On the Automorphism Group of a G-structure[1])

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1. Introduction

Throughout this paper M denotes a paracompact differentiable manifold of dimension n. Let G be a LIE subgroup of $GL(n, R)$. The group of diffeomorphisms of M which leave a G-structure invariant is often a LIE group. We shall give a condition on the LIE algebra $\mathfrak{g}$ of G under which the group of automorphisms of a G-structure is a LIE group. (Cf. Definitions 5 and 6 in Section 3.) The main result is stated as Theorems A and B in Section 5, and examples are given in Sections 8 and 9. To simplify the presentation, 'differentiable' always means 'differentiable of class C^∞'. It is to be remarked that for every specific G-structure however, differentiability of a suitable degree will be sufficient.

H. CARTAN [3] proved in 1935 that the group of all complex analytic transformations of a bounded domain in C^n is a LIE group. S. BOCHNER and D. MONTGOMERY [1] proved in 1946 that the group of all complex analytic transformations of a compact complex manifold is a LIE group. This result was extended in 1963 by W. M. BOOTHBY, S. KOBAYASHI and H. C. WANG [2] to the effect that the automorphism group of an almost complex structure on a compact manifold is a LIE group. By introducing a BERGMAN metric on a bounded domain in C^n, H. CARTAN's result is shown to be a special case of a theorem proved by S. B. MYERS and N. STEENROD [9] in 1939. Their theorem states that the group of isometries of a RIEMANNian manifold, i. e. the automorphism group of an $O(n)$-structure, is a LIE group. In view of the fact that a RIEMANNian manifold has a unique torsion free connection, this result is included in a theorem proved by K. NOMIZU [10], J. HANO and A. MORIMOTO [6]. Their theorem states that the automorphism group of an affinely connected manifold is a LIE group. It will be shown in Section 8 that our main theorem includes all the examples mentioned. Some additional examples will be given in Section 9.

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We now give an outline of the present paper. In Section 3 we construct a sequence of $G^{(i)}$-structures induced by a G-structure on a differentiable manifold M. An automorphism φ of a G-structure will be lifted to automorphisms φ_i of the induced $G^{(i)}$-structures. This construction gives rise to a system of linear partial differential equations for infinitesimal automorphisms of a G-structure (Section 7). The type of the system will depend on the LIE algebra $\mathfrak{g}$ of G. We impose conditions on the LIE algebra $\mathfrak{g}$ so that the vector space of solutions will be finite-dimensional (Section 5). A theorem of R. S. PALAIS [11] shows that if the space of infinitesimal automorphisms is of finite dimension, then the group of automorphisms can be given a LIE group structure.

2. Prolongations of a LIE algebra

Let $\mathfrak{g}$ be a LIE algebra of endomorphisms of a real n-dimensional vector space V. $\mathfrak{g}$ may be regarded as a subspace of the tensor product $V \otimes V^*$, where V^* denotes the dual space of V. *The first prolongation* $\mathfrak{g}^{(1)}$ of $\mathfrak{g}$ is defined to be $\mathfrak{g}^{(1)} = \mathfrak{g} \otimes V^* \cap V \otimes S^2(V^*) \subset V \otimes V^* \otimes V^*$, where $S^2(V^*)$ denotes the space of symmetric tensors of degree two over V^*. With respect to a basis in $V \otimes V^* \otimes V^*$ an element $a \in \mathfrak{g}^{(1)}$ will be given by a matrix $(a^i_{j,k})$. Since $\mathfrak{g} \otimes V^* = \mathrm{Hom}(V, \mathfrak{g})$, an element $a \in \mathrm{Hom}(V, \mathfrak{g})$ is in $\mathfrak{g}^{(1)}$ if and only if

$$a_u(v) = a_v(u) \quad \text{for all} \quad v, u \in V .$$

For each $a \in \mathfrak{g}^{(1)}$ we define an automorphism $\bar{a}$ of $\mathfrak{g} \oplus V$ ($\oplus$ denotes direct sum) as follows.

$$\bar{a}(x) = x \quad \text{for} \quad x \in \mathfrak{g}, \quad \bar{a}(u) = a_u + u \quad \text{for} \quad u \in V .$$

Definition 1. $G^{(1)} = \{\bar{a} \mid a \in \mathfrak{g}^{(1)}\}$. $G^{(1)}$ is a commutative LIE group of automorphisms of the vector space $\mathfrak{g} \oplus V$.

Definition 2. The *k-th prolongation* $\mathfrak{g}^{(k)}$ *of* $\mathfrak{g}$ is defined to be

$$\mathfrak{g}^{(k)} = \mathfrak{g}^{(k-1)} \otimes V^* \cap \mathfrak{g}^{(k-2)} \otimes S^2(V^*) = \mathfrak{g} \otimes V^* \otimes \ldots \otimes V^* \cap V \otimes S^{k+1}(V^*),$$

where $V^* \otimes \ldots \otimes V^*$ denotes the k-fold tensor product. (Note that $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(-1)} = V$.)

Definition 3. A LIE algebra $\mathfrak{g}$ is *of finite type* if $\mathfrak{g}^{(k)} = 0$ for some k.

To $\mathfrak{g}^{(k)}$ will correspond a commutative LIE group $G^{(k)}$ of automorphisms of the vector space $V \oplus \mathfrak{g} \oplus \mathfrak{g}^{(1)} \oplus \ldots \oplus \mathfrak{g}^{(k-1)}$, defined as follows. To $a \in \mathfrak{g}^{(k)}$ define $\bar{a} \in G^{(k)}$ by setting

$$\bar{a}(x) = x \quad \text{for} \quad x \in \mathfrak{g} \oplus \ldots \oplus \mathfrak{g}^{(k-1)},$$

$$\bar{a}(u) = u + a_u \quad \text{for} \quad u \in V .$$

Definition 4. The *annihilator* $\mathfrak{h}^{(k)}$ *of* $\mathfrak{g}^{(k)}$ is defined by:

$$\mathfrak{h}^{(k)} = \{h \mid h \in V^* \otimes S^{k+1}(V)\,,\ \langle h, g\rangle = O \text{ for all } g \in \mathfrak{g}^{(k)}\}\,.$$

The annihilator $\mathfrak{h}^{(k)}$ will be needed in order to state Theorem B.

