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Common Singularities of Commuting Vector Fields on 2-manifolds

by ELON L. LIMA[1]

In this paper, we give the details of our research announcement [5]. We are concerned with vector fields X, Y, ... on a 2-manifold M. These are said to *commute* if the LIE bracket $[X, Y]$ vanishes identically on M. The final result is that any set of pairwise commuting vector fields on a compact 2-manifold with non-zero EULER characteristic has a common singularity.

The proof here presented is, in principle, the same that was sketched in [5], except for style and the correction of a minor mistake that was pointed out to us by H. EDWARDS. The organization of the proof is such that the results of [4] are also included, so that this paper is entirely self-contained. We also single out, in § 2, some lemmas that seem to have independent interest.

A finite collection of commuting vector fields on a compact manifold M is equivalent to a differentiable action of an abelian LIE group on M. The next best thing to an abelian LIE group is a solvable one. The group of affine transformations of the real line is the simplest possible solvable LIE group. In an appendix we show how this group can act without fixed points on a 2-disc, hence on a 2-sphere. It is not difficult to get fixed point free actions of the affine group of the line on a compact cylinder and on a torus. We do not know, however whether or not it may act without fixed points on all compact 2-manifolds. It goes without saying that it would be very interesting to have theorems on common fixed points of vector fields for manifolds of higher dimensions.

I am grateful to S. SMALE for his manifold interest in this work.

1. Preliminaries

All manifolds in this paper are *connected* and may have boundary. The boundary of a compact 2-manifold M is a finite union of disjoint closed curves, which we call the *boundary circles* of M. Throughout this paper we use freely folk theorems on the topology of 2-manifolds. We refer to [1] for explicit proofs of these results.

An *action* of a topological group G on a space M is a *continuous* map $\varphi: G \times M \to M$ such that, for all $g, h \in G$ and $x \in M$, $\varphi(gh, x) = \varphi(g, \varphi(h, x))$ and $\varphi(e, x) = x$, where $e \in G$ is the neutral element. When G is a LIE group

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and M is a differentiable manifold one has also the notion of a differentiable action. A *flow* is an action $\xi : R \times M \to M$ of the additive group of the reals. To give a differentiable flow on a compact manifold M is equivalent to give a differentiable vector field on M (with the additional condition that, if M has a boundary, the field must be tangent to it). We shall tacitly assume this condition, whenever we refer to vector fields.

Let X be a C^1 vector field on a compact manifold M and let $\xi : R \times M \to M$ be the corresponding flow. Given $x \in M$, one has $X(x) = 0$ (x is a singularity of X) if, and only if, $\xi(s, x) = x$ for all $s \in R$, that is, x is a fixed point of the flow ξ. Let Y be another vector field on M, generating a flow η. The condition that the LIE bracket $[X, Y]$ be identically zero means that $\xi(s, \eta(t, x)) = \eta(t, \xi(s, x))$ for all $s, t \in R$ and all $x \in M$. Then we say that the flows ξ, η *commute* and this is why X and Y are said to commute when $[X, Y] \equiv 0$. The pair X, Y generates, then, an action $\varphi : R^2 \times M \to M$ of the additive group of the plane on M, defined by $\varphi(r, x) = \xi(s, \eta(t, x)) = \eta(t, \xi(s, x))$ for $x \in M$ and $r = (s, t) \in R^2$. A point $x \in M$ is fixed under φ if, and only if, it is a common singularity of the commuting vector fields X and Y, that is, $X(x) = Y(x) = 0$. Similarly, any finite number of commuting vector fields $X_1, \ldots, X_n$ on a compact manifold M generate an action $\varphi : R^n \times M \to M$.

Our main result is:

Theorem A. *Every (continuous) action of the additive group R^n on a compact 2-manifold M, with $\chi(M) \neq 0$, has a fixed point.*

Above, $\chi(M)$ denotes the EULER characteristic of M. As a consequence:

Theorem B. *Let $X_1, \ldots, X_n$ be pairwise commuting vector fields of class C^1 on a compact 2-manifold M, with $\chi(M) \neq 0$. There exists a point $x \in M$ such that $X_1(x) = \ldots = X_n(x) = 0$.*

It was remarked to me by R. ELLIS that the above result may be extended to an arbitrary collection $\{X_\alpha\}$ of commuting vector fields on a compact 2-manifold M, with $\chi(M) \neq 0$. In fact, for every finite non-empty subset $A = \{\alpha_1, \ldots, \alpha_n\}$ of the indices α, let $F(A)$ be the set of common singularities of the vector fields $X_{\alpha_1}, \ldots, X_{\alpha_n}$. By Theorem B, each $F(A)$ is a closed non-empty subset of M. Clearly $F(A_1) \cap \ldots \cap F(A_k) = F(A_1 \cup \ldots \cup A_k) \neq \varnothing$, so the family $\{F(A)\}$ has the finite intersection property. Since M is compact, it follows that $F = \bigcap_A F(A)$ is non-empty. But each point $x \in F$ is a common singularity of all the vector fields X_α. Evidently, the same method shows the existence of a fixed point for any action $\varphi : G \times M \to M$ of a topological abelian group G which is generated by subgroups isomorphic to EUCLIDean spaces R^n.

We conclude this introduction with a few definitions.

Given an action $\varphi : G \times M \to M$ of the topological group G on the space M, the *orbit* of a point $x \in M$ is the set $\varphi(G \times x) = \{\varphi(g, x) \in M ; g \in G\}$.

