nu(\overline{\overline{x}})$$

or

$$\overline{\overline{G}}_\nu(\beta \overline{\overline{x}}) - \overline{\overline{\Omega F}}_\nu(\beta \overline{\overline{x}}) = \overline{\overline{G}}_\nu(\overline{\overline{x}}) - \overline{\overline{\Omega F}}_\nu(\overline{\overline{x}})\,.$$

This means that $(\overline{\overline{G}}_\nu - \overline{\overline{\Omega F}}_\nu)(\overline{\overline{x}})$ can be written as $H_\nu(pr\, \overline{\overline{x}})$ where H_ν is a continuous $q \times 1$ matrix in $X - \operatorname{supp} D$ which is meromorphic in $X - S$ and has poles only in $\operatorname{supp} D$. Hence H_ν is meromorphic in all of X. According to our assumptions on G and due to Proposition 1.1, H_ν has divisor $\geq * - D$. Since D is a normalization divisor, we can conclude that H vanishes identically. Thus $\overline{\overline{G}} = \overline{\overline{\Omega F}}$. In particular, G satisfies also the second set of equations (6). However, by substituting (6) into (8′) we come back to (4). Therefore a meromorphic solution of the transmission problem (8) which has divisor $\geq * - \mathrm{D}$ gives rise to a solution of (4). One checks easily that this sets up a bijective correspondence between the set of solutions of (4) and the set of meromorphic solutions of (8) having divisor $\geq * - D$.

In order to get the above statement in the case of families we shall have to define the notion of a HÖLDER continuous function which depends differentiably (holomorphically) on a parameter. Let $\mathfrak{B} \to M$ be a differentiable (holomorphic) family of compact RIEMANN surfaces and S a closed subset of $\mathfrak{B}$ which is locally a differentiable (holomorphic) family of stars of smooth arcs. Then the complex valued function f on S is said to be HÖLDER *continuous and to depend differentiably (holomorphically) on* $t \in M$ if (notation as in § 1, II.) for every point $x \in S$ there exists a coordinate neighborhood U such that

(i) $f \circ \mathfrak{C}_\nu$ restricted to $\pi(U) \times I$ is a continuous function which depends differentiably (holomorphically) on $t \in \pi(U)$

(ii) $f \circ \mathfrak{C}_\nu$ and its first partial derivatives with respect to t restricted to every set $\{t\} \times I$, $t \in \pi(U)$, is HÖLDER continuous with HÖLDER coefficient uniformly bounded in t and HÖLDER exponent uniformly bounded away from zero.

In connection with this definition we should remark that in the case of holomorphic dependence on $t \in M$ it is sufficient to assume that statement (ii) holds only for $f \circ \mathfrak{C}_\nu$ and not for the partial derivatives of this function (see [11], § 2).

A *differentiable (holomorphic) family of singular integral equations* consists a differentiable (holomorphic) family $\mathfrak{B} \to M$ of compact RIEMANN surfaces, and corresponding families of normalization divisors $(D_t : t \in M)$, points $(*_t : t \in M)$, and stars of smooth arcs $(S_t : t \in M)$ together with a family $\Sigma = (\Sigma_t : t \in M)$ of coherent orientations, and the equations

$$A_t(x_1)\, f_t(x_1) + \frac{1}{\pi i} \int_{(S_t, \Sigma_t)} K_t(x_1, x_2)\, f_t(x_2)\, \Omega_t\, (x_2, x_1) = g_t(x_1), \quad t \in M, \tag{4_t}$$

where the matrices $A_t(x_1)$, $K_t(x_1, x_2)$, and $g_t(x_1)$ are HÖLDER continuous and depend holomorphically (differentiably) on $t \in M$. It will be assumed that for each $t \in M$ the conditions I 1) and I 2′) are satisfied. As before we speak of a Ω_t-degenerate singular integral equation if

$$K_t(x_1, x_2) = \sum_{\nu=1}^{n} B_{\nu t}(x_1)\, C_{\nu t}(x_2) \tag{5_t}$$

holds with HÖLDER continuous matrices $B_{\nu t}$ and $C_{\nu t}$ which themselves depend differentiably (holomorphically) on $t \in M$ (note that the height n is assumed to be independent of $t \in M$). In this latter case we assume that for every $t \in M$ the conditions I 1), I 2), and I 3) are satisfied. When we want to put emphasis on the family $\mathfrak{B} \to M$ we speak of a family of singular integral equations over $\mathfrak{B} \to M$.

The transmission problem previously associated with (4_t) shall be denoted by

$$(\overline{\overline{\Omega}}_t F_t)\, (\beta_t \overline{\overline{x}}) = T_t(pr_t \overline{\overline{x}})^{\sigma(\overline{\overline{x}})}\, (\overline{\overline{\Omega}}_t F_t)\, (\overline{\overline{x}}) \quad t \in M\,. \tag{8_t}$$

In the terminology of [11], this constitutes a differentiable (holomorphic) family of transmission problems as can be seen immediately from (9). Clearly, every solution f_t of (4_t) which depends differentiably (holomorphically) on $t \in M$ gives rise to a solution $\overline{\overline{\Omega}}_t F_t$ if the family of transmission problems (8_t) which depends differentiably (holomorphically) on $t \in M$. The latter simply means that $\overline{\overline{\Omega}}_t F_t \circ pr_t$ restricted to $\mathfrak{B} - S$ is a meromorphic function which depends differentiably (holomorphically) on $t \in M$. In addition, [11], Lemma 2.1[6])

[6]) This Lemma is also true for differentiable families of smooth arcs as can be checked easily.

and the procedure and estimates of [7], § 16 show that (notations as in § 1, II., and § 2) for every $\overline{\overline{x}} \in pr^{-1}(S)$ the restriction of $\overline{\overline{\Omega_t F_t}} \circ pr_t$ to $U_{\nu t}$ continuously extended to $\overline{U_{\nu t}}$ yields a function on $\mathfrak{C}_{\nu t}(I) \cup \mathfrak{C}_{\nu+1,t}(I)$ which is HÖLDER continuous and depends differentiably (holomorphically) on t.

Conversely, given a meromorphic solution G_t of (8_t) with divisor $\geq *_t - D_t$ which depends differentiably (holomorphically) on $t \in M$, we get by our previous construction a family f_t of solutions of (4_t). We claim that f_t is HÖLDER continuous and depends differentiably (holomorphically) on $t \in M$. For this purpose we consider the before mentioned local representation $H_t(x)\, h_t(x)$ of $\overline{\overline{G_t}}$. Here, $h_t(x)$ is a holomorphic matrix which depends differentiably (holomorphically) on t. On the other hand, in [11] the matrix $H_t(x)$ was constructed from $T_t(x)$ inductively by
(i) forming the limiting values of $\Omega_t T_t^{(k)}$, $T_t^{(o)}$ being $\exp^{-1}(T_t)$
(ii) defining $T_t^{(k+1)}$ by

$$\exp T_t^{(k+1)} = (\exp \Omega_{t\nu}\, T_t^{(k)})^{-1} \cdot \exp(\Omega_{t\nu}\, T_t^{(k)} + \Omega_{t,\nu+1}\, T_t^{(k)}) \cdot (\exp \Omega_{t,\nu+1}\, T_t^{(k)})^{-1}$$

As mentioned above, (i) preserves the property of being HÖLDER continuous and depending differentiably (holomorphically) on t. (ii) obviously behaves in the same way. Since (see [11], § 2) H_t is the uniform limit of the products

$$\exp(\Omega_t T_t^{(o)}) \cdot \ldots \cdot \exp(\Omega_t T_t^{(k)}) \quad \text{as} \quad k \to \infty$$

the limiting values of H_t also satisfy a HÖLDER condition and depend differentiably (holomorphically) on $t \in M$. Consequently $\overline{\overline{G_t}}$ has the same property. Hence $(7'')$ shows that the solution f_t of (4_t) associated with $\overline{\overline{G_t}}$ depends also differentiably (holomorphically) on $t \in M$.