3. G-structure, torsion tensor

Let M be a differentiable manifold of dimension n. A linear frame u at a point $x \in M$ is an ordered basis $X_1, \ldots, X_n$ of the tangent space $T_x(M)$. Let $L(M)$ be the set of all linear frames at all points of M. Let π be the mapping of $L(M)$ onto M which maps a linear frame u at x into x. $L(M)$ is a principal fiber bundle with structure group $GL(n, R)$. A linear frame u at x can also be defined to be a vector space isomorphism $u: V \to T_x(M)$. The two definitions are related in the following way: Let $e_1, \ldots, e_n$ be a basis in V. $u: V \to T_x(M)$ is defined by $u(e_i) = X_i$. The action of $GL(n, R)$ on $L(M)$ is given by $u \to u \cdot a$, where $u \cdot a: V \xrightarrow{a} V \xrightarrow{u} T_x(M)$. In the sequel we will think of u as the isomorphism $u: V \to T_x(M)$. The notation u^{-1} therefore makes sense.

Definition 5. *A G-structure* on a differentiable manifold M is a *reduction of the structure group $GL(n, R)$ of the bundle of linear frames $L(M)$ to the subgroup G.*

The reduced bundle, a subbundle of $L(M)$, will be denoted by $P(M, G)$:

$$\begin{array}{ccc} P(M, G) & \xrightarrow{\subset} & L(M) \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow[\text{identity}]{} & M \end{array}\,.$$

A diffeomorphism φ of M can be lifted to an automorphism Φ of the bundle $L(M)$.

Definition 6. A diffeomorphism φ of M is called an *automorphism of the G-structure* $P(M, G)$ if Φ maps $P(M, G)$ onto itself. The restriction of Φ to $P(M, G)$ will be denoted by φ_1. (For reference see S. Kobayashi and K. Nomizu [7].)

We now turn to the construction of the *torsion tensors* associated with a G-structure (cf. S. Sternberg [12]). On $L(M)$ define the canonical form ϑ to be the V-valued 1-form

$$\vartheta(X) = u^{-1}\pi_*(X)\,, \quad \text{where} \quad X \in T_u(L(M))\,.$$

The restriction of ϑ to $P(M, G)$ will still be denoted by ϑ. An n-dimensional subspace $H \subset T_u(P(M, G))$ is called a *horizontal subspace* if $\vartheta : H \to V$ is an isomorphism. The exterior derivative $d\vartheta$ of ϑ evaluated at $u \in P(M, G)$ is a bilinear mapping $(d\vartheta)_u : \wedge^2 T_u(P(M, G)) \to V$. In view of the isomorphism $\vartheta : H \to V$, $d\vartheta$ restricted to $H \wedge H$ defines a map $V \wedge V \to V$, i.e. an element $c(u, H) \subset V \otimes V^* \wedge V^*$.

Definition 7. $c(u, H)$ is called the *torsion tensor* corresponding to a pair (u, H).

In the sequel, the dependence of $c(u, H)$ on H will be discussed. The action of G on P, where P stands for $P(M, G)$, induces a homomorphism σ of the LIE algebra $\mathfrak{g}$ of G into the LIE algebra $\mathfrak{X}(P)$ of vector fields on P. For $A \in \mathfrak{g}$, σA is called the fundamental vector field corresponding to A. Since G acts freely on P, the mapping $\sigma(u)$ defined by $A \to (\sigma A)_u$ $((\ )_u =$ evaluation at $u)$ is an isomorphism of the space $\mathfrak{g}$ onto the tangent space at u of the fiber G_u through u. Given a horizontal subspace $H \subset T_u(P)$, we define n vectors Z_i such that $Z_i \in H$ and $\vartheta(Z_i) = e_i$, $i = 1, 2, \ldots, n$, where e_i is the i-th element of a basis in V. For another H' we define Z'_i in the same fashion. For each i there is a unique $A_i \in \mathfrak{g}$ such that $\sigma(u)A_i = Z'_i - Z_i = Y_i$.

Definition 8. $S(H, H')$ is defined to be the map of V into $\mathfrak{g}$ which sends e_i into A_i.

Let $\hat{Y}_i$ be a vector field in a neighborhood of u in P, such that the evaluation of $\hat{Y}_i$ at u is equal to Y_i i.e. $(\hat{Y}_i)_u = Y_i$. Likewise find $\hat{Z}_i$ such that $(\hat{Z}_i)_u = Z_i$. The torsion tensor $c(u, H)$ is a map $V \wedge V \to V$, given by $c(u, H)(e_i, e_j) = d\vartheta(Z_i, Z_j) = \frac{1}{2}\{Z_i\vartheta(\hat{Z}_j) - Z_j\vartheta(\hat{Z}_i) - \vartheta([\hat{Z}_i, \hat{Z}_j])\}$, where $[\ ,\ ]$ denotes the LIE bracket (cf. S. KOBAYASHI and K. NOMIZU [7], p. 36). Likewise we define $c(u, H')$. Hence

$$\begin{aligned}(c(u, H') - c(u, H))(e_i, e_j) &= d\vartheta(Z'_i, Z'_j) - d\vartheta(Z_i, Z_j) \\ &= d\vartheta(Z_i, Y_j) - d\vartheta(Z_j, Y_i) + d\vartheta(Y_i, Y_j).\end{aligned}$$

Since $d\vartheta(Y_i, Y_j)$ is equal to zero (cf. [7], p. 120), we have

$$\begin{aligned}(c(u, H') - c(u, H))(e_i, e_j) = \tfrac{1}{2}\{&Z_i\vartheta(\hat{Y}_j) - Y_j\vartheta(\hat{Z}_i) - (Z_j\vartheta(\hat{Y}_i) - Y_i\vartheta(\hat{Z}_j)) \\ &- (\vartheta([\hat{Z}_i, \hat{Y}_j]) - \vartheta([\hat{Z}_j, \hat{Y}_i]))\}.\end{aligned}$$

Here we note that $Z_i\vartheta(\hat{Y}_j) = 0$ and $Z_j\vartheta(\hat{Y}_i) = 0$ because ϑ maps the vectors tangent to the fiber into zero. Now we choose the vector fields $\hat{Z}_i$ and $\hat{Y}_i$ such that the brackets $[\hat{Z}_i, \hat{Y}_j]$ and $[\hat{Z}_j, \hat{Y}_i]$ vanish, for example, as follows. Let