The *isotropy group* of a point $x \in M$ is the set $G_x = \{g \in G; \varphi(g, x) = x\}$. It is clearly a closed subgroup of G and the map $g \to \varphi(g, x)$ induces, by passage to the quotient, a 1–1 continuous map of the homogeneous space G/G_x onto the orbit of x. When φ is a flow, the isotropy group of a point may be $\{0\}$, a discrete group $\{0, \pm t_0, \pm 2t_0, \ldots; t_0 > 0\}$ or the whole real line. In the first case, x has a non-compact orbit; in the second case, the orbit of x is a simple closed curve, and we say it is a *periodic orbit*, of *period* t_0, and in the third case x is a fixed point. A set $X \subset M$ is said to be *invariant* under φ if $\varphi(g, x) \in X$ for every $g \in G$ and every $x \in X$. A *minimal set* under φ is an invariant, non-empty, closed subset $X \subset M$ which contains no proper subset with these 3 properties. Les $\xi : R \times M \to M$ be a flow and x a point of M. The *ω-limit set* of x is the set of all points $y \in M$ which can be written as $y = \lim\limits_n \xi(t_n, x)$ with $t_n \to +\infty$. The ω-limit set of every point is a closed, invariant, subset of M and, if M is compact, it is also non-empty (and connected). The orbit of x under the flow ξ is said to be *recurrent* if it is not compact and is contained in the ω-limit set of x. Any orbit in a minimal set of a flow is either the whole set or is recurrent. When the group G is abelian, then every point in the same orbit has the same isotropy group. Also, if μ is a minimal set relative to the action of an abelian group G, every point $x \in \mu$ has the same isotropy group.

Finally, we use the terminology 'φ-invariant', 'ξ-orbit', 'φ-minimal', etc., as obvious abbreviations, whenever there are more than one action in the argument.

2. Some lemmas

In this section, we gather together a few general facts, for use in the proof of our main theorem. Some of them are well known, some are new, but all have a certain independent interest.

Lemma 1. *Let $\varphi : G \times M \to M$ be a continuous action of a simply-connected topological group G on the space M. Let $p : (M^*, x_0^*) \to (M, x_0)$ be any covering space of M. There exists a unique continuous action $\varphi^* : G \times M^* \to M^*$ that covers φ, in the sense that the diagram below is commutative:*

$$\begin{array}{ccc} G \times M^* & \xrightarrow{\varphi^*} & M^* \\ \downarrow{\scriptstyle id \times p} & & \downarrow{\scriptstyle p} \\ G \times M & \xrightarrow{\varphi} & M . \end{array}$$

Proof: Let the subscript $\#$ denote the induced homomorphism on the fundamental group. Write $f = \varphi_0 (id \times p)$. The simple connectivity of G implies that the image of $f_{\#} : \pi_1(G \times M^*) \to \pi_1(M)$ equals the image of $p_{\#} : \pi_1(M^*) \to \pi_1(M)$. So, by covering space theory, there exists a unique map $\varphi^* : G \times M^* \to M^*$, making the above diagram commutative and such that $\varphi^*(e, x_0^*) = x_0^*$, $e =$ neutral element of G. Now we show that φ^* is an action. First $\varphi^*(e, x^*) = x^*$ for all $x^* \in M^*$, because both maps $\alpha : x^* \to \to \varphi^*(e, x^*)$ and $\beta : x^* \to x^*$ cover the identity map of M and $\alpha(x_0^*) = x_0^*$. Second, $\varphi^*(g, \varphi^*(h, x^*)) = \varphi^*(gh, x^*)$ for all $g, h \in G$ and $x^* \in M^*$, because both maps $\lambda, \mu : G \times G \times M^* \to M^*$, defined by $\lambda(g, h, x^*) = \varphi^*(g, \varphi^*(h, x^*))$ and $\mu(g, h, x^*) = \varphi^*(gh, x^*)$ cover the map $\nu : G \times G \times M \to M$, given by $\nu(g, h, x) = \varphi(g, \varphi(h, x)) = \varphi(gh, x)$, and $\lambda(e, e, x_0^*) = \mu(e, e, x_0^*) = x_0^*$.

Remark: Given $x^* \in M^*$, and $x = p(x^*)$, then x is a fixed point of φ if, and only if, x^* is a fixed point of φ^*.

The consequence of the preceding lemma is that, for our purposes, all manifolds may be taken to be orientable. In fact, if $\varphi : G \times M \to M$ is an action on a non-orientable manifold M, let $p : M^* \to M$ be its orientable 'double covering.' Lift φ to an action of G on M^*. (In our case, the group G is a vector space, so it is always simply-connected.) M^* is compact if M is, and $\chi(M^*) = = 2 \cdot \chi(M)$, so $\chi(M^*) \neq 0$ if $\chi(M) \neq 0$. Of course, the reduction to M orientable is not really essential but simplifies the consideration of cases. This orientability shall be assumed in the proof of the main theorem, but not in its statement, nor in any of the following lemmas.

Lemma 2. *Let $\varphi : G \times M \to M$ be a continous action of a topological group G on a space M. Every compact invariant subset $X \subset M$ contains a minimal set.*

Proof: ZORN.

Let $\xi : R \times M \to M$ be a flow on a 2-manifold M, and let $y \in M$. A *local cross-section* of ξ *at* y is a subset S of M, homeomorphic to a compact interval $[-a, +a]$, containing y, and such that for some $\epsilon > 0$, the map $[-\epsilon, +\epsilon] \times S \to M$, given by $(t, x) \to \xi(t, x)$ is a homeomorphism onto the closure of an open set containing y. The image of this homeomorphism is called a *rectangular neighborhood* of y (relative to the flow ξ). The point y is an end point of S if, and only if, y is in the boundary of the manifold M.