Summing up we get

Theorem 3.3: *Given a differentiable (holomorphic) family of Ω_t-degenerate singular integral equations over the differentiable (holomorphic) family $\mathfrak{B} \to M$ of compact RIEMANN surfaces, there exists a differentiable (holomorphic) family of transmission problems with values in the trivial vectorbundle (of suitable rank) over $\mathfrak{B} \to M$ such that the set of those solutions of the singular integral equation which depend differentiably (holomorphically) on $t \in M$ is in a bijective correspondence with the set of those solutions of the transmission problem which depend differentiably (holomorphically) on $t \in M$ and have divisor $\geq *_t - D_t$ for every $t \in M$. The rank of that trivial vector bundle equals nq where n denotes the height of the kernel and q the rank of the degenerate integral equation.*

§ 4. The vector bundle associated with an Ω-degenerate singular integral equation

It has been proved in [11] that, given a holomorphic family of transmission problems over $\mathfrak{V} \to M$ with values in a holomorphic vector bundle $\mathfrak{W} \to \mathfrak{V}$, there exists under certain, quite general hypotheses (see [11], § 4, 3) and 4), a) and b)) a holomorphic family of fiber bundles $\hat{\mathfrak{W}} \to \mathfrak{V} \to M$ such that the set of holomorphic (meromorphic) solutions of the transmission problem is mapped bijectively onto the set of holomorphic (meromorphic) sections in $\hat{\mathfrak{W}} \to \mathfrak{V}$[7]. Clearly, the just mentioned requirements 3) and 4) are fulfilled since in our case $\mathfrak{W} \to \mathfrak{V}$ is the trivial vector bundle (or rank nq); the complex Lie group which acts on $\mathfrak{V} \to M$ over S is in our case the general affine group $GA(nq, \mathbf{C})$. Since $\mathfrak{W} \to \mathfrak{V}$ is the trivial vector bundle the hypotheses a) and b) are trivially fulfilled. As pointed out in [11], the same result still holds when we replace the holomorphic family of transmission problems by a differentiable family of transmission problems. In this case, however, we end up with a differentiable family of holomorphic affine bundles $\hat{\mathfrak{W}} \to \mathfrak{V} \to M$ and the statement that the set of those holomorphic (meromorphic) solutions of the transmission problem which depend differentiably on $t \in M$ is mapped bijectively onto the set of those holomorphic (meromorphic) sections in $\hat{W}_t \to V_t$ which depend differentiably on $t \in M$.

But we are looking for meromorphic solutions of the transmission problem which have divisor $= *_t - D_t$, $t \in M$. They correspond to those meromorphic sections in $\hat{W}_t \to V_t$ which have divisor $\geq *_t - D_t$, i.e. which when represented in complex fiber coordinates of $\hat{W}_t \to V_t$ have the property that all their components have divisor $\geq *_t - D_t$. As in [3] we denote by $\{*_t - D_t\}$ the holomorphic line bundle associated with the divisor $*_t - D_t$, i.e. the holomorphic sections in $\{*_t - D_t\}$ correspond to functions which have divisor $\geq *_t - D_t$. From our hypotheses about $\mathfrak{V} \to M$ and the families $(D_t : t \in M)$ and $(*_t : t \in M)$ we conclude easily that $(\{*_t - D_t\} : t \in M)$ is a differentiable (holomorphic) family of line bundles $\mathfrak{L} \to \mathfrak{V} \to M$. From the construction of $\hat{\mathfrak{W}} \to \mathfrak{V}$ as given in [11] we take that except for an arbitrarily chosen neighborhood U of S the transition functions for the bundle $\hat{\mathfrak{W}} \to \mathfrak{V}$ coincide on $\mathfrak{V} - U$ with the transition functions of $\mathfrak{W} \to \mathfrak{V}$. Therefore, by choosing U and the open covering $(U_i : i \in I)$ of $\mathfrak{V}$ appropriately, we may assume that

[7]) Note that the hypothesis of [11], Theorem 4.1, namely that $\mathfrak{V}$ be a normal complex space, is trivially fulfilled in our situation since $\mathfrak{V} \to M$ is a family in the sense of Kodaira-Spencer ([5]) whence $\mathfrak{V}$ is a complex manifold.

(i) the transition functions $(A_{ij}(v), a_{ij}(v))$ – where A_{ij} has values in $GL(nq, \mathbf{C})$ and a_{ij} has values in $\mathbf{C}^{nq}$ – in $U_i \cap U_j$ which describe the bundle $\hat{\mathfrak{W}} \to \mathfrak{V}$ satisfy $A_{ij}(v) = 1$ and $a_{ij}(v) = 0$ whenever $v \in \mathfrak{V} - U$

(ii) the local meromorphic functions d_i in U_i describing the divisor $(*_t - D_t : t \in M)$ and hence the line bundle $\mathfrak{L}$ satisfy $d_i(v) = 1$ whenever $v \in U$.

Thus the functions $(d_i(v)A_{ij}(v)d_j^{-1}(v),\ d_i(v)a_{ij}(v))$ are holomorphic in $U_i \cap U_j$ and depend differentiably (holomorphically) on $t \in M$. Since they fulfill the compatibility relations they define a differentiable (holomorphic) family $\tilde{\mathfrak{W}} \to \mathfrak{V} \to M$ of holomorphic affine bundles. Clearly, the set of those meromorphic sections in $\hat{\mathfrak{W}} \to \mathfrak{V}$ which depend differentiably (holomorphically) on $t \in M$ and have divisor $\geq *_t - D_t$ is mapped bijectively onto the set of those holomorphic sections in $\tilde{\mathfrak{W}} \to \mathfrak{V}$ which depend differentiably (holomorphically) on $t \in M$. Summing up we get therefore

Theorem 4.1: *Given a differentiable (holomorphic) family of Ω_t-degenerate singular integral equations over the differentiable (holomorphic) family $\mathfrak{V} \to M$ of compact* RIEMANN *surfaces, there exists a differentiable (holomorphic) family $\tilde{\mathfrak{W}} \to \mathfrak{V} \to M$ of holomorphic affine bundles such that the set of the solutions of the singular integral equation which depend differentiably (holomorphically) on $t \in M$ is in a bijective correspondence with the set of those holomorphic sections in $\tilde{\mathfrak{W}} \to \mathfrak{V}$ which depend differentiably (holomorphically) on $t \in M$. The rank of the affine bundle $\tilde{\mathfrak{W}} \to \mathfrak{V}$ equals nq where n and q have the same meaning as in Theorem 3.3.*

Corrollary 4.2: *The space of solutions of* (4) *has finite dimension.*

Proof: [3], Satz 15.4.2.

If we pass from (4) resp. (4_t) to the associated homogeneous integral equation, that is if we replace g resp. g_t by the function 0, then we have to replace the affine bundle $\tilde{\mathfrak{W}} \to \mathfrak{V}$ by the one gotten from $\tilde{\mathfrak{W}} \to \mathfrak{V}$ by means of the canonical homomorphism $GA(nq, \mathbf{C}) \to GL(nq, \mathbf{C})$. For various purposes it is important to know the degree of the vector bundle $\tilde{W}_t \to V_t$, $t \in M$, which corresponds to the homogeneous integral equation (4_t). Using the notations of §§ 2 and 3, the previous computation of $\det T_t(x)$, and [11], Satz 6.6, we get in a straight forward way

Proposition 4.3: *Let X be a compact connected* RIEMANN *surface of genus g and $\tilde{W} \to X$ the vector bundle associated with the degenerate integral equation* (4). *Then*

$$\deg \tilde{W} = (g-1)nq + \frac{1}{2\pi} \sum_{\lambda=1}^{l} \sigma^{(\lambda)} \arg_{\mathfrak{C}^{(\lambda)}} (\det K_- \cdot \det K_+^{-1})$$

In the formula contained in Proposition 4.3, $\mathfrak{C}^{(\lambda)}$, $\lambda = 1, \ldots, l$, is the splitting of S with minimal l as described in Proposition 1.4. Furthermore, for the continuous function $\varphi : \mathfrak{C}^{(\lambda)}(I) \to GL(1, \mathbf{C})$, $\arg_{C^{(\lambda)}} \varphi$ is defined to be $(\arg \varphi)(1) - (\arg \varphi)(0)$[8]) where $\arg \varphi : \mathfrak{C}^{(\lambda)}(I) \to R$ is a continuous such that $\varphi = |\varphi| e^{i \arg \varphi}$ holds. Finally, $\sigma^{(\lambda)}$ equals $\sigma(\bar{\bar{x}})$ where $\bar{\bar{x}}$ is any point lying above $\mathfrak{C}_t^{(\lambda)}(I - \{0, 1\})$ for which the filter defining x has a basis consisting of sets which lie on the right side (with respect to the orientation of $\mathfrak{C}_t^{(\lambda)}$) of $\mathfrak{C}_t^{(\lambda)}$.