U be a neighborhood of $\pi(u) \in M$; write $\pi^{-1}(U) = U \times G$ by choosing a cross section $U \to P$ through u tangent to $Z_1, \ldots, Z_n$. Let $x^1, \ldots, x^n$ be a coordinate system in U and let $y^1, \ldots, y^m$ be a coordinate system in a neighborhood of the identity in G such that

$$\left(\frac{\partial}{\partial x^i}, 0\right)_u = Z_i \quad \text{and} \quad \left(0, \frac{\partial}{\partial y^j}\right)_u = Y_j .$$

Set $\hat{Z}_i = \left(\frac{\partial}{\partial x^i}, 0\right)$ and $\hat{Y}_j = \left(0, \frac{\partial}{\partial y^j}\right)$. With this choice we have:

$$(c(u, H') - c(u, H))(e_i, e_j) = \tfrac{1}{2}\{Y_i \vartheta(\hat{Z}_j) - Y_j \vartheta(\hat{Z}_i)\} .$$

We shall prove now that: $Y_j \vartheta(\hat{Z}_i) = -A_j e_i$, where $A_j \in \mathfrak{g}$ is defined by $Y_j = \sigma(u) A_j$. We have

$$\vartheta(\hat{Z}_i)_{ua} = (ua)^{-1} \pi(\hat{Z}_i) = a^{-1} \vartheta(Z_i) = a^{-1} e_i , \; a \in G .$$

Using the definition of the fundamental vector field σA we get

$$Y_j(a^{-1} e_i) = \left(\frac{d}{dt} \exp(-t A_j)\right)_{t=0} e_i = -A_j e_i .$$

Hence we have

Proposition 1. $(c(u, H') - c(u, H))(e_i, e_j) = \frac{1}{2}\{A_j e_i - A_i e_j\}$ where $A_j = S(H, H') e_j$ (cf. Definition 7).

Proposition 2. $c(u, H') - c(u, H) \in \alpha(\mathfrak{g} \otimes V^*) \subset V \otimes V^* \wedge V^*$, where α denotes the alternation in the two covariant factors.

Note that the kernel of α is equal to the first prolongation $\mathfrak{g}^{(1)}$ of $\mathfrak{g}$.

4. Induced $G^{(i)}$-structures

The purpose of this section is to define a sequence of $G^{(i)}$-structures and to show that an *automorphism of a G-structure can be lifted to automorphisms of the successive $G^{(i)}$-structures.*

Let $P(M, G)$ be a G-structure on M (see Definition 5). Let $\mathfrak{g}$ be the Lie algebra of G. We shall choose once and for all a linear subspace $C \subset V \otimes V^* \wedge V^*$ such that $V \otimes V^* \wedge V^* = C \oplus \alpha(\mathfrak{g} \otimes V^*)$. In general there will be no natural way of choosing C. The torsion tensor provides a map of $P(M, G)$ into $V \otimes V^* \wedge V^* = C \oplus \alpha(\mathfrak{g} \otimes V^*)$ (see Definition 7).

Definition 9. The image of $c(u, H)$ in $\alpha(\mathfrak{g} \otimes V^*)$ will be denoted by $k(u, H)$, where $c(u, H)$ is given in Definition 7.

To a *horizontal subspace* H *at* $u \in P$ *we assign a frame*

$$z = (Z_1, \ldots, Z_n, Z_{n+1}, \ldots, Z_{n+m})$$

at u in the following manner. For $i = 1, 2, \ldots, n$, Z_i is defined by $Z_i \in H$ and $\vartheta(Z_i) = e_i$, where e_i is the i-th element of a basis in V. For $j = n+1, \ldots, n+m$, Z_j is defined by $Z_j = \sigma(u) A_{j-n}$, where $A_1, \ldots, A_m$ is a basis in $\mathfrak{g}$.

Definition 10. $P_1(P, G^{(1)})$ is the set of *frames* $z = (Z_1, \ldots, Z_n, Z_{n+1}, \ldots, Z_{n+m})$ corresponding to *horizontal subspaces* $H \subset T_u(P)$ such that $u \in P$ and $k(u, H) = 0$.

Proposition 3. A *G-structure* $P(M, G)$ on M gives rise to a *uniquely defined* $G^{(1)}$*-structure* $P_1(P, G^{(1)})$ on $P = P(M, G)$.

Proof. For $a \in GL(n+m, R)$ and $z \in P_1 = P_1(P, G^{(1)}) \subset L(P)$, $z \cdot a$ is defined by $z \cdot a : V \oplus \mathfrak{g} \xrightarrow{a} V \oplus \mathfrak{g} \xrightarrow{z} T_u(P)$. Proposition 2, Section 3, shows that $z \cdot a$ is in P_1 if and only if $a \in G^{(1)}$. (See Definition 2 and note that $\mathfrak{g}^{(1)}$ is the kernel of the map $\alpha : \mathfrak{g} \otimes V^* \to V \otimes V^* \wedge V^*$.) The following lemma will conclude the proof of Proposition 3.

Lemma. $P_1(P, G^{(1)})$ is locally, in fact globally, trivial.

Proof. We shall construct a distribution of horizontal subspaces $\mathfrak{H}$, in fact a connection, on P such that $k(u, H)$ for $u \in P$ and $H \in \mathfrak{H}$, is zero. Since M is paracompact, the bundle $P(M, G)$ has a connection $\mathfrak{H}'$ giving rise to a differentiable map

$$k(\ , H') : P(M, G) \to \alpha(\mathfrak{g} \otimes V^*) .$$

Since $\mathfrak{g} \otimes V^*$ is isomorphic to kernel $\alpha \oplus \alpha(\mathfrak{g} \otimes V^*)$, we may choose a monomorphism $q : \alpha(\mathfrak{g} \otimes V^*) \to \mathfrak{g} \otimes V^*$. The composition $q \circ k(\ , H)$ maps $P(M, G)$ into $\mathfrak{g} \otimes V^*$. Let $\mathfrak{H}$ be the distribution defined by the vector fields

$$Z_i = Z_i' - \sigma(u)\left((q \circ k(u, H))e_i\right), \ i = 1, 2, \ldots, n .$$

(For definitions of Z_i' and σ see Section 3.) Since $k(u, H)(e_i, e_j) = (k(u, H') - \alpha(q \circ k(u, H')))(e_i, e_j) = 0$, by virtue of Proposition 1, we obtain a global cross section $(Z_1, \ldots, Z_n, Z_{n+1}, \ldots, Z_{n+m})$ of $P_1(P, G^{(1)})$ over $P(M, G)$.