Lemma 3. *Let $\xi : R \times M \to M$ be a flow on a 2-manifold M and let $y \in M$ be a point whose isotropy group is discrete. There exists a local cross-section of ξ at y (so y has a rectangular neighborhood).*

Proof: For a differentiable flow ξ, the cross-section S may be taken as any small transversal segment to the vector field of ξ (see [2], page 392). For an arbitrary flow, the existence of S was established by H. Whitney (see [8], pages 270 and 260).

Lemma 4. *Every flow* $\xi : R \times M \to M$ *on a polyhedron* M, *with* Euler *characteristic* $\chi(M) \neq 0$, *has a fixed point.*

Proof: This is a consequence of the Lefschetz fixed point theorem, according to which every continuous map $f : M \to M$, homotopic to the identity, has a fixed point. For $n = 1, 2, 3, \ldots,$ let $f_n : M \to M$ be defined by $f_n(x) = \xi(1/2^n, x)$. The set F_n of all fixed points of f_n is compact, non-empty, and $F_1 \supset F_2 \supset \ldots$. Then $F = \cap F_n \neq \varnothing$. Clearly F is the set of fixed points of the flow ξ.

We say that a minimal set of a flow is *non-trivial* is it is not (homeomorphic to) a point, a circle or a torus. Denjoy [3] gave an example of a flow of class C^1 on the torus, with a non-trivial minimal set. A modification of this example gives non-trivial minimal sets on any 2-manifold of higher genus [6]. According to a theorem of A. Schwartz [7] and A. Denjoy [3], a flow of class C^2 on a 2-manifold does not admit non-trivial minimal sets. For flows that are merely continuous the best result about non-trivial minimal sets is the lemma below, which was communicated to me by M. Peixoto. It is a generalization of the classical Poincaré-Bendixson theorem:

Lemma 5. *A flow* $\xi : R \times M \to M$ *on a manifold* M *of finite genus* g *has at most* $2g - 1$ *distinct non-trivial minimal sets.*

Proof: Observe that 'distinct' and 'disjoint two by two' are synonimous for minimal sets. We may assume that M is orientable because its double covering M^* would have the same genus, and ξ^* on M^* would admit at least as many non-trivial minimal sets as ξ. The proof goes by induction on the genus g. A preliminary remark: the curves in the boundary of M are submanifolds (with the induced topology) and are orbits of ξ, except when ξ has a fixed point in one of them, so every non-trivial minimal set μ lies in the interior of M. Now consider $g = 0$. Then we may assume $M \subset R^2$, and Lemma 5 is the classical Poincaré-Bendixson theorem: assuming $\mu \subset M$ to be a non-trivial minimal set, we choose $x \in \mu$

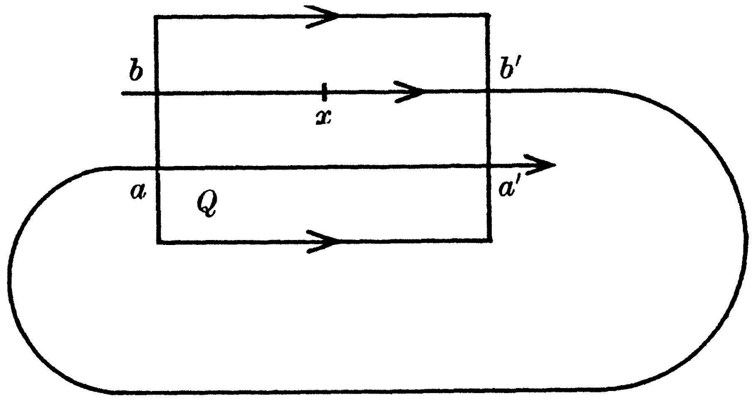

Figure 1.

and take a rectangular neighborhood Q around x. The orbit of x must return to this neighborhood and then it gets trapped in the planar region bounded by the JORDAN curve $a\, a'\, b'\, a$. The ω-limit set of x would be a proper invariant closed subset of μ, a contradiction.

Assume now that $g > 0$ and that the lemma has been proved for all surfaces of genus $< g$. Given a flow $\xi : R \times M \to M$ on a surface of genus g, suppose, by contradiction, that $\mu_1, \ldots, \mu_{2g}$ are $2g$ distinct non-trivial minimal sets of ξ in M. Let Q be a rectangular neighborhood of a point $x \epsilon \mu_{2g}$ (refer to Figure 1). Choose Q so small that it does not intersect the remaining minimal sets $\mu_i, i \leqslant 2g-1$, and $Q \frown \partial M = \varnothing$.

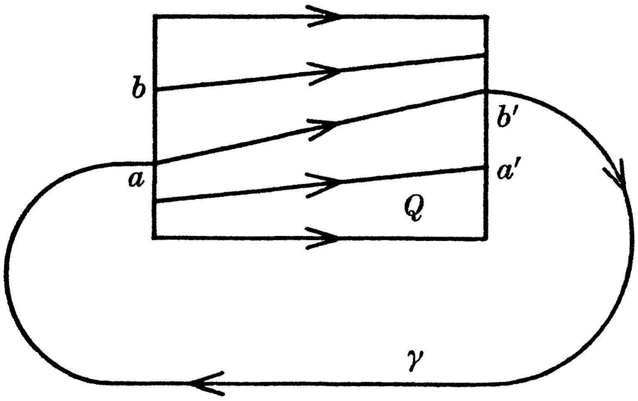

Figure 2.

