On a Certain Property of Closed Hypersurfaces in an EINSTEIN Space

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Introduction.

The following theorem, due to H. LIEBMANN (1900) [1], has been, and still is, the starting point of various interesting investigations within the Differential Geometry in the Large:

The only ovaloids with constant mean curvature H in EUCLIDean space E^3 are the spheres.

The analogous theorem for convex m-dimensional hypersurfaces in E^{m+1} has been proved by W. SÜSS (1929), [2], (cf. also [3], p. 118, and [4]). Recently (1958), A. D. ALEXANDROV has achieved the striking result that the convexity is not necessary for the validity of the LIEBMANN-SÜSS theorem, [5]: the theorem holds for arbitrary closed m-dimensional surfaces (hypersurfaces) without double points in E^{m+1} (i.e., for 1-1-imbeddings of closed m-manifolds). Already previously (1951), H. HOPF had shown that, for $n = 2$ and for surfaces of genus 0, the theorem holds even without the hypothesis that there are no double points (i.e., it holds for immersions, not necessarily one-one, of 2-spheres into E^3), [6]. It remains an open question whether there exists an immersion, not one-one, of a closed surface of higher genus into E^3 such that $H =$ constant.

There are also interesting investigations about generalizing the condition $H =$ constant in LIEBMANN's theorem. But we shall not discuss these problems here.

It is the aim of the present author to investigate the question whether the mentioned theorems, especially the LIEBMANN-SÜSS theorem, are special cases of theorems which hold in more general RIEMANN spaces. One step in this direction has already been made in a previous paper dealing with RIEMANN spaces with constant RIEMANN curvature [7]. The present paper deals with EINSTEIN spaces and generalizes the paper [7], without making use of it. Our result is Theorem 3.1 which, as is easily seen, contains the LIEBMANN-SÜSS theorem as special case (so does, by the way, also the main theorem of [7]).

It is well known that the LIEBMANN-SÜSS theorem is closely related to classical integral formulas of MINKOWSKI (cf. the paper of SÜSS). The base of our proof of Theorem 3.1 is a formula of MINKOWSKI type which holds in arbitrary RIEMANN spaces (formula (I) in § 1). This formula had already been established in [7]; a new proof is given in § 1 below. In § 2, some integral formulas for hypersurfaces with $H =$ constant in EINSTEIN spaces are derived, and in § 3, the main theorem is proved.

The author wishes to express to Professor HEINZ HOPF her very sincere thanks for his valuable advice and suggestions.

§ 1. Another proof of the generalized MINKOWSKI formula (I).

In this section, we shall give a different proof of the generalized MINKOWSKI formula (I) derived in the previous paper ([7], p. 288).

We consider a RIEMANN space $R^{m+1} (m + 1 \geqslant 3)$ of class $C^{\nu} (\nu \geqslant 3)$ which admits an one-parameter continuous group G of transformations generated by an infinitesimal transformation

$$\bar{x}^i = x^i + \xi^i(x)\,\delta\tau \tag{1.1}$$

(where x^i are local coordinates in R^{m+1} and ξ^i are the components of a contravariant vector ξ). We suppose that the paths of these transformations cover R^{m+1} simply and that ξ is everywhere continuous and $\neq 0$. If ξ is a KILLING vector, a homothetic KILLING, a conformal KILLING vector etc. ([8], p. 32), then the group G is called isometric, homothetic, conformal etc., respectively.

We now consider a closed orientable hypersurface V^m of class C^3 imbedded in R^{m+1}, locally given by

$$x^i = x^i(u^\alpha)\ ; \tag{1.2}$$

here and henceforth, Latin indices run from 1 to $m + 1$ and Greek indices from 1 to m.

To the vector ξ introduced above, there belongs a covariant vector $\bar{\xi}$ of V^m with the components

$$\bar{\xi}_\alpha = \frac{\partial x^i}{\partial u^\alpha}\xi_i$$

where ξ_i are the covariant components of ξ; we shall compute its covariant derivatives along V^m: by virtue of the fact that the covariant derivatives of $\frac{\partial x^i}{\partial u^\alpha}$ are

$$\frac{\delta}{\partial u^\beta}\left(\frac{\partial x^i}{\partial u^\alpha}\right) = b_{\alpha\beta} n^i$$

where $b_{\alpha\beta}$ is the second fundamental tensor and n^i is the unit normal vector of V^m, we find

$$\bar{\xi}_{\alpha;\beta} = b_{\alpha\beta} n^i \xi_i + \frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^j}{\partial u^\beta}\xi_{i;j} \tag{1.3}$$

(the symbol " ; " always means the covariant derivative).