Proposition 4. An *automorphism* φ of the G-structure $P(M, G)$ can be *lifted* to an *automorphism* φ_1 of the $G^{(1)}$-structure $P_1(P, G^{(1)})$.

Proof. Let φ_1 be the map introduced in Definition 6, Section 3. The fundamental 1-form ϑ is invariant by φ_1 and hence $d\vartheta(\varphi_{1*}Z_i, \varphi_{1*}Z_j) = d\vartheta(Z_i, Z_j) i$, $j = 1, 2, \ldots, n$, where $z = (Z_1, \ldots, Z_n, Z_{n+1}, \ldots, Z_{n+m}) \epsilon P_1(P, G^{(1)})$. This together with the fact that φ_1 leaves the fundamental vector fields invariant, proves Proposition 4.

Proposition 5. An *automorphism* $\varphi = \varphi_0$ of the *G-structure* $P(M, G)$ can be *lifted* to *automorphisms* $\varphi_1, \varphi_2, \ldots$ of the bundles $P_1, P_2, \ldots$ respectively.

Proof. The bundles and automorphisms are defined inductively by applying Propositions 3 and 4 to P_i and φ_i, $i = 1, 2, \ldots$, instead of applying them to $M = P_0$ and $\varphi = \varphi_0$.

5. Main Theorems

Let G be a (not necessarily closed) Lie subgroup of $GL(n, R)$ and let M be a differentiable manifold of dimension n.

Our main results are

Theorem A[1]**.** *If the* Lie *algebra* $\mathfrak{g}$ *of* G *is of finite type then the automorphism group of a G-structure* $P(M, G)$ *on* M *is a* Lie *group.*

Remark 1. Theorem A applies also to the case where the sequence of bundles starts at the i-th stage. In the case $i = 1$ the theorem reads as follows.

Let $G^{(1)}$ be a Lie subgroup of $GL(n, R)^{(1)}$ (first prolongation). If the Lie algebra $\mathfrak{g}^{(1)}$ of $G^{(1)}$ is of *finite type*, then the group of diffeomorphisms of M whose lifts to $L(M)$ are automorphisms of a $G^{(1)}$-structure on $L(M)$ is a *Lie group.*

Theorem B. *Let* M *be a compact differentiable manifold of dimension* n. *If there is an integer* N *and* n *elements* ${}_l h$, $l = 1, 2, \ldots, n$, *in the annihilator* $\mathfrak{h}^{(N)}$ *of* $\mathfrak{g}^{(N)}$ *such that the determinant of the matrix*

$$
{}_l h_i = \sum_{j_1 \ldots j_{N+1}} {}_l h_i^{j_1 \ldots j_{N+1}} \xi_{j_1} \ldots \xi_{j_{N+1}}
$$

is nonvanishing for every $\xi = (\xi_1, \ldots, \xi_n) \neq 0$, $\xi \epsilon R^n$, *then the automorphism group of a G-structure* $P(M, G)$ *is a* Lie *group.*

Corollary to Theorem B. Let M be a *compact* differentiable manifold of dimension n. If there is a $q = (q^{jk}) \epsilon S^2(V)$ such that $\sum_{jk} q^{jk} \xi_j \xi_k$ is *positive definite* and $V^* \otimes q \subset \mathfrak{h}^{(1)} \subset V^* \otimes S^2(V)$, then the *automorphism group* of a G-structure $P(M, G)$ is a *Lie group.*

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[1]) Added in proof: This theorem is already known. (Cf. S. Sternberg, Lectures on Differential Geometry. Prentice Hall, 1964.)

Remark 2. If a LIE subgroup G of $GL(n, R)$ satisfies the requirements of Theorem A or B, then so does every LIE subgroup of G.

6. Two Lemmas on Partial Differential Equations

In this section, we prepare two lemmas which will be needed in the proof of Theorems A and B.

Consider a system of linear partial differential equations

$$\frac{\partial u^\sigma}{\partial x^j} = \sum_\lambda a^\sigma_{j\lambda}(x^1,\ldots,x^r)\,u^\lambda \qquad \sigma, \lambda = 1, 2,\ldots, s,\ j = 1, 2,\ldots, r, \tag{1}$$

for s functions $u^\sigma = u^\sigma(x^1,\ldots,x^r)$ with initial conditions

$$u^\sigma(0) = u^\sigma_0 . \tag{2}$$

Lemma 1. The system (1) with initial conditions (2) has at most one solution.

Proof. Assume it has two solutions (u^σ) and (v^σ) such that for $x^i = a^i$ we have the following inequality $u^\sigma(a^1,\ldots,a^r) \neq v^\sigma(a^1,\ldots,a^r)$. By setting $x^i = a^i t$ and $u^\sigma = u^\sigma(t)$, a system of ordinary differential equations

$$\frac{du^\sigma}{dt} = \sum_j \frac{\partial u^\sigma}{\partial x^j}\frac{dx^j}{dt} = \sum_{\lambda j} a^\sigma_{\lambda j} a^j u^\lambda ,$$

with initial conditions $u^\sigma(0) = u^\sigma_0$ is obtained. The *uniqueness theorem* on ordinary differential equations implies $u^\sigma(1) = v^\sigma(1)$, i.e. $u^\sigma(a^1,\ldots,a^r) = v^\sigma(a^1,\ldots,a^r)$. This contradiction proves Lemma 1.

In the proof of *Theorem A* the following system of differential equations will occur.

$$\frac{\partial^{d+1}}{\partial x^{i_1}\ldots\partial x^{i_{d+1}}} X^p = \sum_k L^p_{i_1\ldots i_{d+1}k}(x, D)X^k , \tag{1*}$$

where $LP_{i_1\ldots i_{d+1}k}$ is a polynomial in the differential operators $D\colon\left(\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^n}\right)$ of degree smaller than or equal to d with variable coefficients.

Introducing new variables $Y^p_{i_1\ldots i_k} = \frac{\partial^k}{\partial x^{i_1}\ldots\partial x^{i_k}} X^p$, $k = 0, 1,\ldots, d$, we obtain a system of differential equations of first order. By adding the initial conditions $YP_{i_1\ldots i_k}(0) = YP_{i_1\ldots i_k 0}$, we obtain a system (1), (2). According to Lemma 1, this system has at most one solution.