[5], Theorem 2.1, in conjunction with Theorem 4.1 leads immediately to

Theorem 4.4: *Given a differentiable family of Ω_t-degenerate singular integral equations (4_t) over $\mathfrak{B} \to M$. Assume that for every $t \in M$ the dimension $\dim_{\mathbf{C}} L_t$ of the complex affine space L_t of solutions of (4_t) is bigger or equal to 0. Then*

1) $\dim_{\mathbf{C}} L_t$ *is an upper-semicontinuous function of $t \in M$*

2) *if $d = \dim_{\mathbf{C}} L_t$ does not depend on t, then for every $t_0 \in M$ there exists a neighborhood U of t_0 and $d + 1$ solutions $f_t^{(0)}, \ldots, f_t^{(d)}$ of (4_t) which depend differentiably on $t \in U$ and form a basis*[9]) *of L_t for every $t \in U$.*

From Theorem 4.4, 2) it is clear that there is a differentiable affine bundle over M whose differentiable sections over the open subset W of M correspond bijectively to those solutions of (4_t) which are defined for all $t \in W$ and depend there differentiably on t.

In the same way [5], Theorem 18.1, and [11], Theorem 2.3, imply

Theorem 4.5: *Given a holomorphic family of Ω_t-degenerate singular integral equations (4_t) over $\mathfrak{B} \to M$. Assume that for every $t \in M$ the relation $\dim_{\mathbf{C}} L_t \geq 0$ holds. Then*

1) *for every integer j the set $\{t : \dim_{\mathbf{C}} L_t \geq j\}$ is an analytic subset of M*

2) *if $d = \dim_{\mathbf{C}} L_t$ does not depend on t, then for every $t_0 \in M$ there exists a neighborhood U of t_0 and $d + 1$ solutions $f_t^{(0)}, \ldots, f_t^{(d)}$ of (4_t) which depend holomorphically on $t \in U$ and form a basis of L_t for every $t \in U$.*

Finally it should be remarked that in the case of homogeneous Ω_t-degenerate singular integral equations for which S is contained in the RIEMANNian sphere one can define the so called component indices (see [7], § 127). As carried out in [11] for the transmission problem, it can be shown that they are nothing but the exponents turning up in the GROTHENDIECK splitting ([4]) of the vector bundle associated with the singular integral equation. Therefore the statement of [11], Satz 6.9, holds also in the present situation.

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[8]) The orientation of $\mathfrak{C}^{(\lambda)} : I \to X$ is assumed to be chosen in such a way that 0 is the initial point of $\mathfrak{C}^{(\lambda)}$.

[9]) A subset $a_0, \ldots, a_d$ of a complex affine space L is called a basis if every element of A can be written in exactly one way as $\sum_{\nu=0}^{d} \alpha_\nu a_\nu$ where $\alpha_\nu \in \mathbf{C}$, $\nu = 0, \ldots, d$, and $\sum_{\nu=0}^{d} \alpha_\nu = 1$ holds.

§ 5. Adjoint equation and dual bundle

Starting out from the singular integral equation (4), the adjoint equation is defined to be

$$A^t(x_1)\eta(x_1) + \frac{1}{\pi i} \int_{(S,\Sigma)} K^t(x_2, x_1)\, \Omega(x_1, x_2)\, \eta(x_2) = 0 \tag{4*}$$

where $\eta(x_2)$ is a differential form on S. Since every integrable solution of (4*) is HÖLDER continuous (see footnote[3]) we have only to look for HÖLDER continuous solutions η of (4*). One sees very easily that (4*) satisfies the condition I 1), provided (4) does. Moreover, Proposition 3.1 shows that (4*) is Ω-degenerate if (4) is Ω-degenerate; yet it might be that (4*) does not fulfill the analog of I 2). If the original Ω-degenerate singular integral equation has kernel (5), then we write the kernel of (4*) in the form

$$K^t(x_2, x_1) = \sum_{\nu=1}^{n} C^t_\nu(x_1)\, B^t_\nu(x_2) \tag{5*}$$

whence we shall have to make the additional assumption that

I*2) $\qquad |\sigma_{x_2}|\, B_\nu(x_2) = 0$ *for every* ν *and* $x_2 \in S$.

Since (4*) satisfies I 3) whenever (4) does, we can apply the results of § 4 to (4*). In passing from (4) to (4*) we have to interchange σ and $-\sigma$, B_ν and C^t_ν, C_ν and B^t_ν, K_+ and K^t_+, and F and H. Therefore the transmission problem associated with the adjoint equation turns out to be

$$(\overline{\overline{\Omega H}})(\beta \overline{\overline{x}}) = T^*(pr\,\overline{\overline{x}})^{\sigma(\overline{\overline{x}})}(\overline{\overline{\Omega H}})\,(\overline{\overline{x}}) \quad {}^{10)} \tag{8*}$$

where T^* denotes $(T^t)^{-1}$. Furthermore, Proposition 1.2 shows that $\overline{\overline{\Omega H}}$ has divisor $\geq D - *$ on $X - S$. Conversely, in the same way as in § 3 it can be shown that every solution of the transmission problem (8*) which has divisor $\geq D - *$ on $X - S$ gives rise to a solution of (4*).

The transmission problem (8*) calls for meromorphic differential forms on $X - S$. This means in the terminology of [11] that we are dealing with a transmission problem with values in $nqK_X = K_X \oplus \ldots \oplus K_X$ (nq-times). Therefore the construction which associates with the transmission problem (8*) a holomorphic vector bundle (see [11], § 4) leads in this case to the bundle $\hat{W}^* \otimes K_X \to X$ where $\hat{W} \to X$ is the vector bundle associated with the homogeneous equation (4). Thus the set of all solutions of (4) corresponds bijectively to the set of all those meromorphic sections in $\hat{W} \overset{*}{\longrightarrow} K_X$ which have divisor

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[10]) Since we deal with homogeneous equations (4), T assumes values in $GL(nq, \mathbf{C})$.

$\geq D - *$. This and the remarks in § 4 leading up to the construction of the bundle $\tilde{W} \to X$ show that the vector space of solutions of (4*) is mapped bijectively onto the vector space of all holomorphic sections in the vector bundle $\tilde{W}^* \otimes K_X \to X$. Summing up we get

Theorem 5.1: *Let $\tilde{W} \to X$ be the vector bundle associated with the homogeneous Ω-degenerate singular integral equation* (4) *according to Theorem* 4.1. *Then the vector bundle associated with the adjoint equation equals $\tilde{W}^* \otimes K_X \to X$.*

Denoting the vector space of all solutions of the homogeneous equation (4) by L (of (4*) by L^*) we get as an immediate corollary of Theorem 5.1, using Proposition 4.3 and the RIEMANN-ROCH theorem for vector bundles over compact RIEMANN surfaces (see [3]).

Corollary 5.2 (Index Theorem):

$$\dim_{\mathbf{C}} L - \dim_{\mathbf{C}} L^* = \sum_{\lambda=1}^{l} \sigma^{(\lambda)} \arg_{\mathfrak{C}^{(\lambda)}} (\det K_+^{-1} \cdot \det K_-) .$$

Obviously Corollary 5.2 corresponds to two of F. NOETHER's theorems on singular integral equations ([7], § 131, Theorem II and Theorem III). The remaining theorem of F. NOETHER is thus

Proposition 5.3: *The inhomogeneous Ω-degenerate singular integral equation* (4) *has a solution if and only if for every solution η of the adjoint equation*

$$\int_{(S,\Sigma)} \eta^t(x)\, g(x) = 0$$

holds.

