A Note on the Intrinsic Join of Stiefel Manifolds

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Introduction

In [1] Bott defines a map,

$$\lambda^E : S(O_{n,k} \wedge O_{m,k}) \to O_{n+m,k} ,$$

of the suspension of the reduced join $O_{n,k} \wedge O_{m,k}$ of the real (complex, quaternionic) Stiefel manifolds of the orthonormal k-frames in real (complex, quaternionic) n- and m-space to $O_{n+m,k}$, the Stiefel manifold of orthonormal k-frames in real (complex, quaternionic) $(n+m)$-space. James defines ([2], or [1], p. 256) a map

$$\lambda^J : O_{n,k} * O_{m,k} \to O_{n+m,k} ,$$

of the join of $O_{n,k}$ and $O_{m,k}$ to $O_{n+m,k}$. λ^E and λ^J have the same range; and since $S(O_{n,k} \wedge O_{m,k})$ and $O_{n,k} * O_{m,k}$ are of the same homotopy type, λ^E and λ^J have equivalent domains of definition as well. Bott showed in [1] that λ^E and λ^J are equivalent when $k = 1$ and asked about their relation in general. The aim of this note is to answer his question, by first defining an auxiliary map λ^B (in (1.2)) and showing that it is equivalent to λ^E, and then proving that λ^B is equivalent to λ^J. The precise statement is given in Proposition (4.1) below.

1. Suppose that K is the real, complex or quaternionic field. Let K^{n+m} be the vector space of $(n+m)$-tuples over K with the usual inner product. Let $a_1, \dots, a_{n+m}$ be the usual basis. We shall find it convenient to write $W(i_1, \dots, i_k)$ for the subspace of K^{n+m} generated by the basic vectors $a_{i_1}, \dots, a_{i_k}$. Denote by O_{n+m} the group of orthogonal transformations of K^{n+m}. Thus O_{n+m} is the real, unitary or symplectic group when K is the real, complex or quaternionic field. Let $i : O_n \to O_{n+m}$ be the imbedding induced by the map which takes K^n onto $W(1, \dots, n)$ and $i' : O_m \to O_{n+m}$, the imbedding induced by the map which takes K^m onto $W(n-k+1, \dots, n+m-k)$. Define χ_t to be the orthogonal transformation which leaves $W(1, \dots, n-k) + W(n+1, \dots, n+m-k)$ pointwise fixed and is a rotation through an angle of $\frac{\pi}{2}t$ in each of the planes $W(n-k+j, n+m-k+j)$ for $j = 1, \dots, k$. Thus

$$\chi_t \cdot a_{n-k+j} = \cos\frac{\pi}{2}t \; a_{n-k+j} + \sin\frac{\pi}{2}t \; a_{n+m-k+j}$$

$$\chi_t \cdot a_{n+m-k+j} = -\sin\frac{\pi}{2}t\ a_{n-k+j} + \cos\frac{\pi}{2}t\ a_{n+m-k+j}$$

for $j = 1, \ldots, k$.

Suppose g^o is the frame $(a_{n+m-k+1}, \ldots, a_{n+m})$. Then $O_{n+m,k}$, the STIEFEL manifold of k-frames in K^{n+m}, can be identified with O_{n+m}/O_{n+m-k} by making the coset xO_{n+m-k} correspond to the frame xg^o. We wish to define a map of $O_m/O_{m-k} * O_n/O_{n-k}$, the join of O_m/O_{m-k} and O_n/O_{n-k} into $O_{n+m,k}$. So let us recall the necessary notions.

Definition (1.1). Let A and B be two countable CW-complexes, and consider the disjoint union $A \cup A \times B \times I \cup B$, where I is the unit interval $[0, 1]$. Identify $(a, B, 0)$ with a in A and $(A, b, 1)$ with b in B. The resulting space is the *join* $A * B$. Following JAMES we denote the image of (a, b, t) in $A * B$ by the same symbol. If a_0 and b_0 are the basepoints of A and B, then we take $(a_0, b_0, 1)$ to be the basepoint in $A * B$. There is an alternative way of looking at the join. By $A \wedge B$ one understands the space obtained from $A \times B$ by collapsing the sum $A \vee B = A \times b_0 \cup a_0 \times B$ to a point. $A \wedge B$ is called the *reduced join*. The suspension of $A \wedge B$ is obtained from $(A \wedge B) \times I$ by collapsing the subset $(A \wedge B) \times 0 \cup (A \wedge B) \times 1 \cup (a_0 \wedge b_0) \times I$ to a point. The natural map $q: A * B \to S(A \wedge B)$ is a homotopy equivalence.