Let M be a differentiable manifold and let $\mathfrak{X}$ be a vector space of infinitesimal transformations on M such that every point of M has a coordinate neighborhood with a system (1*) of differential equations which is satisfied by all infinitesimal transformations X in $\mathfrak{X}$. Let r be the number of linearly independent initial conditions $(Y_{i_1\ldots i_k}(0))$ at an arbitrary point $0 \in M$.

Lemma A. The dimension of the vector space $\mathfrak{X}$ is smaller than or equal to r.

Proof. Locally use Lemma 1 and note that the continuation of a solution along a curve in M is unique if it exists at all.

In order to prove *Theorem B*, a lemma on *elliptic partial differential equations* will be needed (cf. A. Douglis and L. Nirenberg [4]). Let

$$\sum_{j=1}^{n} l_{ij} X^j = 0 \tag{3}$$

be a system of linear partial differential equations in n independent variables $x^1,\ldots, x^n$ and n functions $X^1,\ldots, X^n$. The l_{ij}'s are linear differential operators which may be expressed as polynomials, $l_{ij}(x, D)$, in the differential operators $D:(\partial/\partial x^1,\ldots,\partial/\partial x^n)$ with variable coefficients, $a_{ij,\varrho}$. Let $l'_{ij}(x, D)$ represent the sum of the terms in $l_{ij}(x, D)$ which are of order S, where S denotes the order of the system (3). For arbitrary scalars $\xi = (\xi_1,\ldots,\xi_n)$ the *characteristic matrix* of (3) is defined to be the $n \times n$ matrix $l'_{ij}(x, \xi)$. The determinant, $L(x, \xi)$, is a homogeneous polynomial in ξ of degree $n \cdot S$. The system is called elliptic if $L(x, \xi)$ is nonvanishing for every $\xi \neq 0$.

Lemma 2. Assume that

(i) $L(x, \xi) \geq K \cdot |\xi|^{n \cdot S}$ for some $K > 0$;

(ii) There exists a constant L_1 such that

$$\left| \frac{\partial^k a_{ij,\varrho}}{\partial x^{i_1}\ldots\partial x^{i_k}} \right| < L_1 \quad \text{for} \quad k = 0, 1, 2;$$

(iii) $X = (X^1,\ldots, X^n)$ is a solution of (3) in a domain D and there exists a constant L_2 such that

$$\left| \frac{\partial^k X^p}{\partial x^{i_1}\ldots\partial x^{i_k}} \right| < L_2 \quad \text{for} \quad k = 0, 1,\ldots, S+1 \,.$$

Then for any compact subset $F \subset D$ there exists a constant C depending only on L_1, L_2, and K such that

$$\left| \frac{\partial^{S+1} X^p(P)}{\partial x^{i_1}\ldots\partial x^{i_{S+1}}} - \frac{\partial^{S+1} X^p(Q)}{\partial x^{i_1}\ldots\partial x^{i_{S+1}}} \right| < C \cdot |P - Q| \,,$$

where P and Q are arbitrary points in F.

Let $\mathfrak{X}' = \{(XP)\}$ be a family of functions subject to the conditions in Lemma 2. The family of functions $\mathfrak{X}'$ and their partial derivatives through the $S+1$-st order is bounded and equicontinuous in F. By Arzela's theorem every sequence in X', if restricted to F, has a subsequence which is convergent with respect to the topology of uniform convergence of functions together with their partial derivatives through the $S+1$-st order.

Let M be a compact differentiable manifold and let $\mathfrak{X}$ be a vector space of infinitesimal transformations on M such that every point of M has a coordinate neighborhood with a system (3) of partial differential equations which is satisfied by all infinitesimal transformations X in $\mathfrak{X}$. In addition, X is subject to the assumptions in Lemma 2. In local coordinates $X \in \mathfrak{X}$ is given by $X = \sum_{p=1}^{n} X^p \frac{\partial}{\partial x^P}$. By choosing an arbitrary RIEMANNian metric we define

$$||X|| = \max_{P \in M} |X| + \ldots \max_{P \in M} | \nabla^{S+1} X|,,$$

where ∇ denotes the covariant derivative defined by this metric and $|\ |$ denotes the norm obtained by extending the RIEMANNian metric.

The norm $||\ ||$ makes $\mathfrak{X}$ into a BANACH space. Since convergence in this norm is equivalent to uniform convergence of functions together with their partial derivatives through the $S+1$-st order, and since M is compact, the following lemma is obtained.

Lemma B. The BANACH space $\mathfrak{X}$ is locally compact and hence *finite-dimensional.*

7. Proof of the Main Theorems

The proof consists of the following steps. First a system of linear partial differential equations for infinitesimal automorphisms of a G-structure is established. Under the assumptions of Theorems A and B we shall prove that the *space of solutions* is *finite-dimensional.* Then a theorem of R. S. PALAIS [11] shows that in this case the *group of automorphisms* is a *LIE group.*

Let the vector fields $\{Z_j, j = 1, 2, \ldots, n+m\}$ be a cross section of the bundle $P_1(P, G^{(1)})$. By Proposition 4, Section 4, an automorphism φ of $P(M, G)$ can be lifted to an automorphism φ_1 of $P_1(P, G^{(1)})$. Hence, for $u \in P(M, G)$, $\{\varphi_{1*}((Z_j)_u), j = 1, 2, \ldots, n+m\}$ will be an element of $P_1(P, G^{(1)})$ at $\varphi_1(u)$. Recalling the proof of Proposition 3, Section 4, we have

Proposition 6. For each $u \in P$ there is an $a \in \mathfrak{g}^{(1)}$ such that

$$\varphi_{1*}((Z_j)_u) = (Z_j)_{\varphi_1(u)} + \sigma(\varphi_1(u))(a e_j), \quad \text{for } j = 1, 2, \ldots, n.$$

Note that $\sigma(\varphi_1(u))(a e_j)$ is the fundamental vector field, evaluated at $\varphi_1(u)$, which corresponds to $a e_j \in \mathfrak{g}$. Let $(x^1, \ldots, x^n)$ be a coordinate system in $U \subset M$. With respect to the coordinates (x^i, x^i_j) in $U \times GL(n, R)$,