Proof: The necessity of this finite set of conditions can be proved as follows. If (4) has a solution f, then (7) gives for $pr\,\bar{\bar{x}} = x$, denoting $\sigma(\bar{\bar{x}})$ by σ

$$g(x) = K_+(x)\, f(x) + \sum_{\nu=1}^{n} B_\nu(x) \{(1 - \sigma)\, \bar{\bar{\Omega}} F_\nu(\beta \bar{\bar{x}}) + (1 + \sigma)\, \bar{\bar{\Omega}} F_\nu(\bar{\bar{x}})\}$$

For the same reason we get for every solution η of (4*)

$$\eta^t(x) = - \sum_{\nu=1}^{n} \{(1 + \sigma)\, \bar{\bar{\Omega}} H_\nu^t(\beta \bar{\bar{x}}) + (1 - \sigma)\, \bar{\bar{\Omega}} H_\nu^t(x)\}\, C_\nu(x)\, K_+^{-1}(x)$$

and according to (6)

$$- \sigma\, \eta^t(x)\, B_\nu(x) = \bar{\bar{\Omega}} H_\nu^t(\beta \bar{\bar{x}}) - \bar{\bar{\Omega}} H_\nu^t(\bar{\bar{x}}) .$$

Hence

$$\eta^t(x) g(x) = - \sum_{\nu=1}^{n} \{(1 + \sigma)\, \bar{\bar{\Omega}} H_\nu^t(\beta \bar{\bar{x}}) + (1 - \sigma)\, \bar{\bar{\Omega}} H_\nu^t(\bar{\bar{x}})\}\, C_\nu(x)\, f(x) +$$

$$+ \eta^t(x) \sum_{\nu=1}^{n} B_\nu(x) \{(1 - \sigma)\, \bar{\bar{\Omega}} F_\nu(\beta \bar{\bar{x}}) + (1 + \sigma)\, \bar{\bar{\Omega}} F_\nu(\bar{\bar{x}})\}$$

A simple calculation leads then to

$$\eta^t(x)\, g(x) = -\, 2\, \sigma(\bar{\bar{x}}) \sum_{\nu=1}^{n} \{ \bar{\bar{\Omega H}}{}^t_\nu(\beta \bar{\bar{x}})\, \bar{\bar{\Omega F}}_\nu(\beta \bar{\bar{x}}) - \bar{\bar{\Omega H}}{}^t_\nu(\bar{\bar{x}})\, \bar{\bar{\Omega F}}_\nu(\bar{\bar{x}}) \}$$

Therefore

$$\int_{(S,\Sigma)} \eta^t(x)\, g(x) = \sum_{j\in J} \int_{bdy\, X_j} \bar{\bar{\Omega H}}{}^t \cdot \bar{\bar{\Omega F}}$$

where $(X_j : j \in J)$ is the family of connected components of $X - S$. Since $\bar{\bar{\Omega H}}$ has divisor $\geq D - *$ and $\bar{\bar{\Omega F}}$ has divisor $\geq * - D$, CAUCHY's integral theorem shows that the right side of the last equation vanishes.

In order to see that the converse is also true, we show that there is a family of HÖLDER continuous differential forms $(\zeta_\lambda : \lambda \in \Lambda)$ on S which depend only on the left side of (4) such that (4) has a solution if and only if

$$\int_{(S,\Sigma)} \zeta^t_\lambda(x)\, g(x) = 0 \quad \text{for all } \lambda \in \Lambda\, .$$

After having established this auxiliary result, the proof of Proposition 5.3 can be concluded very quickly as carried out in [7], § 53.

Thus it remains to be shown that the above auxiliary result, which is usually proved by employing reducing operators and FREDHOLM theory (see [7]), is valid in our case. Here we want to give a proof using only theory of functions and thus establishing F. NOETHER's theorems for Ω-degenerate singular integral equations exclusively within the framework of theory of functions. For this purpose let $\bar{\bar{N}}{}^t$ be an $nq \times nq$ matrix such that

(i) each column of $\bar{\bar{N}}{}^t$ constitutes a meromorphic solution of the transmission problem associated with the adjoint equation, i.e.

$$\bar{\bar{N}}{}^t(\beta \bar{\bar{x}}) = T^*(pr\, \bar{\bar{x}})^{\sigma(\bar{\bar{x}})}\, \bar{\bar{N}}{}^t(\bar{\bar{x}})$$

(ii) det $\bar{\bar{N}}$ does not vanish identically

(iii) $\bar{\bar{N}}(\bar{\bar{x}}) = 0$ for all points $\bar{\bar{x}}$ which lie above points of order $\neq 2$.

Such matrices exist due to Theorem 4.1 and well known statements about holomorphic vector bundles over compact RIEMANN surfaces. Then we get for each meromorphic soluiton $\bar{\bar{G}}$ of (8), putting $pr\, \bar{\bar{x}} = x$,

$$\sigma(\bar{\bar{x}})\, \{ N(\beta \bar{\bar{x}})\, \bar{\bar{G}}(\beta \bar{\bar{x}}) - \bar{\bar{N}}(\bar{\bar{x}})\, \bar{\bar{G}}(\bar{\bar{x}}) \} =$$

$$= -\, \tfrac{1}{2}\{(1 + \sigma(\bar{\bar{x}}))\, \bar{\bar{N}}(\beta \bar{\bar{x}}) + (1 - \sigma(\bar{\bar{x}}))\bar{\bar{N}}(\bar{\bar{x}})\} \begin{pmatrix} C_1(x)\, K_+^{-1}(x)\, g(x) \\ \vdots \\ C_n(x)\, K_+^{-1}(x)\, g(x) \end{pmatrix}$$

as can be checked easily. Due to (iii) and I 2), the right side of this equation depends only on x. Hence we have for a suitable Hölder continuous matrix Z on S

$$\sigma(\bar{\bar{x}})\,\{\bar{\bar{N}}(\beta\bar{\bar{x}})\,\bar{\bar{G}}(\beta\bar{\bar{x}}) - \bar{\bar{N}}(\bar{\bar{x}})\,\bar{\bar{G}}(\bar{\bar{x}})\} = Z(x)\begin{pmatrix} g(x) \\ \vdots \\ g(x) \end{pmatrix}$$

Conversely, setting

$$H(x_2) = \frac{1}{2\pi i}\int_{(S,\Sigma)} Z(x_1)\begin{pmatrix} g(x_1) \\ \vdots \\ g(x_1) \end{pmatrix}\Omega(x_1, x_2),$$

we get by Theorem 2.2

$$\sigma(\bar{\bar{x}})\,\{\bar{\bar{H}}(\beta\bar{\bar{x}}) - \bar{\bar{H}}(\bar{\bar{x}})\} = Z(x)\begin{pmatrix} g(x) \\ \vdots \\ g(x) \end{pmatrix}$$

whence $\bar{\bar{N}}^{-1}(\bar{\bar{x}})\,\bar{\bar{H}}(\bar{\bar{x}})$ is a meromorphic solution of (8). Therefore the general meromorphic solution of (8) is of the form

$$\bar{\bar{G}}(\bar{\bar{x}}) = \bar{\bar{N}}^{-1}(\bar{\bar{x}})\,(\bar{\bar{H}}(\bar{\bar{x}}) + L(pr\,\bar{\bar{x}}))$$

where L is meromorphic on X. In order to find all those solutions $\bar{\bar{G}}$ whose divisor is $\geq * - D$, we may restrict ourselves to meromorphic matrices L on X whose divisor is $\geq D_0$ where D_0 is a certain divisor that can be computed easily from N and D. Assuming that $L_1(x), \ldots, L_m(x)$ constitutes a basis for the vector space of all those meromorphic matrices on X which have divisor $\geq D_0$, we have to find complex numbers $c_1, \ldots, c_m$ such that

$$N^{-1}(x)(H(x) + c_1 L_1(pr\,x) + \ldots + c_m L_m(pr\,x))$$

has divisor $\geq * - D$. Necessary and sufficient for this is that certain coefficients in the Laurent series of this function vanish, i.e. that a certain inhomogeneous system of linear equation in the c_μ's has a solution. Since the inhomogeneous terms of those linear equations are of the form $\int_{(S,\Sigma)} \zeta^t(x)\,g(x)$ where $\zeta(x)$ are suitable Hölder continuous differential forms on S, the necessary and sufficient conditions for the solubility of that inhomogeneous system of linear equations are precisely of the form that we stated previously. This finishes the proof of the remaining theorem of F. Noether.

Summing up we can say that we arrived at proofs for F. Noether's theorems for Ω-degenerate singular integral equations which use only results and techniques of theory of functions.