Consider the following homeomorphism $h : Q \to Q$. It is the identity on all sides of Q but the vertical right side, where h is linear and $h(a') = b'$. In the interior of Q, h is defined by mapping linearly each horizontal segment yy' onto the segment $yh(y')$.

We define a new flow $\xi' : R \times M \to M$ by requiring that, outside Q, the orbits of ξ' are the same as those of ξ, and traversed in the same way. Inside Q, the orbits of ξ, are the images by h of the ξ-orbits. (It would be rather tedious but not hard to describe ξ' formally.) Under the new flow ξ', the point a has a closed orbit γ. Observe also that $\mu_1, \ldots, \mu_{2g-1}$ are still minimal sets of ξ'.

The curve γ cannot separate M : if this happened, the closed curve $a\, a'\, b'\, a$, of Fig. 1, would also separate M and, arguing with ξ, this would lead to a contradiction, just as in the case of genus zero. Therefore, by cutting M along γ, we obtain one manifold of genus $g - 1$, on which the flow ξ' is defined and admits at least the non-trivial minimal sets $\mu_1, \ldots, \mu_{2g-1}$, a contradiction, since $2g - 1 > 2(g - 1) - 1$. This concludes the proof of Lemma 5.

Remark: It was actually proved above that, in a manifold of genus g, there are at most $2g - 1$ recurrent orbits with pairwise disjoint closures. (In particular, there are no recurrent orbits in a manifold of genus zero.) Notice that this result gives information also in the differentiable case, since recurrent orbits exist even for C^∞ actions.

Lemma 6. *Let $\xi : R \times M \to M$ be a flow on a 2-manifold M. Suppose that $x \epsilon M$ is such that $x = \lim x_n$ where each x_n has a closed orbit μ_n, of period t_n. Then:*

(a) *If the genus of M is finite, the orbit of x cannot be recurrent:*

(b) *If x has a closed orbit μ, of period t_0, one must have* $\lim t_n = a \cdot t_0$ ($a = 1$ *or* 2, *according to whether μ is two-sided or one-sided in M), and* $\mu = \{y \in M; y = \lim y_n, y_n \in \mu_n, n = 1, 2, 3 \ldots\}$.

Proof: To prove (a) let Q be a rectangular neighborhood of x. Clearly no boundary point of M may have a recurrent orbit, so we may assume that Q is disjoint from the boundary of M. Consider first the special case in which, for infinitely many values of n, there exists an open set $A_n \subset M$ whose point set boundary ∂A_n satisfies the condition $\partial A_n \cap Q \subset \mu_n \cap Q$. (For instance, if μ_n separates M this situation occurs. This is actually the only case we use in the proof of the main theorem.)

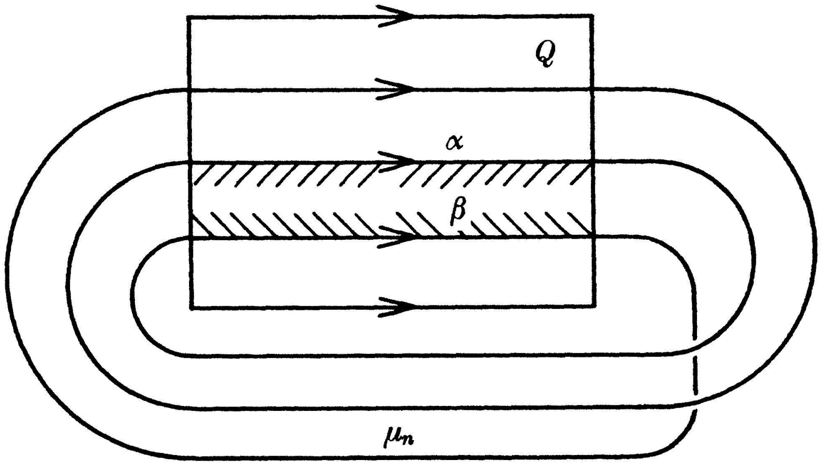

Figure 3.

Suppose then that the above condition holds and that, by contradiction, the orbit of x is recurrent. There exists a value t_0 sufficiently large such that the orbit of x crosses Q at least 3 times for $0 \leqslant t \leqslant t_0$. The orbits of all points sufficiently near x will have exactly the same property. So, we may choose μ_n and the corresponding open set A_n, with $\partial A_n \cap Q \subset \mu_n \cap Q$, such that μ_n crosses Q at least 3 times. The segments of $\mu_n \cap Q$ are precisely the intersections of Q with ∂A_n. Since there are at least 3 segments as these, there must be two consecutive ones such that Q enters A_n at one of them and leaves A_n at the other. Such thing, however, cannot happen because those segments are equally oriented in Q and μ_n, on the other hand, has its own orientation (see Figure 4).

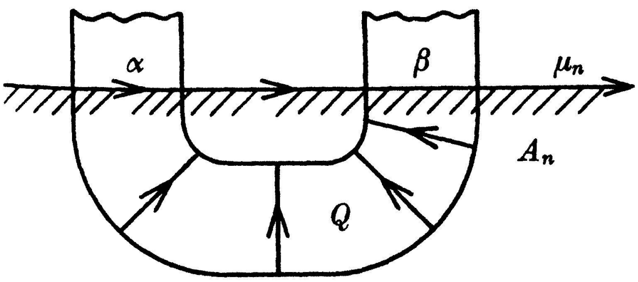

Figure 4.