Multiplying (1.3) by the contravariant metric tensor $g^{\alpha\beta}$ of V^m and contracting, we get

$$g^{\alpha\beta}\bar{\xi}_{\alpha;\beta} = mH_1 n^i \xi_i + \frac{1}{2} g^{\alpha\beta} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} \cdot \underset{\xi}{\mathcal{L}} g_{ij} , \tag{1.4}$$

where H_1 is the first mean curvature $\frac{1}{m} g^{\alpha\beta} b_{\alpha\beta}$ of V^m and $\underset{\xi}{\mathcal{L}} g_{ij}$ is the LIE derivative of the fundamental tensor g_{ij} of R^{m+1} with respect to the infinitesimal transformation (1.1) (cf. [8], p.5). If we put

$$\frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} \underset{\xi}{\mathcal{L}} g_{ij} = \underset{\xi}{\mathcal{L}} g_{\alpha\beta}$$

then (1.4) rewritten is as follows:

$$\frac{1}{m} \bar{\xi}^\alpha{}_{;\alpha} = H_1 n^i \xi_i + \frac{1}{2m} g^{\alpha\beta} \underset{\xi}{\mathcal{L}} g_{\alpha\beta} .$$

dA being the area element of V^m, there holds

$$\underset{V^m}{\int \cdots \int} \bar{\xi}^\alpha{}_{;\alpha} dA = 0$$

because V^m is closed and orientable ([9], p.31). Thus we obtain the integral formula

$$\underset{V^m}{\int \cdots \int} H_1 n^i \xi_i dA + \frac{1}{2m} \underset{V^m}{\int \cdots \int} g^{\alpha\beta} \underset{\xi}{\mathcal{L}} g_{\alpha\beta} dA = 0 \tag{I}$$

which is nothing but the formula (I) of the previous paper [7], p.288.

Let the group G be conformal, that is, ξ^i satisfy the equation

$$\underset{\xi}{\mathcal{L}} g_{ij} \equiv \xi_{i;j} + \xi_{j;i} = 2\Phi g_{ij}$$

(cf. [8], p.32), then (I) becomes

$$\underset{V^m}{\int \cdots \int} H_1 n^i \xi_i dA + \underset{V^m}{\int \cdots \int} \Phi dA = 0 ; \tag{I$_c$}$$

let G be homothetic, that is, $\Phi \equiv C =$ constant, then

$$\underset{V^m}{\int \cdots \int} H_1 n^i \xi_i dA + C \underset{V^m}{\int \cdots \int} dA = 0 ; \tag{I$_h$}$$

and let G be isometric, that is, $C = 0$, then

$$\underset{V^m}{\int \cdots \int} H_1 n^i \xi_i dA = 0 . \tag{I$_i$}$$

§ 2. Some integral formulas for a closed hypersurface with H_1 = constant in an EINSTEIN space.

Hereafter we shall assume that the RIEMANN space R^{m+1} is an EINSTEIN space and V^m is a closed orientable hypersurface with H_1 = constant.

If we take the covariant vector of the hypersurface V^m, defined by

$$\eta_\alpha = n^i{}_{;\alpha}\xi_i$$

and calculate its covariant derivatives along V^m, we have

$$\eta_{\alpha;\beta} = n^i{}_{;\alpha;\beta}\xi_i + n^i{}_{;\alpha}\xi_{i;j}\frac{\partial x^j}{\partial u^\beta}.$$

Remembering the following formulas for hypersurfaces

$$n^i{}_{;\alpha} = -b^\gamma_\alpha \frac{\partial x^i}{\partial u^\gamma}, \tag{2.1}$$

$$\frac{\delta}{\partial u^\beta}\left(\frac{\partial x^i}{\partial u^\gamma}\right) = b_{\beta\gamma}n^i, \tag{2.2}$$

where b^γ_α means $g^{\gamma\delta}b_{\alpha\delta}$ ([10] p. 136, p. 127), we find that

$$\eta_{\alpha;\beta} = -\left(\xi_i b^\gamma_{\alpha;\beta}\frac{\partial x^i}{\partial u^\gamma} + b^\gamma_\alpha b_{\gamma\beta}\xi_i n^i + \xi_{i;j}b^\gamma_\alpha \frac{\partial x^i}{\partial u^\gamma}\frac{\partial x^j}{\partial u^\beta}\right).$$

Multiplying by $g^{\alpha\beta}$ and contracting, we obtain

$$g^{\alpha\beta}\eta_{\alpha;\beta} = -g^{\alpha\beta}\left(\xi_i b^\gamma_{\alpha;\beta}\frac{\partial x^i}{\partial u^\gamma} + b^\gamma_\alpha b_{\gamma\beta}\xi_i n^i + \xi_{i;j}b^\gamma_\alpha \frac{\partial x^i}{\partial u^\gamma}\frac{\partial x^j}{\partial u^\beta}\right). \tag{2.3}$$