§ 6. General singular integral equations

After having found proofs of F. Noether's theorems for Ω-degenerate singular integral equations, one might ask whether it is possible to reduce the proofs of these theorems in the general case to that for Ω-degenerate kernels, much in the same way as in the theory of FREDHOLM equations. We shall briefly outline that this is indeed possible and proceed for this purpose essentially as in [2], III, § 3. Of course we shall restrict ourselves to singular integral equations (4) which are subject to the conditions I 1), I 2'), and

$$\text{I}^* \, 2') \qquad |\sigma_{x_2}| \, K^t(x_2, x_1) = 0 \quad \textit{holds on} \quad S \times S .$$

In addition we shall assume that for each $x \in S$ the relation $|\sigma_x| = 0$ holds; this relation, of course, implies both, I 2') and I* 2'). We want to show, however, that the assumption $|\sigma_x| = 0$ for all $x \in S$ is no loss of generality. Since $\sum\limits_{x \in S} |\sigma_x| = 0$ we can imbed the oriented, 1-dimensional complex (S, Σ) in a finite, oriented, 1-dimensional abstract complex (S', Σ') which satisfies $|\sigma_{x'}| = 0$ for each $x' \in S'$; for this purpose one has only to attach finitely many 1-simplices to (S, Σ) in a suitable way. (S', Σ') in turn can be imbedded in a compact RIEMANN surface X' such that the imbedding of (S, Σ) in (S', Σ') can be extended to a diffeomorphism of some neighborhood of S in X to some neighborhood of its image in X'. Now the matrices A and K which are defined on S resp. $S \times S$ shall be extended to S' resp. $S' \times S'$ as follows. Let τ be a 1-simplex of S' which is not contained in S; then its end points t_0 and t_1 are in S and $|\sigma_{t_0}| \neq 0$ and $|\sigma_{t_1}| \neq 0$ whence, due to I 1) and I 2'), $\det A(t_0) \neq 0$ and $\det A(t_1) \neq 0$ holds. Extend the matrix A to τ as a differentiable mapping of τ into $GL(q, \mathbf{C})$ which maps t_0 into $A(t_0)$ and t_1 into $A(t_1)$. This way we get a HÖLDER continuous matrix A' on S'. The matrix K' on $S' \times S'$ extending K is determined by the property that its restriction to $(S' - S) \times S' \cup S' \times (S' - S)$ is the zero matrix; K' is HÖLDER continuous due to I 2') and I* 2'). Clearly there is a natural bijective correspondence between the space of solutions of the original system and the system formed by means of A', K', and (S', Σ'), provided we use the same CAUCHY kernel in both cases. Since the indices for both systems are the same we may deal with the latter system instead of with the original one.

Defining the norm $|A|$ of the matrix $A = (a_{ij})$ by $|A|^2 = \sum\limits_{i,j} |a_{ij}|^2$, we observe firstly that every matrix $K(x_1, x_2)$ of HÖLDER continuous functions which satisfies in a compact subset S' of $\mathbf{C}^2$ the inequality

$$|K(x_1', x_1') - K(x_1'', x_2'')| \leq H(|x_1' - x_1''|^h + |x_2' - x_2''|^h) \qquad (10)$$

can be approximated uniformly by sums

$$\sum_{\nu=1}^{n} B_\nu(x_1) \, C_\nu(x_2)$$

such that the matrices $B_\nu(x_1)$, $C_\nu(x_2)$, $\nu = 1, \ldots, n$, and $\sum\limits_{\nu=1}^{n} B_\nu(x_1) \, C_\nu(x_2)$

satisfy an inequality analogous to (10) with the same HÖLDER exponent, but the HÖLDER coefficient H replaced by aH where a depends only on the geometric situation and not on the degree of approximation. For this purpose we simply regularize each component of $K(x_1, x_2)$ and approximate the resulting C^∞-functions including their first partial derivatives as well as we want.

Secondly, a simple argument (see [7], §§ 49–51) shows that every solution of (4) satisfies a HÖLDER condition with HÖLDER exponent $h' = \frac{h}{2}$, provided all given functions in (4) satisfy a HÖLDER condition with exponent h. Furthermore, denoting by $H(h')$ the complex vector space of those HÖLDER continuous $q \times 1$ matrices on S which have HÖLDER exponent h', it is easy to prove that (see [7], § 49) $H(h')$ being equipped with the norm

$$||f|| = \sup\{|f(x)| : x \in S\} + \sup\left\{\frac{|f(x_1) - f(x_2)|}{|t_\nu(x_1) - t_\nu(x_2)|^{h'}} : x_1, x_2 \in U_\nu, \nu = 1, \ldots, N\right\}$$

where $(U_\nu : \nu = 1, \ldots, N)$ is a covering of S by the coordinate neighborhoods U_ν of the complex coordinate t_ν, is a complex BANACH space. Moreover, the operator which asigns to $f \in H(h')$ the function (note that under the above hypothesis the relation (2) is trivially fulfilled)

$$(\mathfrak{K}f)(x_1) = A(x_1) f(x_1) + \frac{1}{\pi i} \int_{(S,\Sigma)} K(x_1, x_2) f(x_2)\, \Omega(x_2, x_1), \quad x_1 \in S$$

maps $H(h')$ into itself and is a bounded linear operator whose norm depends only on the geometric situation and the HÖLDER constants of A and K (see [7], § 49, and [11], § 1). In particular, if we approximate $\mathfrak{K}$ by operators given by

$$(\mathfrak{K}^{(m)}f)(x_1) = A(x_1) f(x_1) + \frac{1}{\pi i} \int_{(S,\Sigma)} \sum_{\nu=1}^{n} B_\nu^{(m)}(x_1)\, C_\nu^{(m)}(x_2) f(x_2)\, \Omega(x_2, x_1) \qquad (11)$$

in such a way that the approximating matrices $B_\nu^{(m)}$ and $C_\nu^{(m)}$ satisfy the previously stated conditions, then we arrive at a uniform bound for the norm of the operators $\mathfrak{K}$ and $\mathfrak{K}^{(m)}$, $m = 1, \ldots$.

Obviously, the approximating operators $\mathfrak{K}^{(m)}$ can be chosen such that each of them satisfies the conditions I 1), I 2) (which in the present situation is trivially satisfied), and I 3), whence the results of the previous sections apply to them.

Now the fundamental theorems of F. NOETHER can be proved in more or less the same way as it is done in [2], III, § 3, in the case of FREDHOLM equations. In particular, the argument of [2], pp. 118–119, can be repeated literally, however replacing (ϱ_n, ϱ_n) of [2], III, by $||\varrho_n||$. From that follows as in [2], III, § 3, the validity of Proposition 5.3 in the general case, provided the homogeneous integral equation $\mathfrak{K}f = 0$ has only the trivial solution.

In order to geht the remaining statements, one could work out a proof for the general CAUCHY-kernel Ω. This, however, seems to lead to lengthy computations and arguments. Therefore it is advisable to show that we can restrict ourselves to a quite special form of Ω from which we then derive the desired auxiliary results by an easy calculation.

In the discussion of the singular integral equation (4) we are only interested in properties of Ω on $S \times S$. This means that we may use a different CAUCHY-kernel Ω on X or even change the complex structure on X. Rewriting the original integral equation (4) in terms of the new CAUCHY-kernel Ω, we have to replace the kernel $K(x_1, x_2)$ by $k(x_1, x_2)\, K(x_1, x_2)$, where $k(x_1, x_2)$ is chosen such that on $S \times S$ the relation $\Omega(x_1, x_2) = k(x_1, x_2)\, \Omega'(x_1, x_2)$ holds. This leads then to an integral equation of the previously considered type, provided $\Omega'(x_1, x_2)$ has no zero on $S \times S$ and $k(x_1, x_2)$ is HÖLDER continuous on $S \times S$.

It is well known that, given a compact RIEMANN surface X of genus g and mutually distinct points $x_1^*, \ldots, x_{2g+2}^*$ of X, there is a diffeomorphism of X onto the RIEMANN surface X' defined by

$$w^2 - (z - a_1) \cdot \ldots \cdot (z - a_{2g+1}) = 0 \tag{12}$$

which sends the point x_λ^* into the point of X' lying above a_λ, $\lambda = 1, \ldots, 2g+1$, and x_{2g+2}^* into the point of X' lying above ∞. The points $a_1, \ldots, a_{2g+1}$ are subject only to the condition of being mutually distinct. If we now choose $x_1^*, \ldots, x_{2g+2}^*$ in $X - S$, then the image of S in X' will not contain any of the ramification points of $z : X' \to P^1$. Denoting the function

$$\sqrt{\frac{z - a_1}{z - a_2}, \ldots, \frac{z - a_{2g-1}}{z - a_{2g}} \cdot (z - a_{2g+1})}$$

on X' by $s(x)$ where $z = z(x)$ is the image of x under the projection $z : X' \to P^1$, one can check immediately that the CAUCHY-kernel on X' to $* = \infty$ and the normalization divisor $D = (a_2) + \ldots + (a_{2g})$ equals

$$\frac{1}{2}\left(1 + \frac{s(x_2)}{s(x_1)}\right) \frac{dz(x_1)}{z(x_1) - z(x_2)} \tag{13}$$

Moreover one verifies easily that this CAUCHY-kernel has no zeros on $S \times S$ since S does not meet the ramification points. Consequently we may assume from now on that X is the RIEMANN surface defined by (12) and that $\Omega(x_1, x_2)$ is given by (13).