The general case of (a) reduces to the first one as follows: suppose that only finitely many of the μ_n satisfy our special condition with respect to Q. Neglecting them, we may assume that no μ_n satisfies that condition. Then μ_1 does not separate M so, cutting M along μ_1, we obtain a surface M_1, of smaller genus than M, and the result is proved by induction on the genus g. (For $g = 0$, it is true because there are no recurrent orbits then.)

To prove (b), let $\varepsilon > 0$ be given. We try to find n_0 such that $|t_n - t_0| < \varepsilon$ for all $n > n_0$. Take a rectangular neighborhood Q of x, such that $\mu \cap Q$ is a segment and it takes a time $< \varepsilon$ for any point to go from one end of Q to the other by the flow ξ. Take also a tubular neighborhood V of the closed orbit μ, not wider than Q. Choose n_0 so large that, for $n > n_0$, x_n is close enough to x, so that $\xi(t, x_n) \in V$ for $0 \leqslant t \leqslant 2t_0$, that the orbit of x_n hits Q for the first time at $t = t'$, where $|t' - t_0| < \epsilon/2$, and for the second time at $t = t''$, with $|t'' - t_0| < \varepsilon/2$ also. Consider initially the case where μ is 2-sided, so V is a cylinder. Clearly $\xi(t', x_n)$ must lie on the same horizontal segment of Q as x_n, since one of the semi-orbits of x_n is trapped to remain forever in V and if the contrary were true, μ_n would never close. Therefore, it takes no more time than $\varepsilon/2$ for $\xi(t', x_n)$ to reach x_n for the first time, so $t' < t_n < t' + \varepsilon/2$ and $|t_n - t_0| < \varepsilon$. Now suppose that μ is one-sided, so V is a MOEBIUS band whose equator is μ. Then $\xi(t, x_n)$ returns to Q for the first time at a point on the other side of μ (relatively to x_n in Q) but, on the second return, it must close, under penalty of remaining forever open. Therefore, $\lim t_n = 2t_0$ in this case. To conclude the proof we first observe that the inclusion $\mu \subset \{y \in M;\ y = \lim y_n,\ y_n \in \mu_n,\ n = 1, 2, \ldots\}$ is obvious. Conversely, let $y = \lim y_n$, $y_n \in \mu_n$. Then $y_n = \xi(s_n, x_n)$, where $0 \leqslant s_n \leqslant t_n$, so the sequence $\{s_n\}$ is bounded. Passing to a convergent subsequence, we may write $s_n \to s$. Then $y = \lim \xi(s_n, x_n) = \xi(s, x) \in \mu$.

For the next lemma, we consider the following situation: M is a 2-manifold and Γ is a boundary circle of M; $\{\mu_\alpha\}$ is a family of disjoint simple closed curves in M, none of them touching the boundary of M. We assume that for each α there exists a compact cylinder $\bar{C}_\alpha \subset M$, whose boundary circles are Γ and μ_α. For convenience, we let $\Gamma \subset C_\alpha$, but $C_\alpha \cap \mu_\alpha = \varnothing$, so that each C_α is open in M. The point set boundary of C_α in M is μ_α and we write $\partial C_\alpha = \mu_\alpha$ to denote this. Finally, $\bar{C}_\alpha = C_\alpha \cup \mu_\alpha$ is the closure of C_α in M.

Lemma 7. With the above notations:

(a) *The family $\{C_\alpha\}$ is linearly ordered by inclusion, unless M is a disc with boundary Γ;*

(b) *The union of any given linearly ordered subfamily of $\{C_\alpha\}$ is a cylinder C, open in M, containing Γ, that may be written as $C = \bigcup C_n$, $C_1 \subset C_2 \subset \cdots$ where, for each $n = 1, 2, \ldots, C_n = C_\alpha$ for some α;*

(c) *If the point set boundary of C in M is a simple closed curve μ then, either* *the closure $\overline{C}$ is a compact cylinder with boundary circles Γ and μ, or else $\overline{C} = M =$* MOEBIUS *band.*

Proof: (a) Let $C_\alpha \neq C_\beta$ be chosen arbitrarily. Assume that M is not a disc. Denote by $h : S^1 \times [0, 1] \to \overline{C}_\alpha$ a homeomorphism, with $S^1 =$ unit circle and $h(S^1 \times 0) = \Gamma$. Let $x = h(z, t)$ be a point in $\overline{C}_\alpha \cap \overline{C}_\beta$ with $t \in [0, 1]$ greatest possible. Since $C_\alpha \cap C_\beta$ is open, x must belong to μ_α, or to μ_β, but not to both. To fix ideas, say $x \in \mu_\beta$ but $x \notin \mu_\alpha$. Then $x \in \mu_\beta \cap C_\alpha$. The connected set μ_β, then, meets C_α but is disjoint to $\mu_\alpha = \partial C_\alpha$, so $\mu_\beta \subset C_\alpha$. Observe that μ_β cannot bound a disc D in C_α since this would imply $C_\beta \cup D$ to be a disc in M, with boundary circle equal to Γ. The disc $C_\beta \cup D$ would then be open and closed in M, so $M = C_\beta \cup D =$ disc, contrary to the assumption. Therefore $\overline{C}_\alpha - \mu_\beta = X \cup Y$, the disjoint union of 2 connected components, with $\Gamma \subset X$, $\mu_\alpha \subset Y$ and $\partial X = \partial Y = \mu_\beta$. The connected set C_β intersects X along Γ but does not meet $\partial X = \mu_\beta$, so $C_\beta \subset X \subset C_\alpha$. If we had assumed $x \in \mu_\alpha$, the conclusion would be that $C_\alpha \subset C_\beta$. This proves statement (a).