We shall first calculate the first term of the right-hand side of (2.3):

$$g^{\alpha\beta}\xi_i b^\gamma_{\alpha;\beta}\frac{\partial x^i}{\partial u^\gamma} = g^{\alpha\beta}g^{\gamma\delta}b_{\alpha\delta;\beta}\xi_i\frac{\partial x^i}{\partial u^\gamma}. \tag{2.4}$$

As well-known, an hypersurface in a RIEMANN space has the following property

$$b_{\alpha\delta;\beta} - b_{\alpha\beta;\delta} = -R_{ijkl}\frac{\partial x^i}{\partial u^\alpha}n^j\frac{\partial x^k}{\partial u^\delta}\frac{\partial x^l}{\partial u^\beta}, \quad \text{([10] p. 138)}$$

where R_{ijkl} is the curvature tensor of R^{m+1}. Multiplying both sides of this equation by $g^{\alpha\beta}$ and contracting, we get

$$g^{\alpha\beta}(b_{\alpha\delta;\beta} - b_{\alpha\beta;\delta}) = -R_{ijkl}\frac{\partial x^i}{\partial u^\alpha}n^j\frac{\partial x^k}{\partial u^\delta}\frac{\partial x^l}{\partial u^\beta}g^{\alpha\beta}; \tag{2.5}$$

substituting

$$\frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^l}{\partial u^\beta}g^{\alpha\beta} = g^{il} - n^i n^l$$

into the right-hand side of (2.5), we obtain

$$g^{\alpha\beta}(b_{\alpha\delta;\beta} - b_{\alpha\beta;\delta}) = - R_{jk} n^j \frac{\partial x^k}{\partial u^\delta} \tag{2.6}$$

R_{ik} being the Ricci tensor of $R^{m+1} (R_{jk} = g^{il} R_{ijkl})$. Because R^{m+1} is an EINSTEIN space and V^m has the property H_1 = constant, the right-hand side of (2.6) and the second term of the left-hand side vanish; it follows that

$$g^{\alpha\beta} b_{\alpha\delta;\beta} = 0 \,. \tag{2.7}$$

Therefore, and with respect to (2.4), the first term of the right-hand side of (2.3) is equal to zero.

Next, we discuss the second term of the right-hand side of (2.3):

$$g^{\alpha\beta}\, b^\gamma_\alpha\, b_{\gamma\beta}\, n^i\, \xi_i = g^{\alpha\beta}\, g^{\gamma\delta}\, b_{\alpha\delta}\, b_{\gamma\beta}\, n^i\, \xi_i \,. \tag{2.8}$$

Let $k_1, k_2, \ldots, k_m$ be the principal curvatures at a point P of V^m, and let H_2 be the second mean curvature of V^m at the point P which is defined to be the second elementary symmetric function of $k_1, k_2, \ldots, k_m$ divided by the number of terms, that is.

$$\binom{m}{2} H_2 = \sum_{(\alpha,\beta)} k_\alpha k_\beta \quad (\alpha < \beta) \,;$$

since, furthermore the following relation holds

$$\frac{1}{2}\,(g^{\alpha\delta} g^{\gamma\beta} b_{\alpha\delta} b_{\gamma\beta} - g^{\alpha\beta} g^{\gamma\delta} b_{\alpha\delta} b_{\gamma\beta}) = \binom{m}{2} H_2 \,,$$

(2.8) can be written as follows

$$g^{\alpha\beta} b^\gamma_\alpha b_{\gamma\beta} n^i \xi_i = \{m^2 H_1^2 - 2 \binom{m}{2} H_2\}\, n^i \xi_i \,. \tag{2.9}$$

At last, for the third term of the right-hand side of (2.3), we calculate as follows

$$\begin{aligned} g^{\alpha\beta} b^\gamma_\alpha \frac{\partial x^i}{\partial u^\gamma} \frac{\partial x^j}{\partial u^\beta}\, \xi_{i;j} &= g^{\alpha\beta}\, g^{\gamma\delta}\, b_{\alpha\delta} \frac{\partial x^i}{\partial u^\gamma} \frac{\partial x^j}{\partial u^\beta}\, \xi_{i;j} \\ &= \frac{1}{2}\, g^{\alpha\beta}\, g^{\gamma\delta}\, b_{\alpha\delta} \frac{\partial x^i}{\partial u^\gamma} \frac{\partial x^j}{\partial u^\beta} (\xi_{i;j} + \xi_{j;i}) \qquad (2.10) \\ &= \frac{1}{2} H^{\beta\gamma} \underset{\xi}{\mathcal{L}}\, g_{\beta\gamma} \end{aligned}$$

where $H^{\beta\gamma}$ denotes $b_{\alpha\delta} g^{\alpha\beta} g^{\gamma\delta}$.