Consider the map

$$\lambda: O_m \times O_n \times I \to O_{n+m,k}$$

such that $\lambda(y, x, t) = \chi_t y' \chi_t^{-1} x\, \chi_t y'^{-1} \chi_t^{-1} x^{-1} \cdot g^o$, where x stands for ix and y' stands for $i'y$. Notice that $\lambda(y, x, t)$ does not vary when y is changed by an element in O_{m-k}, the subgroup of O_m which leaves $W(n-k+1, \ldots, n)$ pointwise fixed. Also $\lambda(y, x, t)$ does not vary when x is changed by an element in O_{n-k}, the subgroup of O_n which leaves $W(n-k+1, \ldots, n)$ pointwise fixed. Moreover, $\lambda(y, x, 0)$ is independent of x and $\lambda(y, x, 1)$ is independent of y, since x and $\chi_1 y' \chi_1^{-1}$ commute. Hence λ induces a map

$$\lambda^B: O_m/O_{m-k} * O_n/O_{n-k} \to O_{n+m,k}. \tag{1.2}$$

Take e_m, the identity in O_m, to be the basepoint, and its image, $\tilde{e}_m$, to be the basepoint in O_m/O_{m-k}; hence $(\tilde{e}_m, \tilde{e}_n, 1)$ is the basepoint in $O_m/O_{m-k} * O_n/O_{n-k}$, and λ^B takes $(\tilde{e}_m, \tilde{e}_n, 1)$ to $\tilde{e}_{n+m}$. We claim that λ^B is equivalent to the BOTT map λ^E of [1]. Our argument is as follows: it is easy to see that λ also induces a map $\tilde{\lambda}^B: S(O_m/O_{m-k} \wedge O_n/O_{n-k}) \to O_{n+m,k}$, such that $\tilde{\lambda}^B \circ q = \lambda^B$, where $q: O_m/O_{m-k} * O_n/O_{n-k} \to S(O_m/O_{m-k} \wedge O_n/O_{n-k})$ is the natural equivalence. Consider now the map

$$r: O_n \times O_m \times I \to O_m \times O_n \times I$$

such that $r(x, y, t) = (y, x, t)$. Then $\lambda \circ r$ induces the map

$$\lambda^E : S(O_n/O_{n-k} \wedge O_m/O_{m-k}) \to O_{n+m,k},$$

which BOTT introduced in [1] (p. 252). Observe that $\lambda^E = \tilde{\lambda}^B \circ r$ where $\tilde{r} : S(O_n/O_{n-k} \wedge O_m/O_{m-k}) \to S(O_m/O_{m-k} \wedge O_n/O_{n-k})$ is the homeomorphism induced by r. Hence

$$\lambda^B = \lambda^E \circ \tilde{r}^{-1} \circ q, \text{ and } q \text{ and } \tilde{r}^{-1} \text{ are} \tag{1.3}$$

homotopy equivalences.

2. Consider next the homeomorphism

$$T : O_n/O_{n-k} * O_m/O_{m-k} \to O_m/O_{m-k} * O_n/O_{n-k} \tag{2.1}$$

which takes (x, y, t) to $(y, x, 1-t)$. (Observe that T does not preserve basepoints; but, since $(e_m, e_n, 1)$ and $(e_m, e_n, 0)$ are connected by a contractible segment, T can (in most cases) be regarded as preserving basepoints. Hence

$$\lambda^B \circ T : O_n/O_{n-k} * O_m/O_{m-k} \to O_{n+m,k}$$

is induced by the map of $O_n \times O_m \times I \to O_{n+m,k}$, which takes (x, y, t) to

$$\chi_{1-t} y' \chi_{1-t}^{-1} x \chi_{1-t} y'^{-1} \chi_{1-t}^{-1} x^{-1} \cdot g^o = \chi_t^{-1} y'' \chi_t x \chi_t^{-1} y''^{-1} \chi_t x^{-1} \cdot g^o$$

where $y'' = \chi_1 y' \chi_1^{-1}$. Notice that $\lambda^B \circ T$ preserves basepoints although T does not.

Let us consider now the map

$$F : (O_n \times O_m \times I) \times I \to O_{n+m,k}$$

such that $F((x, y, t), s) = \chi_{(1-s)t}^{-1} y'' \chi_{(1-s)t} x \chi_t^{-1} y''^{-1} \chi_{s+(1-s)t} x^{-1} g^o$. Observe that $F((x, y, t), s)$ does not change when x and y are varied by an element in O_{n-k} and O_{m-k} respectively. Moreover, $F((x, y, 0), s)$ is independent of y since x and y'' commute, and $F((x, y, 1), s)$ is independent of x since $x \chi_1^{-1} y''^{-1} \chi_1 x^{-1} \cdot g^o = g^o$. Thus F induces a homotopy

$$\tilde{F}_s : O_n/O_{n-k} * O_m/O_{m-k} \to O_{n+m,k} \tag{2.2}$$

such that $\tilde{F}_0 = \lambda^B \circ T$, and the basepoint is stationary during the homotopy, i.e., $F(\tilde{e}_n, \tilde{e}_m, 1) = g^o$, for $0 \leq s \leq 1$.

