

# On the Local Triviality of the Restriction Map for Embeddings.

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# On the Local Triviality of the Restriction Map for Embeddings

by ELON L. LIMA<sup>1)</sup>

Let  $V, M$  be  $C^\infty$  manifolds,  $V$  compact. A map  $f: M \rightarrow M$  is said to have *compact support* if it agrees with the identity outside of a compact set. For  $1 \leq r \leq \infty$ , we consider the following spaces endowed with the  $C^r$ -topology:  $\mathcal{E}^r(V, M) =$  all  $C^\infty$  embeddings of  $V$  in  $M$ ;  $\mathcal{C}^r(M) =$  all  $C^\infty$  maps, with compact support, of  $M$  into  $M$ ;  $\mathcal{D}^r(M) =$  all  $C^\infty$  diffeomorphisms, with compact support, of  $M$  onto  $M$ . We remark that  $\mathcal{D}^r(M)$  is an open subset of  $\mathcal{C}^r(M)$ .

R. PALAIS proved [1] that if  $V$  is a submanifold of  $W$  then the restriction map  $j: \mathcal{E}^r(W, M) \rightarrow \mathcal{E}^r(V, M)$  is a locally trivial fibration. Previously, R. THOM had observed [2] that  $j$  has the covering homotopy property for polyhedra. The local triviality of  $j$  follows easily from the theorem below, (see [1], or Remark 2), at the end of this note), which was also proved by J. CERF [3]. We present here a very simple proof of this theorem. For implications and applications, see the bibliography.

**Theorem:** *Given  $f \in \mathcal{E}^r(V, M)$ , there is a neighborhood  $U$  of  $f$  and a continuous map  $\xi: U \rightarrow \mathcal{D}^r(M)$  such that  $g = \xi(g) \circ f$  for every  $g \in U$ .*

*Proof:* We may assume that  $V$  is a submanifold of  $M$ ,  $f =$  inclusion, and  $M$  is embedded in some euclidean space  $R^k$ . Let  $\pi': T' \rightarrow M$  be a tubular neighborhood of  $M$  in  $R^k$  and  $\pi: T \rightarrow V$  a tubular neighborhood, of radius  $\varepsilon > 0$ , of  $V$  in  $R^k$ , with  $T \subset T'$ . Denote by  $\frac{1}{2}T$  the tubular neighborhood of  $V$  with radius  $\varepsilon/2$ . Since the shortest line from a point in  $R^k$  to  $V$  is a normal segment, any line segment of length  $< \varepsilon/2$  which intersects  $\frac{1}{2}T$  lies entirely within  $T$ . Choose a neighborhood  $U'$  of  $f$  in  $\mathcal{E}^r(V, M)$  so small that  $|g(y) - y| < \varepsilon/2$  for all  $g \in U'$  and all  $y \in V$ . Let  $\lambda: R \rightarrow [0, 1]$  be a  $C^\infty$  function with  $\lambda(t) = 1$  for  $|t| \leq \varepsilon/4$  and  $\lambda(t) = 0$  for  $|t| \geq \varepsilon/2$ . Define a map  $\xi': U' \rightarrow \mathcal{C}^r(M)$  as follows. Given  $g \in U'$ , put  $\xi'(g)(x) = x$ , if  $x \in M - T$ , and  $\xi'(g)(x) = \pi' \{x + \lambda(|x - \pi x|) \cdot [g(\pi x) - \pi x]\}$  if  $x \in T$ . One sees that  $\xi'$  is continuous and  $\xi'(f)$  is the identity map of  $M$ , so  $\xi'(f) \in \mathcal{D}^r(M)$ . Since  $\mathcal{D}^r(M)$  is open in  $\mathcal{C}^r(M)$ , a smaller neighborhood  $U$  of  $f$  can be chosen so that  $\xi'(U) \subset \mathcal{D}^r(M)$ . Put  $\xi = \xi'|_U$ .

*Remarks:* 1) Let  $\mathcal{D}'_0(M) \subset \mathcal{D}^r(M)$  be the subset of  $C^\infty$  diffeomorphisms, with compact support, that are diffeotopic to the identity. It is known that

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$\mathcal{D}_0^r(M)$  is open in  $\mathcal{D}^r(M)$ . (This can be seen by a construction similar to, and simpler than, the above one.) So, if needed,  $U$  may be taken such that  $\xi(U) \subset \mathcal{D}_0^r(M)$ .

2) Given  $f \in \mathcal{D}^r(V, M)$ , take  $\xi$  and  $U$  as in the theorem, let  $F = j^{-1}(f)$  and define a homeomorphism  $\psi: F \times U \rightarrow j^{-1}(U)$  by  $\psi(\bar{f}, g) = \xi(g) \circ \bar{f}$ , for  $\bar{f} \in F$ ,  $g \in U$ . This shows that  $j$  is a locally trivial fibration.

3) When  $r = \infty$ ,  $\mathcal{E}^\infty(V, M)$  and  $\mathcal{D}^\infty(J)$  are  $C^\infty$  (infinite dimensional) manifolds, locally homeomorphic with FRÉCHET spaces. The reason why  $\xi$  is continuous is that, in the last analysis, it is obtained as a series of compositions of the variable map  $g$  with fixed  $C^\infty$  maps. Now, composition is a differentiable map in the  $C^\infty$  topology. (See [4], pages 182, 183.) So, by the same token,  $\xi$  is a  $C^\infty$  map when  $r = \infty$ . It follows from this and Remark 2) above that  $j: \mathcal{E}^\infty(W, M) \rightarrow \mathcal{E}^\infty(V, M)$  would be a  $C^\infty$  fibration, in the sense that the local trivializing maps  $\psi: F \times U \rightarrow j^{-1}(U)$  are  $C^\infty$ , provided one could show that  $F$  is a differentiable manifold.

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