In order to proceed with the proof of F. NOETHER's theorems, we remark first that the space of solutions of the homogeneous singular integral equation $\mathfrak{K} f = 0$ has finite dimension over $\mathbf{C}$. The proof of this statement is simply a repetition of [7], § 53, p. 140. Now let $\psi_1, \ldots, \psi_r$ be a $\mathbf{C}$-basis of the vector space of solutions of $\mathfrak{K} f = 0$. We want to show that for a suitable choice of the CAUCHY-kernel (13) the $q \times 1$ matrices

$$\psi_1(x), \ldots, \psi_r(x), s(x)\,\psi_1(x), \ldots, s(x)\,\psi_r(x), \quad x \in S \tag{14}$$

are linearly independent over $\mathbf{C}$. For that purpose choose the sets

$$\{a_1^{(\varrho)}, \ldots, a_{2g+1}^{(\varrho)}\}, \ \varrho = 1, \ldots, r,$$

of mutually distinct points in such a way that for every $\lambda = 1, \ldots, 2g+1$ the points $a_\lambda^{(\varrho)}$, $\varrho = 1, \ldots, r$, are in a sufficiently small neighborhood of a_λ and that the sets $\{a_1^{(\varrho)}, \ldots, a_{2g+1}^{(\varrho)}\}$ are mutually disjoint. By replacing each a_λ in $s(x)$ by $a_\lambda^{(\varrho)}$ we get a function $s_\varrho(x)$ as used in the construction of the CAUCHY-kernel (13). We claim that for at least one index ϱ the functions corresponding to (14) are linearly independent over $\mathbf{C}$. Assume that for some complex numbers $b_{\varrho\sigma}$, $b'_{\varrho\sigma}$, $\varrho, \sigma = 1, \ldots, r$,

$$\psi_1(x)\,(b_{1\sigma} + b'_{1\sigma}\, s_\sigma(x)) + \ldots + \psi_r(x)\,(b_{r\sigma} + b'_{r\sigma}\, s_\sigma(x)) = 0 \tag{15}$$

for all $x \in S$. Let us consider

$$\det(b_{\varrho\sigma} + b'_{\varrho\sigma}\, s_\sigma(x))_{\varrho,\sigma=1,\ldots,r}\,, \quad x \in X\,. \tag{16}$$

If (16) would not vanish identically, then (15) would imply that for each point x on S which does not belong to the divisor of (16) $\psi_1(x) = \ldots = \psi_r(x) = 0$ holds. By continuity the ψ_ϱ's would therefore vanish identically on S contradicting the assumed linear independence. Therefore the rank m of (16) is smaller than r. We choose now the $b_{\varrho\sigma}$ and $b'_{\varrho\sigma}$ such that m is maximal. If this maximal m equals 0, then for any choice of ϱ the functions corresponding to (14) will be linearly independent over C. If $m > 0$ holds, then we may assume that

$$\det\,(b_{\varrho\sigma} + b'_{\varrho\sigma}\, s_\sigma(x))_{\varrho,\sigma=1,\cdots,m}$$

does not vanish identically. Then (15) implies the existence of rational functions

$$f_{\kappa,m+1}(w_1, \ldots, w_m), \ldots, f_{\kappa,r}(w_1, \ldots, w_m), \quad \varkappa = 1, \ldots, m$$

such that

$$\psi_\kappa(x) = \sum_{\mu=m+1}^{r} f_{\kappa\mu}(s_1(x), \ldots, s_m(x))\,\psi_\mu(x), \quad \varkappa = 1, \ldots, m \tag{17}$$

holds. Because the rank of (16) equals m, we have for $\mu = m+1, \ldots, r$

$$0 = b_{\mu,m+1} + b'_{\mu,m+1} s_{m+1}(x) + \sum_{\kappa=1}^{m} f_{\kappa\mu}(s_1(x), \ldots, s_m(x))\,(b_{\kappa,m+1} + b'_{\kappa,m+1} s_{m+1}(x)) =$$

$$= \{b_{\mu,m+1} + \sum_{\kappa=1}^{m} f_{\kappa\mu}(s_1(x), \ldots, s_m(x))\, b_{\kappa,m+1}\} +$$

$$+ \{b'_{\mu,m+1} + \sum_{\kappa=1}^{m} f_{\kappa\mu}(s_1(x), \ldots, s_m(x))\, b'_{\kappa,m+1}\} s_{m+1}(x)$$

This shows that both terms $\{\ldots\}$ of the preceding line have to be identically zero, because $s_{m+1}(x)$ has ramification points where neither of the terms $\{\ldots\}$ has. Therefore using (17) we arrive at

$$0 = \sum_{\mu=m+1}^{r} \{b_{\mu,m+1} + \sum_{\kappa=1}^{m} f_{\kappa\mu}(s_1(x), \ldots, s_m(x))\, b_{\kappa,m+1}\}\, \psi_\mu(x) =$$
$$= \sum_{\mu=m+1}^{r} b_{\mu,m+1}\, \psi_\mu(x) + \sum_{\kappa=1}^{m} \{ \sum_{\mu=m+1}^{r} f_{\kappa\mu}(s_1(x), \ldots, s_m(x))\, \psi_\mu(x)\}\, b_{\kappa,m+1} =$$
$$= \sum_{\varrho=1}^{r} b_{\varrho,m+1}\, \psi_\varrho(x)$$

whence $b_{1,m+1} = \ldots = b_{r,m+1} = 0$ due to the linear independence of $\psi_1, \ldots, \psi_r$. A corresponding argument shows that also $b'_{1,m+1} = \ldots = b'_{r,m+1}$ holds. Thus a suitable choice of the CAUCHY-kernel (13) makes the functions (14) linearly independent over $\mathbf{C}$, as claimed before.

The next step is to replace the approximating sequence operators $\mathfrak{K}^{(m)}$ by a sequence of Ω-degenerate operators $\hat{\mathfrak{K}}^{(m)}$ which has the properties previously stated for the sequence $\mathfrak{K}^{(m)}$ and fulfills the additional requirement $\hat{\mathfrak{K}}^{(m)}\psi_\varrho = 0$ for $\varrho = 1, \ldots, r$.

Since the $q \times 1$ matrices (14) are linearly independent over $\mathbf{C}$ the SCHMIDT orthonormalization process gives the existence of a non-singular $2r \times 2r$ matrix C' with entries complex numbers such that

$$\int_{(S,\Sigma)} \overline{C'}^t \begin{pmatrix} \overline{\psi_1}^t(x_2) \\ \vdots \\ \overline{\psi_r}^t(x_2) \\ \overline{s(x_2)}\; \overline{\psi_1}^t(x_2) \\ \vdots \\ \overline{s(x_2)}\; \overline{\psi_r}^t(x_2) \end{pmatrix} (\psi_1(x_2), \ldots, \psi_r(x_2), s(x_2)\,\psi_1(x_2), \ldots, s(x_2)\,\psi_r(x_2))\; C'\; |dz(x_2)| =$$

holds, where 1 designates the $2r \times 2r$ unit matrix. Therefore we get a $2r \times 2r$ matrix C with entries complex numbers such that

$$\int_{(S,\Sigma)} C \begin{pmatrix} \overline{\psi_1}^t(x_2) \\ \vdots \\ \overline{s(x_2)}\; \overline{\psi_r}^t(x_2) \end{pmatrix} (\psi_1(x_2), \ldots, s(x_2)\,\psi_r(x_2))\; |dz(x_2)| = 1\,. \tag{18}$$