To prove (b), change notation and write the given subfamily still as $\{C_\alpha\}$. By LINDELOF's theorem, there exists a sequence $C_{\alpha_1}, \ldots, C_{\alpha_n}, \ldots$ such that $\bigcup C_{\alpha_n} = \bigcup C_\alpha$. Let $C_n = C_{\alpha_1} \cup \ldots \cup C_{\alpha_n}$. Since the C_α's are linearly ordered by inclusion, each $C_n = C_\alpha$ for some α. Clearly $C_1 \subset C_2 \subset \ldots$ and $\bigcup C_n = \bigcup C_\alpha = C$. If $C = C_n$ for some n, then clearly C is a cylinder, open in M. Otherwise, $\overline{C}_n \subset C_{n+1}$ for infinitely many values of n, so we may as well assume that this happens for every n. Each $\overline{C}_{n+1} - C_n$ is a closed cylinder. Consider the standard cylinder $K = \{(x, y, z) \in R^3;\ x^2 + y^2 = 1, z \geqslant 0\}$. Let $K_n = \{(x, y, z) \in K;\ z < n\}$, so $K = \bigcup K_n, \overline{K}_n \subset K_{n+1}$. For each n, let $k_n : \overline{K}_n - K_{n-1} \to \overline{C}_n - C_{n-1}$ be a homeomorphism such that $k_n(\overline{K}_{n-1} - K_{n-1}) = \overline{C}_{n-1} - C_{n-1}$. For convenience, let $K_0 = C_0 = \varnothing$. Now we define a homeomorphism $h : K \to C$, thus showing that C is a cylinder. We start by letting $h \mid K_1 = k_1$ and proceed by induction. Supposing $h : \overline{K}_n \approx \overline{C}_n$ defined, extend it to $\overline{K}_{n+1} - K_n$ by putting $h(x, y, n + z) = k_{n+1}[k_{n+1}^{-1} h(x, y, n) + (0, 0, z)]$, $0 \leqslant z \leqslant 1$. This definition fits with the previous one and $h: K \approx C$ is constructed.

Finally, we prove (c). Observe that, C being open in M, $\varnothing = C \cap \partial C = C \cap \mu \supset \Gamma \cap \mu$, so μ has no points in common with Γ. Suppose first that there exists a 'collar' $\overline{A} = S^1 \times [0, 1]$ in M, with $\mu = S^1 \times \{0\}$. Write $A = S^1 \times (0, 1)$. A collar always exists when μ touches the boundary M, or when μ is 2-sided in M. In the first case, any collar meets C and, in the second case, there are collars in both sides of μ, but we shall always choose $\overline{A}$ in the side such that $A \cap C \neq \varnothing$. Since A is connected and disjoint from μ, it follows that $A \subset C$. Clearly we may choose $\overline{A}$ so narrow that $A_1 = S^1 \times \times (0, 1] \subset C$, so that $\nu = S^1 \times \{1\}$ is a closed curve in C, disjoint from Γ. There are 2 *a priori* possibilities. The first one is that ν bounds a disc in C. This can't really happen: take the one-point compactification of C. It is a disc $\hat{C} = C \cup \omega$, containing the disc $\hat{A}_1 = A_1 \cup \omega$, which is also the one-point compactification of A_1. These discs have in common the interior point ω, so $\partial \hat{A}_1 = \nu$ cannot bound in $\hat{C} - \omega$. We are left with the second possibility: ν does not bound in C. Then ν and Γ, together, bound a compact cylinder $\overline{B}$, such that $\overline{A} \cap \overline{B} = \nu$. Any sequence of points of C, tending to a point in $\mu = \overline{C} - C$, must eventually get inside A, so $\overline{C} = \overline{A} \cup \overline{B}$ and therefore $\overline{C}$ is a compact cylinder, bordered by μ and Γ. Next, suppose that μ is one-sided in M. Then μ does not touch the boundary of M. Take a tubular neighborhood V of μ, $\overline{V} \cap \partial M = \varnothing$. Then $\overline{V}$ is a Moebius band, with boundary ν. $V - \mu$ is connected, meets C but not $\mu = \partial C$, so $V \subset C$. Then $\overline{C} = C \cup \mu$ is open and closed in M, so $\overline{C} = M$ is a 2-manifold with boundary Γ and, cutting M along μ, one obtains a 2-manifold M' with two boundary circles μ, Γ, such that omitting the circle μ from M' one obtains the cylinder C. Hence M' is a compact cylinder and M is a Moebius band. This finishes the proof of Lemma 7.