Z_j, for $j = 1, 2, \ldots, n$, is given by $Z_j = \left(\sum_i c_j^i(u) \frac{\partial}{\partial x^i}, \sum_{kl} c_{lj}^k(u) \frac{\partial}{\partial x_l^k}\right)$. For $u = (x^i, x_j^i)$, $\varphi_1(u)$ is expressed by $\left(\varphi^i(x), \sum_q \frac{\partial \varphi^i(x)}{\partial x^q} \cdot x_j^q\right)$. Proposition 6 yields

$$\left(\sum_{ik} c_j^i(u) \frac{\partial \varphi^k}{\partial x^i} \frac{\partial}{\partial x^k}, \sum_{pl}\left(\sum_{iq} c_j^i(u) \frac{\partial^2 \varphi^p(x)}{\partial x^q \partial x^i} x_l^q + \sum_q c_{lj}^q(u) \frac{\partial \varphi^p(x)}{\partial x^q}\right) \frac{\partial}{\partial x_l^p}\right) =$$

$$= \left(\sum_i c_j^i(\varphi_1(u)) \frac{\partial}{\partial x^i}, \sum_{kl} c_{lj}^k(\varphi_1(u)) \frac{\partial}{\partial x_l^k}\right) + \sigma(\varphi_1(u))(a e_j). \qquad (1)$$

As in Section 2, an element $a \in \mathfrak{g}^{(1)}$ is given by a matrix $a = (a_{l,j}^p)$. For $u = (x^i, \delta_j^i)$ (δ_j^i = KRONECKER delta) we get $\sigma(u)(a e_j) = \sum_{pl} a_{l,j}^p \frac{\partial}{\partial x_l^p}$. Since this is true for $u = (x^i, \delta_j^i)$ only, we evaluate the above equation at $w = (y^i, y_j^i) = \left(\left(\varphi^{-1}(x)\right)^i, \sum_k \left(\frac{\partial \varphi^k(x)}{\partial x^j}\right)^{-1} \delta_k^i\right)$. Thus we have $\varphi_1(w) = (x^i, \delta_j^i)$. The components in the fiber direction of equation (1) yield

$$\sum_{iq} c_j^i(w) \frac{\partial^2 \varphi^p(y)}{\partial x^q \partial x^i} y_l^q + \sum_q c_{lj}^q(w) \frac{\partial \varphi^p(y)}{\partial x^q} = c_{lj}^p(x^i, \delta_j^i) + a_{l,j}^p, \qquad (2)$$

where the $a_{l,j}^p$'s depend on $\varphi(x)$.

In order to obtain a system of differential equations, satisfied by all automorphisms φ, we let an element $h = (h_p^{lj}) \in \mathfrak{h}^{(1)}$ operate on equation (2). (For $\mathfrak{h}^{(1)}$ see Definition 4.) Thus we get

$$\sum_{plj} h_p^{lj}\left(\sum_{iq} c_j^i(w) \frac{\partial^2 \varphi^p(y)}{\partial x^q \partial x^i} y_l^q + \sum_q c_{lj}^q(w) \frac{\partial \varphi^p(y)}{\partial x^q} - c_{lj}^p(x)\right) = \sum_{plj} h_p^{lj} a_{l,j}^p = 0. \qquad (3)$$

Let X be a vector field on M. In a coordinate neighborhood U write $X = \sum_p X^p \frac{\partial}{\partial x^p}$. X generates a local 1-parameter group of local transformations $\varphi: U' \times I_\varepsilon \to U$, where U' is an open subset of U and I_ε is an open neighborhood of zero in R. X is called an *infinitesimal automorphism* of the *G-structure* $P(M, G)$ if for $t \in I_\varepsilon$ $\varphi(x, t)$ is an isomorphism of $P(U', G)$ onto $P(\varphi_t(U'), G)$.

In this case, $\varphi(x, t)$ for $t \in I_\varepsilon$ is a solution of equation (3). By taking the derivative of equation (3) with respect to t for $t = 0$, and noting that

$$\varphi^p(x, 0) = x^p \text{ and } \frac{\partial^{k+1} \varphi(x, 0)}{\partial t \partial x^{i_1} \ldots \partial x^{i_k}} = \frac{\partial^k X^p}{\partial x^{i_1} \ldots \partial x^{i_k}} \text{ for } k = 0, 1, \ldots,$$

we get

$$\sum_{ijlp} h_p^{lj} c_j^i(x) \frac{\partial^2 X^p}{\partial x^l \partial x^i} + \sum_{jp} h_p^j(x) \frac{\partial X^p}{\partial x^j} + \sum_p h_p(x) X^p = 0\,, \tag{4}$$

where $c_j^i(x), h_p^j(x)$, and $h_p(x)$ depend on $x = (x^1, \ldots, x^n)$ only, while h_p^{lj} is a constant. Thus we see that every element $h = (h_p^{lj}) \in \mathfrak{h}^{(1)}$ gives rise to an equation (4). By lifting an automorphism φ to a $G^{(d)}$-structure, we see that an element $h = (h_p^{j_1 \ldots j_{d+1}}) \in \mathfrak{h}^{(d)}$ gives rise to a linear partial differential equation of order $d + 1$. The highest order term will be

$$\sum_{ij_1 \ldots j_{d+1}p} h_p^{j_1 \ldots j_{d+1}} c_{j_{d+1}}^i(x) \frac{\partial^{d+1} X^p}{\partial x^i \partial x^{j_1} \ldots \partial x^{j_d}}\,. \tag{5}$$

If the LIE algebra $\mathfrak{g}$ of G is *of finite type*, then there is an integer d such that $\mathfrak{g}^{(d)} = 0$, i. e. $\mathfrak{h}^{(d)}$ equals $V^* \otimes S^{d+1}(V)$. This yields the system of linear partial differential equations:

$$\frac{\partial^{d+1}}{\partial x^{j_1} \ldots \partial x^{j_{d+1}}} X^p = \sum_k L^p_{j_1 \ldots j_{d+1} k}(x, D) X^k\,.$$

(For a definition of D see equation [1*] in Section 6.) By Lemma A, Section 6, the *vector space* of all *infinitesimal automorphisms* of a G-structure $P(M, G)$ is of *finite dimension*. An upper bound of this dimension is given by the following: $n + \dim \mathfrak{g} + \dim \mathfrak{g}^{(1)} + \ldots + \dim \mathfrak{g}^{(d)}$.