Putting
$$\chi_\varrho^{(m)}(x_1) = -\,(\mathfrak{K}^{(m)}\psi_\varrho)(x_1) \quad \varrho = 1, \ldots, r,\; m = 1, \ldots \tag{19}$$
we note that $|\chi_\varrho^{(m)}(x_1)|$ converges uniformly on S to 0 as m tends to infinity, due to the previously stated properties of the approximating sequence $\mathfrak{K}^{(m)}$. Furthermore we know that the matrices $\chi_\varrho^{(m)}$ are HÖLDER continuous on S with HÖLDER constants which are uniformly bounded. Defining now the $q \times q$ matrix $L^{(m)}(x_1, x_2)$ on $S \times S$ by

$$\pi i\,(\chi_1^{(m)}(x_1),\dots,\chi_r^{(m)}(x_1),\,s(x_1)\,\chi_1^{(m)}(x_1),\dots,\,s(x_1)\;\chi_r^{(m)}(x_1))\;C\begin{pmatrix}\overline{\psi_1}^t(x_2)\\ \vdots\\ \overline{s(x_2)}\;\overline{\psi_r}^t(x_2)\end{pmatrix}\cdot$$
$$\cdot\,\frac{s(x_2)}{s(x_1)}\;\frac{(z(x_2)-z(x_1))\,|dz(x_2)|}{dz(x_2)} \tag{20}$$

the relation (18) implies that

$$\frac{1}{\pi i}\int_{(S,\Sigma)} L^{(m)}(x_1,x_2)\,(\psi_1(x_2),\dots,\psi_r(x_2))\;\Omega(x_2,x_1)=$$
$$=\frac{1}{2}\int_{(S,\Sigma)}(\chi_1^{(m)}(x_1),\dots,s(x_1)\;\chi_r^{(m)}(x_1))\;C\begin{pmatrix}\overline{\psi_1}^t(x_2)\\ \vdots\\ \overline{s(x_2)}\;\overline{\psi_r}^t(x_2)\end{pmatrix}(\psi_1(x_2),\dots,\psi_r(x_2))\;\cdot$$
$$\cdot\left(\frac{s(x_2)}{s(x_1)}+1\right)|dz(x_2)|=$$
$$=\frac{1}{2}\,(\chi_1^{(m)}(x_1),\dots,s(x_1)\,\chi_r^{(m)}(x_1))\cdot\left\{\frac{1}{s(x_1)}\begin{pmatrix} & 0 & \\ \cdots & \cdots & \cdots\\ 1 & & 0\\ & \ddots & \\ 0 & & 1\end{pmatrix}+\begin{pmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\\ \cdots & \cdots & \cdots\\ & 0 & \end{pmatrix}\right\}(\chi_1^{(m)}(x_1),\dots,\chi_r^{(m)}(x_1)).$$

Thus, using (19), we find that the operator $\hat{\mathfrak{K}}^{(m)}=\mathfrak{K}^{(m)}+\mathfrak{L}^{(m)}$ where

$$\mathfrak{L}^{(m)}f(x_1)=\frac{1}{\pi i}\int_{(S,\Sigma)} L^{(m)}(x_1,x_2)\,f(x_2)\;\Omega(x_2,x_1)$$

satisfies the equations

$$\hat{\mathfrak{K}}^{(m)}\psi_\varrho=0\quad\text{for}\quad \varrho=1,\dots,r,\;m=1,\dots$$

In addition (20) shows that $\mathfrak{L}^{(m)}$ and therefore $\hat{\mathfrak{K}}^{(m)}$ is Ω-degenerate and that $\hat{\mathfrak{K}}^{(m)}$ satisfies the conditions I 1) and I 3) since $L^{(m)}(x_1,x_1)=0$ on S. Switching from the original approximating sequence to the one which has just been constructed we may now assume that the solutions $\psi_1,\dots,\psi_r$ of $\mathfrak{K}f=0$ are also solutions of $\mathfrak{K}^{(m)}f=0$.

We claim finally that the vector space of solutions of $\mathfrak{K}^{(m)}f=0$ has for sufficiently large m exactly dimension r over $\mathbf{C}$. So far we know that this dimension is at least r. If for infinitely many m this dimension were bigger than r, then we could select a subsequence such that every integral equation of this subsequence has a solution space of dimension $\geq r+1$. Therefore we could find for every m a solution $\psi_{r+1}^{(m)}$ of $\mathfrak{K}^{(m)}f=0$ which does not depend linearly upon $\psi_1,\dots,\psi_r$. Furthermore we may assume that for $\varrho=1,\dots,r$

$$\int_{(S,\Sigma)}\overline{\psi_\varrho}^t(x_1)\,\psi_{r+1}^{(m)}(x_1)\,|dz(x_1)|=0\quad\text{and}\quad\int_{(S,\Sigma)}\overline{\psi_{r+1}^{(m)}}^t(x_1)\,\psi_{r+1}^{(m)}(x_1)\,|dz(x_1)|=1$$

hold. As can be seen very easily (see [2], III, § 1, and [7], §§ 45 and 51) the latter condition implies that the $\psi_{r+1}^{(m)}$ form a uniformly bounded and equicontinuous family. Hence ARZELA's theorem shows the existence of a uniformly convergent subsequence, whose limit ψ_{r+1} is not identically zero, is orthogonal to $\psi_1, \ldots, \psi_r$, and constitutes a solution of $\mathfrak{K} f = 0$ which is impossible. Since the index of a singular integral equation as defined by the right side of Corollary 5.2 is an integer and since the sequence $\mathfrak{K}^{(m)}$ approximates $\mathfrak{K}$ in the previously stated manner, the index i_m of $\mathfrak{K}^{(m)}$ equals the index i of $\mathfrak{K}$ for sufficiently large m. Therefore, denoting the space of solutions of $\mathfrak{K}^{(m)} f = 0$ (the adjoint equation $\mathfrak{K}^{(m)*} f = 0$) by L^m (L_m^*) and the corresponding spaces for $\mathfrak{K}$ by L (L^*), we get from Corollary 5.2

$$i = i_m = \dim_{\mathbf{C}} L_m - \dim_{\mathbf{C}} L_m^* = \dim_{\mathbf{C}} L - \dim_{\mathbf{C}} L_m^*$$

for sufficiently large m. Because the sequence $\mathfrak{K}^{(m)*}$ approximates $\mathfrak{K}^*$ in. the same way as the sequence $\mathfrak{K}^{(m)}$ approximates $\mathfrak{K}$, we get $\dim_{\mathbf{C}} L_m^* \leq \dim_{\mathbf{C}} L^*$e If, however, $\dim_{\mathbf{C}} L_m^* < \dim_{\mathbf{C}} L$ would hold for arbitraryly large m, then we could interchange the roles of $\mathfrak{K}$ and $\mathfrak{K}^*$ and thus find that the dimension of the solution space of $\mathfrak{K} f = 0$ had to be bigger than $\dim_{\mathbf{C}} L$ which is a contradiction. Therefore Corollary 5.2 (Index Theorem) has been proved in the general case.

It remains to be shown that Proposition 5.3 too is valid in the general case. The necessity of the conditions in Proposition 5.3 is essentially trivial (see [7], § 53). The sufficiency can be shown as follows. According to our previous construction we may assume that the approximating sequence is chosen in such a way that $\dim_{\mathbf{C}} L_m^* = \dim_{\mathbf{C}} L^* = r^*$ holds for all m. Furthermore we may assume that for each n the matrix valued differential forms $\eta_1^{(m)}, \ldots, \eta_r^{(m)}$ constitute a $\mathbf{C}$-basis of L_m^* such that

(i)
$$\int\limits_{(S,\Sigma)} \left(\frac{\eta_\varrho^{(m)}(x_1)}{|dz(x_1)|}\right)^t \overline{\left(\frac{\eta_\sigma^{(m)}(x_1)}{|dz(x_1)|}\right)} |dz(x_1)| = \begin{cases} 0 & \text{if } \varrho \neq \sigma \\ 1 & \text{if } \varrho = \sigma \end{cases}$$

(ii) each sequence $\eta_\varrho^{(m)}$, $m = 1, \ldots,$ converges uniformly to η_ϱ.

Then the matrix differential forms $\eta_1, \ldots, \eta_{r^*}$ form a $\mathbf{C}$-basis von L^*. If now

$$\int\limits_{(S,\Sigma)} \eta_\varrho^t(x_1)\, g(x_1) = 0 \quad \varrho = 1, \ldots, r^*$$

holds, then for each ϱ the sequence

$$\int\limits_{(S,\Sigma)} \eta_\varrho^{(m)t}\, (x_1)\, g(x_1)$$

converges to zero, whence the sequence

$$g^{(m)}(x_2) = g(x_2) - \sum_{\varrho=1}^{r^*} \int\limits_{(S,\Sigma)} \eta_\varrho^{(m)t}(x_1)\, g(x_1) \cdot \overline{\left(\frac{\eta_\varrho^{(m)}(x_2)}{|dz(x_2)|}\right)}$$

converges uniformly to $g(x_1)$. Moreover (i) implies that for $\varrho = 1, \ldots, r^*$

$$\int_{(S,\Sigma)} \eta_\varrho^{(m)t}(x_1)\, g^{(m)}(x_1) = 0$$

Thus Proposition 5.3 gives the existence of a solution $f^{(m)}$ of the integral equation $\mathfrak{K}^{(m)} f = g^{(m)}$. An argument analogous to the one in [2]. III, § 3, permits then to pass to the limit in $\mathfrak{K}^{(m)} f^{(m)} = g^{(m)}$ and thus to establish the existence of a solution of the original integral equation $\mathfrak{K} f = g$.