§ 4. Proof of Theorem A

We use induction on n. For $n = 1$, the theorem is contained in Lemma 4. Let $n > 1$ and suppose that it has been proved for actions of R^{n-1}. Consider a continuous action $\varphi : R^n \times M \to M$. Our first task is to establish the following two auxiliary results:

Sublemma I. *Let $\mathcal{M}$ be the collection of all φ-minimal sets in M. Either φ has a fixed point or else $\mathcal{M}$ is uncountable, and all but a finite number of its elements are circles (1-dimensional closed orbits of φ).*

Proof: For every hyperplane $Z \subset R^n$ (through the origin), let $\varphi \mid Z$ denote the action of Z on M induced by restriction of φ. By the inductive hypothesis, $\varphi \mid Z$ has at least one fixed point z in M. For each fixed point $z \in M$ of the action $\varphi \mid Z$, let $K(z)$ denote the closure of its φ-orbit $\varphi(R^n \times z)$. All points in $K(z)$ are left fixed by Z, so the orbit of every one of them is the same as its orbit under any line through the origin of R^n, provided that line is not contained in Z. $K(z)$ being closed and φ-invariant, contains at least one

φ-minimal set. Let $\mathcal{M}(Z)$ denote the collection of all φ-minimal subsets of M which are left pointwise fixed by Z. Given any 1-dimensional subspace $l \subset R^n$ not contained in Z, every $\mu \in \mathcal{M}(Z)$ is also a minimal set of the flow $\varphi \mid l$, induced by restriction of the action φ to the line l. We know that each $\mathcal{M}(Z)$ is non-empty. Moreover, if Z, W are distinct hyperplanes (through the origin of R^n) then $\mathcal{M}(Z) \cap \mathcal{M}(W) = \varnothing$ for if these two collections had an element μ in common, each point $x \in \mu$ would be fixed under Z and under W. Since R^n is spanned by Z and W, x would be a fixed point of φ, contrary to the assumption. Therefore $\mathcal{M} = \bigcup \mathcal{M}(Z)$, where Z describes all the homogeneous hyperplanes of R^n, is an uncountable collection. Moreover, $\mathcal{M}$ contains all φ-minimal sets. In fact, given a φ-minimal set μ, the orbit of no point $x \in \mu$ may contain an interior point: otherwise, the point set boundary of the orbit (which is not empty because M is not a torus) would be a proper, closed invariant subset of the minimal set μ. So, the orbit of $x \in \mu$ is 1-dimensional and, consequently, the isotropy group of x contains a hyperplane Z. So does the isotropy group of all other points of μ, hence $\mu \in \mathcal{M}(Z)$. Let $l_1, \ldots, l_n$ denote the axis of R^n and write $\mathcal{M}_i = \bigcup\{\mathcal{M}(Z);\ Z \cap l_i = 0\}$. Every set $\mu \in \mathcal{M}_i$ is minimal under l_i. So, by Lemma 5, the sets $\mu \in \mathcal{M}_i$ are all circles, except a finite number of them. But clearly $\mathcal{M} = \mathcal{M}_1 \cup \ldots \cup \mathcal{M}_n$, because a given hyperplane cannot contain all the axis of R^n. Therefore, all sets in $\mathcal{M}$ are circles, with a finite number of exceptions. (Notice that if the genus of M is zero, there are no exceptions.)

Sublemma II. *Either φ has a fixed point in M or else we may find a closed 1-dimensional orbit of φ, which is disjoint from the boundary circles of M and does not bound a cylinder together with any of them.*

Proof: Start with a boundary circle Γ. Consider the collection $\{C_\alpha\}$ of all cylinders which are open in M, contain Γ as one of their edges and such that the point set boundary (relative to M) is $\partial C_\alpha = \mu_\alpha$, a closed 1-dimensional orbit of φ. If M is not a disc, the collection $\{C_\alpha\}$ is linearly ordered by inclusion (Lemma 7). If M is a disc, we apply the HAUSDORFF maximal principle and, changing notation, let $\{C_\alpha\}$ stand for a maximal linearly ordered subfamily. At any rate, by Lemma 7 again, the union $C = \bigcup C_\alpha$ is an open cylinder, that may be written as $C = \bigcup C_n$, with each $C_n = C_\alpha$ for some α and $C_1 \subset C_2 \subset \ldots$. Moreover, we may conclude that $\overline{C}$ is a compact cylinder with edges Γ and μ, provided we show that its point set boundary is a closed curve $\mu = \partial C$. In order to do that we remark first that $x \in \partial C$ if, and only if, $x = \lim x_n$, $x_n \in \mu_n = \partial C_n$, $n = 1, 2, \ldots$. Clearly ∂C is φ-invariant and closed. Let $\mu \subset \partial C$ be a φ-minimal set. The isotropy group of (every point of) μ contains a certain hyperplane Z. Choose a line l through the origin of R^n, which does not lie in the isotrophy group of any of the μ_n, or of μ either. Let $\xi : R \times M \to M$ be the flow induced by restriction of φ to the line l. The curves μ_n are orbits of ξ and μ is a minimal set of ξ. Now we use Lemma 6. By (a), no point of μ may have a recurrent orbit, so μ is actually a circle. By (b), $\mu = \partial C$. The compact cylinder $\overline{C}$ is such that its point set boundary (relative to M) μ is disjoint from the other boundary curves of M (since M is not a cylinder) and $M - C$ is a compact 2-manifold, homeomorphic with M, with μ replacing Γ as a boundary circle. $M - C$ is invariant under φ, and no closed 1-dimensional orbit of φ in $M - C$ may bound a cylinder together with Γ. Repeat the same construction with all the other boundary circles of M. A manifold $M' \subset M$ is obtained, which is homeomorphic with M, invariant under φ and no closed 1-dimensional φ-orbit in M' may bound a cylinder together with any boundary circle of M. But $\chi(M') \neq 0$ so there is a closed 1-dimensional orbit ν of φ in M', obtained by the Sublemma I. Since $\mathcal{M}$ is infinite, ν may be chosen disjoint from the boundary circles of M', so ν proves Sublemma II.