The condition on the LIE algebra $\mathfrak{g}$ of G in *Theorem B* is to insure that condition (i) of Lemma 2, Section 6 is fulfilled. Note that it is possible to choose a coordinate system such that $c_j^i(0) = \delta_j^i$. Condition (ii) can be satisfied by restricting the vector field X to a smaller neighborhood if necessary. Lemma B therefore shows that under the assumptions in Theorem B the *vector space* of *infinitesimal automorphisms* of a G-structure $P(M, G)$ is of *finite dimension*.

An application of the following theorem of R. S. PALAIS [11] concludes the proof of Theorems A and B. Let H be a group of differentiable transformations acting on a differentiable manifold M. Let $\mathfrak{h}'$ be the set of all vector fields on M which generate a global 1-parameter group of transformations belonging to H. Let $\mathfrak{h}$ be the subalgebra generated by $\mathfrak{h}'$ in the LIE algebra $\mathfrak{X}(M)$ of all differentiable vector fields on M.

Theorem. If $\mathfrak{h}$ is *finite-dimensional*, then H admits a *LIE group structure* (such that the map $H \times M \to M$ is differentiable), and $\mathfrak{h} = \mathfrak{h}'$. The LIE algebra of H is naturally isomorphic to $\mathfrak{h}$.

Now let H be the group of all automorphisms of a G-structure $P(M, G)$. Then $\mathfrak{h}'$ and $\mathfrak{h}$ are contained in the LIE algebra of *infinitesimal automorphisms* of $P(M, G)$. This LIE algebra has been proved to be of finite dimension under the assumptions of Theorems A and B respectively.

8. Examples Mentioned in the Introduction

1. RIEMANNian manifold.

Let $O(n)$ denote the group which leaves a given nondegenerate symmetric bilinear form $(\, , \,)$ on V (of arbitrary signature) invariant. A RIEMANNian structure is an *$O(n)$-structure* on M. We will show that the LIE algebra $\mathfrak{o}(n)$ of $O(n)$ is *of finite type*, in fact, $\mathfrak{o}(n)^{(1)} = 0$. Thus *Theorem A* will apply. The maximal dimension of the automorphism group is $n + \dim \mathfrak{o}(n) = \frac{n(n+1)}{2}$. The following computation is taken from V. W. GUILLEMIN and S. STERNBERG [5]. A linear transformation a of V is in $\mathfrak{o}(n)$ if and only if $(au, v) + (u, av) = 0$; for all $u, v \in V$. For any $a \in \mathfrak{o}(n)^{(1)}$ and any $u, v, w \in V$ we have

$$(awv, u) = (avw, u) = -(avu, w) = -(auv, w) = (auw, v) =$$
$$= (awu, v) = -(awv, u).$$

Thus $(awu, v) = 0$, which implies $a = 0$ because $(\, , \,)$ is nonsingular.

2. Conformal structure on a manifold of dimension $n \geq 3$.

Let $(\, , \,)$ be as in Example 1. $\mathfrak{co}(n)$ denotes its *conformal algebra.*

$$a \in \mathfrak{co}(n) \text{ if and only if } (au, v) + (u, av) = \lambda \cdot (u, v) \text{ for all } u, v \in V,$$

where λ is a scalar depending on a. V. W. GULLEMIN and S. STERNBERG [5] show that $\mathfrak{co}(n)^{(1)}$ is of dimension n by establishing a vector space isomorphism $\mathfrak{co}(n)^{(1)} \to V^*$. For $a \in \mathfrak{co}(n)^{(2)}$ and u, v, x, y in V we get

$$(auvx, y) + (x, auvy) = (\Lambda uv)(x, y),$$

where Λ is an element of $V^* \otimes V^*$ which depends on a. By a computation similar to that used in Example 1, Λ is shown to be zero (cf. [5]). This implies that $a = 0$ because in this case $a \in \mathfrak{o}(n)^{(2)} = 0$. Hence $\mathfrak{co}(n)^{(2)} = 0$, the conformal structure is *of finite type* and *Theorem A* applies.

3. Manifold with an affine connection.

An affine connection is a $G^{(1)}$-structure on the bundle of frames $L(M)$, where $G^{(1)}$ consists of the identity in $GL(n, R)^{(1)}$ alone, i.e., $\mathfrak{g}^{(1)} = 0$ (cf. Remark 1, Section 5). *Theorem A* applies; hence, the maximal dimension of the automorphism group of an affine connection will be

$$N = n + \dim \mathfrak{gl}(n, R) = n + n^2.$$

4. Almost complex structure

Let M be a compact differentiable manifold of dimension $n = 2m$, and let $G = GL(m, C) \subset GL(2m, R)$.

$$a = (a^p_q) \in \mathfrak{g} \text{ if and only if } a^i_j = a^{i+m}_{j+m},\ a^{m+i}_j = -a^i_{m+j};$$
$$i, j = 1, 2, \ldots, m;\ p, q = 1, 2, \ldots, 2m.$$

Computation of $\mathfrak{g}^{(1)}$ (cf. Section 2):

$$a^i_{j,k} = a^{i+m}_{j+m,k} = a^{i+m}_{k,j+m} = -a^i_{k+m,j+m} = -a^i_{j+m,k+m},$$
$$a^{i+m}_{j,k} = -a^i_{j+m,k} = -a^i_{k,j+m} = -a^{i+m}_{k+m,j+m} = -a^{i+m}_{j+m,k+m}.$$

Since $a^p_{i,j} + a^p_{i+m,j+m} = 0$, we obtain

$$h = (h^{ql}_p) = (a_p \cdot \delta^{ql}) \subset \mathfrak{h}^{(1)} \quad \text{(cf. Definition 4).}$$

The KRONECKER Delta, δ^{ql}, is a unit matrix and hence, positive definite; therefore, the corollary to *Theorem B* applies. The *automorphism group* of an *almost complex structure* on a *compact* differentiable manifold is a *LIE group.*

9. Further Examples

1. Tensor product structure on a manifold M.

Let M be a manifold of dimension $p \cdot q$ where $p, q \geq 2$. Let G be the LIE subgroup of $GL(p \cdot q, R)$ whose LIE algebra is given by the tensor product representation of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ on $V_1 \otimes V_2$. The action of $\mathfrak{g}$ on $V = V_1 \otimes V_2$ is given by

$$(a, b)(v_1 \otimes v_2) = av_1 \otimes v_2 + v_1 \otimes bv_2,$$

where $a = (a^i_j) \in \mathfrak{g}_1$ and $b = (b^k_l) \in \mathfrak{g}_2$.