Therefore we have proved

Theorem 6.1: *Assuming that the singular integral equation* (4) *satisfies* I 1), I 2′), *and* I* 2′), *the statements of* Corollary 4.2, Corollary 5.2, *and* Proposition 5.3 *are valid for the singular integral equation* (4).

§ 7. Final remarks

The procedures of § 3 together with the results of [11] lend themselves to a treatment of certain types of nonlinear singular integral equations analogous to the one developed in the preceding sections. Unfortunately, only little is known about holomorphic sections in holomorphic fiber bundles other than vector bundles over compact Riemann surfaces. This puts a severe limitation on the applicability of this method to nonlinear singular integral equations.

The singular integral equation (4) which satisfies (5) is of the general form (see the argument following (6))

$$f(x_1) = \tag{21}$$
$$= \Phi\left(C_1(x_1)\, f(x_1), \ldots, C_n(x_1)\, f(x_1), \frac{1}{\pi i} \int_{(S,\Sigma)} C_1(x_2)\, f(x_2)\; \Omega(x_2, x_1), \ldots\right)$$

Assuming that f is a Hölder continuous solution[11]) of (21) we introduce as previously

$$F_\nu(x_1) = \frac{1}{2\pi i} \int_{(S,\Sigma)} C_\nu(x_2)\, f(x_2)\; \Omega(x_2, x_1), \quad x_1 \in X - S$$

and find as in § 3

$$f(pr\, \overline{\overline{x}}) =$$
$$= \Phi(\sigma(\overline{\overline{x}})\{\overline{\overline{\Omega}} F_1(\beta \overline{\overline{x}}) - \overline{\overline{\Omega}} F_1(\overline{\overline{x}})\}, \ldots, \sigma(\overline{\overline{x}})\{\overline{\overline{\Omega}} F_n(\beta \overline{\overline{x}}) - \overline{\overline{\Omega}} F_n(\overline{\overline{x}})\}, \overline{\overline{\Omega}} F_1(\beta \overline{\overline{x}}) + \overline{\overline{\Omega}} F_1(\overline{\overline{x}}), \ldots$$

Therefore we have for $\nu = 1, \ldots, n$

$$\overline{\overline{\Omega}} F_\nu(\beta \overline{\overline{x}}) - \overline{\overline{\Omega}} F_\nu(\overline{\overline{x}}) =$$
$$= \sigma(\overline{\overline{x}})\, C_\nu(pr\, \overline{\overline{x}})\, \Phi(\sigma(\overline{\overline{x}})\, \{\overline{\overline{\Omega}} F_1(\beta \overline{\overline{x}}) - \overline{\overline{\Omega}} F_1(\overline{\overline{x}})\}, \ldots, \overline{\overline{\Omega}} F_1(\beta \overline{\overline{x}}) + \overline{\overline{\Omega}} F_1(\overline{\overline{x}}), \ldots)$$

[11]) Contrary to linear singular integral equations, (21) may have integrable solutions which are not Hölder continuous.

or

$$\bar{\bar{\Omega}} F(\beta \bar{\bar{x}}) - \bar{\bar{\Omega}} F(\bar{\bar{x}}) =$$ (2

$$= \sigma(\bar{\bar{x}}) \begin{pmatrix} C_1(pr\, \bar{\bar{x}}) \\ \vdots \\ C_n(pr\, \bar{\bar{x}}) \end{pmatrix} \Phi\,(\sigma(\bar{\bar{x}})\, \{\bar{\bar{\Omega}} F_1(\beta \bar{\bar{x}}) - \bar{\bar{\Omega}} F_1(\bar{\bar{x}})\}, \ldots, \bar{\bar{\Omega}} F_1(\beta \bar{\bar{x}}) + \bar{\bar{\Omega}} F_1(\bar{\bar{x}}), \ldots$$

Under the assumption that

N) *there exists a complex* LIE *group* G *acting on* $\mathbf{C}^{nq}$ *as a group of complex automorphisms, together with a* HÖLDER *continuous mapping* $T: S \to G$ *such that for any two elements* $y^{(1)}, y^{(2)}$ *in* $\mathbf{C}^{nq}$ [12]) *the equations*

$$y^{(1)} - y^{(2)} =$$

$$= \sigma(\bar{\bar{x}}) \begin{pmatrix} C_1(pr\, \bar{\bar{x}}) \\ \vdots \\ C_n(pr\, \bar{\bar{x}}) \end{pmatrix} \Phi(\sigma(\bar{\bar{x}})\, \{y_1^{(1)} - y_1^{(2)}\}, \ldots, \sigma(\bar{\bar{x}})\, \{y_n^{(1)} - y_n^{(2)}\}, y_1^{(1)} + y_1^{(2)}, \ldots)$$

and

$$y^{(1)} = T(pr\, \bar{\bar{x}})^{\sigma(\bar{\bar{x}})} \cdot y^{(2)}$$

are equivalent,

any HÖLDER continuous solution f of (21) leads via (22) to a solution $\bar{\bar{\Omega}} F$ of the transmission problem

$$\bar{\bar{G}}(\beta \bar{\bar{x}}) = T(pr\, \bar{\bar{x}})^{\sigma(\bar{\bar{x}})} \cdot \bar{\bar{G}}(\bar{\bar{x}}) \tag{23}$$

for which F is meromorphic in $X - S$ and has divisor $\geq * - D$. If one now imposes on Φ a condition analogous to I 3) — which is of course empty, provided S is a disjoint union of simple closed curves —, then one can show as in § 3, that every meromorphic solution of the above transmission problem which has divisor $\geq * - D$ gives rise to a HÖLDER continuous solution of the singular integral equation (21). Therefore one arrives at an obvious generalization of Theorem 3.3. Hence, using [11], one can again associate with (21) a holomorphic fiber bundle over X with fiber $\mathbf{C}^{nq}$ and structure group G such that the HÖLDER continuous solutions of (21) correspond bijectively to those meromorphic sections in the bundle which have divisor $\geq * - D$ (see proof of Theorem 4.1)[13]).

As mentioned previously there are virtually no general theorems concerning the existence of holomorphic sections in the fiber bundles we finally arrive at. However, the following statement is true (for a similar, yet more special case see [9]; the proof given there can be generalized so as to cover the situation

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[12]) We put for $y \in \mathbf{C}^{nq}$ $y^t = (y_1^t, \ldots, y_n^t) \in \mathbf{C}^q \oplus \ldots \oplus \mathbf{C}^q = \mathbf{C}^{nq}$.

[13]) The notation «meromorphic section in the fiber bundle which has divisor $\geq * - D$» makes only sense for the specific transition functions $\hat{g}_{ij}$ constructed in [11], § 4, since the transmission problem has values in a trivial vector bundle. Therefore this notion is not an invariant of the fiber bundle, but only of the coordinate bundle.

here). Given a covering $(U_i : i \in I)$ of P^1 by open subsets which are homeomorphic to a disk, let the holomorphic fiber bundle $W \to P^1$ with connected structure group G be described by transition function g_{ij} in $U_i \cap U_j$; then $W \to P_1$ is holomorphically trivial, provided the transition functions g_{ij} assume only values in a (sufficiently small) neighborhood of the neutral element in G which depends only on the given covering $(U_i : i \in I)$. On the other hand, one sees easily from [11], § 4, that the fiber bundle we are interested in satisfies these requirements, provided T assumes only values in a sufficiently small neighborhood of the neutral element of G. Therefore we get

Theorem 7.1: *Suppose that Φ maps Hölder continuous functions into Hölder continuous functions, that* N) *is satisfied, and that S is a disjoint union of simple closed curves which are contained in P^1. Then the integral equation* (21), *considered for $X = P^1$, has exactly one Hölder continuous solution, provided the transmission function T assumes on S values which are contained in a (sufficiently small) neighborhood of the neutral element of G which depends only on the choice of S.*

It is clear that the whole argument of this section can again be carried over to the discussion of families of nonlinear singular integral equations. The details are left to the reader.

As an example of the type of integral equation dealt with in this section, we assume again S to be a disjoint union of simple closed curves. Then for any complex Lie group G acting on $\mathbf{C}^q$ as a group of complex automorphisms and for every Hölder continuous mapping $T: S \to G$, the nonlinear integral equation

$$f(x_1) = \frac{1}{\pi i} \int_{(S,\Sigma)} f(x_2)\, \Omega(x_2, x_1) + T(x_1)\left(f(x_1) + \frac{1}{\pi i} \int_{(S,\Sigma)} f(x_2)\, \Omega(x_2, x_1)\right)$$

where f has values in $\mathbf{C}^q$ is of the type we have considered in this section. Thus the above results apply to it.

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