Accordingly, from (2.7), (2.9), and (2.10), (2.3) becomes

$$\frac{1}{m} \eta^\alpha{}_{;\alpha} = - \{(m H_1^2 - (m-1) H_2)\, n^i \xi_i + \frac{1}{2m} H^{\beta\gamma} \underset{\xi}{\mathcal{L}}\, g_{\beta\gamma}\} \,.$$

And also on making use of

$$\int_{V^m}\cdots\int \eta^{\alpha}{}_{;\alpha} dA = 0$$

by virtue of V^m being closed orientable, we finally reach the integral formula

$$\int_{V^m}\cdots\int \{mH_1^2 - (m-1)H_2\} n^i \xi_i dA + \frac{1}{2m}\int_{V^m}\cdots\int H^{\alpha\beta} \underset{\xi}{\mathcal{L}} g_{\alpha\beta} dA = 0 . \tag{II}$$

If the group G of transformation is conformal, (II) becomes

$$\int_{V^m}\cdots\int \{mH_1^2 - (m-1)H_2\} n^i \xi_i dA + \int_{V^m}\cdots\int \Phi H_1 dA = 0 ; \tag{II$_c$}$$

if G is homothetic (i.e. $\Phi \equiv$ constant $= C$),

$$\int_{V^m}\cdots\int \{mH_1^2 - (m-1)H_2\} n^i \xi_i dA + C\int_{V^m}\cdots\int H_1 dA = 0 ; \tag{II$_h$}$$

and if G is isometric (i.e. $\Phi \equiv 0$),

$$\int_{V^m}\cdots\int \{mH_1^2 - (m-1)H_2\} n^i \xi_i dA = 0 . \tag{II$_i$}$$

§ 3. Closed orientable hypersurfaces with H_1 = constant in an EINSTEIN space.

In this section, we shall prove the following theorem:

Theorem 3.1. *Let R^{m+1} be an EINSTEIN space, V^m a closed orientable hypersurface with H_1 = constant in R^{m+1}; we suppose that there exists a continuous one-parameter group G of conformal transformations of R^{m+1} such that the scalar product $\tilde{p} = n^i \xi_i$ of the normal vector n of V^m and the vector ξ belonging to G does not change the sign (and is not $\equiv 0$) on V^m. Then every point of V^m is umbilic.*

Proof. Multiplying the formula (I)$_c$ in § 1 by H_1(= const.), we obtain

$$\int_{V^m}\cdots\int H_1^2 \tilde{p}\, dA + \int_{V^m}\cdots\int \Phi H_1 dA = 0 ,$$

and subtracting this formula from the formula (II)$_c$ in § 2, we find

$$\int_{V^m}\cdots\int (m-1)(H_1^2 - H_2)\tilde{p}\, dA = 0 . \tag{3.1}$$

From

$$H_1^2 - H_2 = \frac{1}{m^2}\cdot(\Sigma k_\alpha)^2 - \frac{2}{m(m-1)}\sum_{\alpha,\beta} k_\alpha k_\beta = \frac{1}{m^2(m-1)}\Sigma(k_\alpha - k_\beta)^2 \tag{3.2}$$

(with $\alpha \neq \beta$) we see that

$$H_1^2 - H_2 \geqq 0 . \tag{3.3}$$

From (3.1), (3.3) and the fact that $\tilde{p}$ has a fixed sign we conclude that

$$H_1^2 - H_2 = 0$$

and therefore, because of (3.2), that

$$k_1 = k_2 = \dots = k_m$$

at each point of V^m. This means, that each point of V^m is umbilic.

We wish now to show that the LIEBMANN-SÜSS Theorem is a special case of our Theorem 3.1. Because in euclidean E^{m+1} an hypersurface is a sphere if all its points are umbilical we have only to verify that, for a convex V^m in E^{m+1}, there exists a vector field ξ having the properties formulated in Theorem 3.1. We take a point in the interior of V^m as origin of the euclidean coordinates x^i and attach to each point x the vector $\xi(x)$ with the components $\xi^i = x^i$ (i.e. the position vector of x). Then, the transformations (1.1) are homothetic, thus conformal; furthermore, for $x \in V^m$, $\tilde{p}(\mathrm{x})$ is the support function and, because V^m is convex, $\tilde{p}(x) \neq 0$.

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