V has a basis $e_{(i,k)} = e_i \otimes f_k$, where (e_i) and (f_k) are bases in V_1 and V_2 respectively.

An element in $\mathfrak{g}$ is denoted by $A = (A^{(i,k)}_{(j,l)}) = (a^i_j \delta^k_l + \delta^i_j b^k_l)$. This gives rise to the following four equations:

$$\text{For } i \neq j,\quad k \neq l,\quad A^{(i,k)}_{(j,l)} = 0; \tag{1}$$

$$\text{For } i \neq j,\quad A^{(i,n)}_{(j,n)} = A^{(i,m)}_{(j,m)};\ A^{(m,i)}_{(m,j)} = A^{(n,i)}_{(n,j)}; \tag{2}$$

$$A^{(i,m)}_{(i,m)} - A^{(j,m)}_{(j,m)} = A^{(i,n)}_{(i,n)} - A^{(j,n)}_{(j,n)}; \tag{3}$$

$$A^{(i,m)}_{(i,m)} - A^{(i,n)}_{(i,n)} = A^{(j,m)}_{(j,m)} - A^{(j,n)}_{(j,n)}. \tag{4}$$

We show that $\mathfrak{g}^{(2)} = 0$. Let $A = (A^{(h,l)}_{(i,m),(j,k),(k,o)})$ be an element in $\mathfrak{g}^{(2)}$. It is easy to show that unless all index pairs coincide, the corresponding component of A vanishes as the following computation illustrates. Assume $l \neq m, i \neq j, k \neq l$.

$$A^{(i,l)}_{(i,m),(i,l),(i,l)} = A^{(j,l)}_{(j,m),(i,l),(i,l)} = A^{(j,l)}_{(i,l),(j,m),(i,l)} = A^{(j,k)}_{(i,k),(j,l),(i,l)} = 0.$$

This is true because of equations (1) and (2). If all index pairs coincide, equation (3) or (4) is used to reduce this case to the previously solved case. The *group of automorphisms* of a tensor product structure is therefore, by *Theorem A*, a *Lie group.*

2. G-structures for which the Lie algebra g of G acts irreducibly on V.

Let $\mathfrak{g}$ be an *irreducible Lie algebra of endomorphisms* of a real vector space V of dimension n. There are *six classes of Lie algebras* which are of *infinite type* (see Y. Matsushima [8]):

$\mathfrak{g} = \mathfrak{gl}(n, R)$ Lie algebra of all endomorphisms of V; (1)

$\mathfrak{g} = \mathfrak{sl}(n, R)$ Lie algebra of all endomorphisms of V of trace zero; (2)

$\mathfrak{g} = \mathfrak{sp}(2m, R)$ $n = 2m$, $\mathfrak{g}$ is the Lie algebra of endomorphisms of V which leave the following skew symmetric bilinear form of maximal rank, $Q(x, y)$, invariant.

$$Q(x, y) = x_1 y_2 - x_2 y_1 + \ldots + x_{n-1} y_n - x_n y_{n-1}; \qquad (3)$$

$\mathfrak{g} = \mathfrak{sp}(2m, R) + Z$, where Z = center of $\mathfrak{gl}(2m, R)$; (4)

$\mathfrak{g} = \mathfrak{sl}(m, C) + U \subset \mathfrak{gl}(2m, R)$, where U is a certain real subspace of the center of $\mathfrak{gl}(m, C)$; (5)

$\mathfrak{g} = \mathfrak{sp}(2m, C) + U \subset \mathfrak{gl}(4m, R)$, where U is a certain real subspace of the center of $\mathfrak{gl}(2m, C)$. (6)

Let M be a compact manifold of dimension $n \geq 2$, and let $P(M, G)$ be a G-structure on M. If the Lie algebra $\mathfrak{g}$ of G is one of the Lie algebras in (5) or (6), it follows immediately from Example 4, Section 8, that the automorphism group of $P(M, G)$ is a Lie group. If the Lie algebra $\mathfrak{g}$ of G is one of the Lie algebras in (1), (2), (3), and (4), the group of automorphisms of a G-structure $P(M, G)$ is *not* in general a *Lie group.*

Counterexample. All groups corresponding to (1), (2), (3), and (4) contain $SL(2, R) \times I_{n-2}$ as a Lie subgroup. $SL(2, R)$ denotes the special linear group acting on the space spanned by the first two elements of a basis in V; I_{n-2} is the identity on the space spanned by the last $n - 2$ elements of the same basis. Let T be the torus obtained from R^2 by identifying the points (x, y) and $(x + p, y + q)$, where p and q are integers. The frame field $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ defines a $SL(2, R)$ structure on T. The vector field $f(y) \frac{\partial}{\partial x}$, where $f(p) = f(q)$, p, q integers, is an infinitesimal automorphism of this $SL(2, R)$ structure. The vector space of infinitesimal automorphisms is of *infinite* *dimension.* The automorphism group is therefore *not a* LIE *group.* (Since T is compact, every infinitesimal automorphism generates a 1-parameter group of automorphisms.)

Additional applications of Theorem B can be obtained by considering non-irreducible LIE algebras.

Let M be a compact differentiable manifold of dimension $n = 2m$ and let $\mathfrak{g}$ be a commutative LIE algebra of endomorphisms on a $2m$-dimensional vector space V such that for any element $a = (a^p_q) \in \mathfrak{g}$ the following equations hold:

$$a^i_j = -a^{i+m}_{j+m}, \quad a^{i+m}_j = a^i_{j+m}; \quad i, j = 1, 2, \ldots, m.$$

Let G denote the LIE subgroup of $GL(n, R)$ whose LIE algebra is equal to $\mathfrak{g}$. Computation of $\mathfrak{g}^{(1)}$:

$$a^i_{j,k} = -a^{i+m}_{j+m,k} = -a^{i+m}_{k,\,j+m} = -a^i_{k+m,j+m} = -a^i_{j+m,k+m},$$

and

$$a^{i+m}_{j,\,k} = a^i_{j+m,k} = a^i_{k,j+m} = -a^{i+m}_{k+m,j+m} = -a^{i+m}_{j+m,k+m}.$$

For every $a = (a_p) \in V^*$, $h = (a_p \cdot \delta^{ql}) \in \mathfrak{h}^{(1)}$. The corollary to *Theorem B* applies. The *automorphism group* of a G-structure is a *LIE group.*

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