The proof that $\varphi : R^n \times M \to M$ has a fixed point will be given by a second induction, now on the genus g of the manifold M. To start the induction, we prove this for a manifold of genus 0. Consider, in first place, the case $M = D$ wo prove this for a manifold of genus 0. Consider, in first place, the case $M = D = $ 2-disc. Assuming that $\varphi : R^n \times D \to D$ does not have a fixed point leads us to the conclusion of Sublemma II and this is obviously contradictory when M is a disc. Next, take the case $M = S^2 =$ 2-sphere. Given $\varphi : R^n \times S^2 \to S^2$, choose one of the circles $\mu \subset S^2$ given by Sublemma I. There is a disc $D \subset S^2$, bounded by μ. D is invariant under φ. By the previous case, φ has a fixed point in D, hence in S^2. The case of manifolds of genus zero is completed by a third induction, this time on the number b of its boundary circles. We have established it for $b = 0$ (sphere) and $b = 1$ (disc). It is false for $b = 2$ (cylinder), when $\chi(M) = 0$, anyway, so we start this induction with $b = 3$. Our manifold M is then a sphere with 3 holes or, as we prefer to think of it, a disc with 2 inner holes. Sublemma II provides us with a closed 1-dimensional orbit of φ which, in this case, has to bound a disc in M. Such disc is φ-invariant and thus contains a fixed point. Now suppose that M has $b > 3$ boundary curves and again take a closed 1-dimensional φ-orbit ν given by Sublemma II. Considered as a plan JORDAN curve, ν contains b' boundary circles of M in its interior and b'' in its exterior, with $b = b' + b''$ and $b', b'' \geqslant 2$. Therefore, cutting M along ν, wo obtain 2 compact 2-manifolds of genus zero, both with less boundary curves than M, both φ-invariant and none of them a cylinder. The induction hypothesis (on b) provides a fixed point. This concludes the case $g = 0$.

We return now to the induction on the genus of M. Let $g > 0$, suppose the assertion proved for manifolds of genus $< g$ and pick a curve ν, provided by Sublemma II. Cut M along g. We either obtain a manifold of lower genus and same EULER characteristic as M or a pair of manifolds, both with lower genus and non-zero EULER characteristics, adding up to that of M. In either case, the theorem is proved.

APPENDIX

The affine group of the line acts without fixed points on a disc

The affine group of the real line is the set G of all maps of the form $x \to ax + b$, $a > 0$, of R onto itself, with the group structure given by composition of maps. Abstractly, G is the set of all pairs (a, b) of real numbers, such that $a > 0$, and the operation in G is defined by $(a, b)_* (c, d) = (ac, ad + b)$. G has a natural topology, which makes it homeomorphic with the open right half plane, so it is a simply-connected 2-dimensional LIE group. Its LIE algebra has a basis $\{e_1, e_2\}$ for which the only non-zero bracket product is $[e_1, e_2] = e_1$. In order to obtain an action of G on a compact manifold M, it suffices then to give a pair of vector fields X, Y on M, such that $[X, Y] = X$.

In order to define such vector fields X, Y on the unit disc

$$D = \{(x, y) \in R^2;\ x^2 + y^2 \leqslant 1\},$$

we start with the fields A, B on the plane R^2, which are given by $A(x, y) = (0, 1)$ and $B(x, y) = (x, y)$. Clearly $[A, B] = A$. We shall then define a diffeomorphism $h : R^2 \to \operatorname{int} D$, of the plane onto the interior of D. This yields two vector fields $X = h_*(A)$ and $Y = h_*(B)$ on $\operatorname{int} D$. Finally, we show that X and Y may be continuously extended to the boundary $S^1 = \partial D$, in such a way that $X = 0$ and Y is the unit vector tangent to S^1. Then $[X, Y] = X$ all over D and nowhere X and Y vanish simultaneously. Therefore, the action of G on D defined by X and Y has no fixed points.

The diffeomorphism $h : R^2 \to \operatorname{int} D$ is given by the formula below, in which we identify each $(x, y) \in R^2$ with the complex number $z = x + iy$:

$$h(z) = \frac{z}{(1 + |z|^2)^{1/2}} \cdot \exp[if(|z|)],$$

where $f : R \to R$ is a differentiable function such that $f(t) = 0$ for $t \leqslant 1$ and $f(t) = \log t$ for $t \geqslant 2$. The geometrical meaning of h is better shown by its expression in polar coordinates:

$$h(\varrho e^{i\theta}) = \varrho(1 + \varrho^2)^{-1/2} \exp[i\varphi + if(\varrho)],\ h(0) = 0.$$

After computation, we arrive at the following expressions for $X = h_*(A)$ and $Y = h_*(B)$ in the interior of the disc D:

$$Y(z) = z(1 - |z|^2) + iz\,,$$

$$X(h(z)) = h(z)\left[\frac{i}{z} + \frac{iy}{|z|^2} - \frac{y}{1+|z|^2}\right], \quad z = x + iy\,.$$

We see then that X and Y may be extended to the closed disc D, in such a way that $X(w) = 0$ and $Y(w) = iw$ for every $w \in S^1$. This is clear for Y. As for X, just notice that if $h(z) \to w_0 \in S^1$, then $z \to \infty$ so, in the expression of X, the first factor is bounded, whereas each summand in the second factor tends to zero.

As a corollary of the preceding construction, we obtain an action of G on the sphere S^2 without fixed points. Just glue two copies of D along the boundary and let G act on each disc as before.

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