Define $\mu = \tilde{F}_1$. Then

$$\mu : O_n/O_{n-k} * O_m/O_{m-k} \to O_{n+m,k} \tag{2.3}$$

is induced by the map of $O_n \times O_m \times I \to O_{n+m,k}$ which takes (x, y, t) to $y'' x \chi_t^{-1}(y''^{-1} \chi_1 g^o) = y'' x \chi_t^{-1}(\chi_1 g^o)$, since

$$y''^{-1} \chi_1 = \chi_1 y'^{-1} \chi_1^{-1} \cdot \chi_1$$
$$= \chi_1 y'^{-1}, \text{ and } y'^{-1} g^o = g^o. \text{ Summing up we have}$$

Proposition (2. 2). The maps $\mu, \lambda^B \circ T : O_n/O_{n-k} * O_m/O_{m-k} \to O_{n+m,k}$ are homotopic. Here λ^B is the map defined in (1.2) and T and μ are the maps defined above in (2.1) and (2.3).

3. Let $O_{n,k}$ and $O_{m,k}$ be the STIEFEL manifolds of orthonormal k-frames in $W(1, \ldots, n) = K^n$ and $W(n+1, \ldots, n+m) = K^m$, respectively. Suppose that f^o is the k-frame which consists of the last k basic vectors of the image of K^n, i.e., $f^o = (a_{n-k+1}, \ldots, a_n)$. Then $O_{n,k}$ can be identified with O_n/O_{n-k} by making the coset xO_{n-k} correspond to the k-frame xf^o. Similarly $O_{m,k}$ can be identified with O_m/O_{m-k} by making the coset yO_{m-k} correspond to the k-frame $y'' g^o$, where $y'' = \chi_1 y' \chi_1^{-1} = \chi_1 i' y \chi_1^{-1}$. We claim

Proposition (3. 1.) If $f = (f_1, \ldots, f_k) \in O_{n,k}$ and $g = (g_1, \ldots, g_k) \in O_{m,k}$, and if $x \in O_n$ and $y \in O_m$ be such that $xf^o = f$ and $y''g^o = g$, then

$$\mu(\tilde{x}, \tilde{y}, t) = \left(-\cos\frac{\pi}{2}t\right)f + \left(\sin\frac{\pi}{2}t\right)g,$$

where $\tilde{x}$ and $\tilde{y}$ are the images of x and y in O_n/O_{n-k} and O_m/O_{m-k}.

Proof. By the definition of μ in (2.3)

$$\mu(\tilde{x}, \tilde{y}, t) = y'' x \chi_t^{-1}(\chi_1 g^o) = -y'' x \chi_t^{-1} f^o,$$

since $\chi_1 g^o = -f^o = (-a_{n-k+1}, \ldots, -a_n)$. Notice that $x \cdot g^o = g^o$ and $y'' \cdot f^o = f^o$. Hence

$$\mu(\tilde{x}, \tilde{y}, t) = \left(-\cos\frac{\pi}{2}t\right)f + \left(\sin\frac{\pi}{2}t\right)g.$$

4. Suppose that ϱ_0 is the element of O_n which has -1 along the diagonal and zeroes everywhere else. Then ϱ_0 induces a homeomorphism of $O_n/O_{n-k} * O_m/O_{m-k}$ by sending $(\tilde{x}, \tilde{y}, t)$ to $(\varrho_0\tilde{x}, \tilde{y}, t)$. Denote this homeomorphism (which is related to what JAMES calls in [2] a "row operation") by the same letter ϱ_0. Proposition (3.1) implies that the JAMES map λ^J (the map h' of [2], p. 513) is $\mu\varrho_0$. Now we can conclude

Proposition (4. 1). If $\lambda^B : O_m/O_{m-k} * O_n/O_{n-k} \to O_{n+m,k}$ is the map of (1.2) and

$$\lambda^J : O_n/O_{n-k} * O_m/O_{m-k} \to O_{n+m,k}$$

is the JAMES map ([2], p. 513), then $\lambda^B \circ T$ is homotopic to $\lambda^J \varrho_0$.

Since λ^B and λ^E are quivalent (see (1.3)), Proposition (4.1) implies that λ^E and λ^J are equivalent.

Remark. If $k = 1$, and if d is the dimension of K over the real field, then $O_{n,1}$, $O_{m,1}$ and $O_{n+m,1}$ become the unit spheres S^{dn-1}, S^{dm-1}, and $S^{d(n+m)-1}$

in K^n, K^m, and K^{n+m}, respectively. Moreover, λ^J reduces to the usual join of sphere

$$h' : S^p * S^q \to S^{p+q+1} ,$$

where h' takes (x, y, t) to $\left(x \cos \frac{\pi}{2} t, y \sin \frac{\pi}{2} t\right)$ ([2], p. 512). If h' is used to orient $S^{dm-1} * S^{dn-1}$ and $S^{dn-1} * S^{dm-1}$, then T becomes a map of degree $(-1)^{(dn)(dm)}$ ([2], p. 512) and ϱ_0 is a map of degree $(-1)^{(dn)}$. Hence when K is the complex or quaternionic field, λ^B and λ^J have the same degree. In the real case they also agree if $n(m+1)$ is even; otherwise they agree up to a sign.

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