Zeitschrift: Commentarii Mathematici Helvetici

Band: 39 (1964-1965)

Artikel: Uniform distribution in locally compact groups.

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DOI: https://doi.org/10.5169/seals-29884

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Uniform distribution in locally compact groups

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In [12], WEYL introduced and studied the notion of uniform distribution mod 1, that is, uniform distribution on the circle group. Later, ECKMANN [2] extended the notion to compact topological groups, and obtained analogues of some of WEYL's results. Recently, NIVEN [7] introduced the notion of uniform distribution on the integers Z, and obtained some results that were later extended by S. UCHIYAMA [11]. We show that all these notions of uniform distribution are special cases of the more general notion of uniform distribution on a locally compact group, and show how some of the results of NIVEN and UCHIYAMA follow easily from the general theory.

We also define a notion of monogenic group that generalizes VAN DANTZIG's notion of monothetic group. Both "monothetic" and "monogenic" are extensions of "cyclic". But "monothetic" involves the topological structure, whereas "monogenic" emphasizes the measure-theoretic structure.

Throughout this paper, we use the word "group" to denote locally compact topological groups. For the basic facts we use about groups, the book of PONTRYAGIN [8] will serve as reference, while the book of RUDIN [9] will serve as reference for ABELian groups. Our bibliography includes several items that are not referred to in the text.

We now briefly recall some of the theory of uniform distribution on compact groups, essentially following [2]. We denote by m the unique (left) Haar measure on G, normalized so that $m(G) = 1$.

Definition (ECKMANN). *If G is a compact group, the sequence $\{x_\nu\}$, $\nu = 1, 2, 3, \ldots$, of elements of G is said to be uniformly distributed provided that for each continuous complex-valued function f on G,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\nu=1}^{N} f(x_\nu) = \int_G f\,dm.$$

Proposition (ECKMANN). *If G is a compact group, then $\{x_\nu\}$ is uniformly distributed in G if and only if*

$$\lim_{N\to\infty} \frac{N(A)}{N} = m(A)$$

for each measurable set $A \subseteq G$ such that $m(\partial A) = 0$, where $N(A)$ is the number of x_ν contained in A for $\nu = 1, 2, \ldots, N$.

[1] This research was supported in part by an Air Force research grant.

We remark that the important condition that $m(\partial A) = 0$ was omitted in [2].

Theorem (ECKMANN). *If G is compact, then $\{x_\nu\}$ is uniformly distributed in G if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\nu=1}^{N} D(x_\nu) = 0$$

for each non-trivial irreducible representation D of G.

Theorem (ECKMANN). *If G is compact and ABELian, then $\{x_\nu\}$ is uniformly distributed in G if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\nu=1}^{N} (x_\nu, \gamma) = 0 \tag{1}$$

for each non-trivial continuous character γ in the PONTRYAGIN dual group Γ.

We shall refer to (1) as the WEYL criterion. We now present NIVEN's [7] definition of uniform distribution of a sequence of integers.

Definition (NIVEN). *The sequence $\{x_\nu\}$, $\nu = 1, 2, 3, \ldots$, of integers is said to be uniformly distributed* mod (n), *where $n \geq 2$ is an integer, provided that*

$$\lim_{N\to\infty} \frac{1}{N} A(N, j, n) = \frac{1}{n}$$

for each j, $1 \leq j \leq n$, where $A(N, j, n)$ is the number of x_ν, $\nu = 1, 2, \ldots, N$, that are congruent to j mod (n). *Further, the sequence $\{x_\nu\}$ is said to be uniformly distributed if it is uniformly distributed* mod (n) *for each $n = 2, 3, 4, \ldots$.*

We may now turn to the general situation.

Definition. *Let H be a closed normal subgroup of the group G. We say that H is of compact index to mean that G/H is compact.*

We recall that the topology on G/H is the weakest topology in which the natural homomorphism $\varphi_H : G \to G/H$ is continuous, so that the open sets in G/H are just the images, under φ_H, of the open sets in G.

Definition. *If G is a locally compact group, we say that $\{x_\nu\}$ is uniformly distributed in G provided that $\varphi_H(x_\nu)$ is uniformly distributed in G/H for each closed normal subgroup H of compact index.*

Proposition. *If G is compact, then the above definition coincides with the standard one.*

Proof. Suppose that $\{x_\nu\}$ is uniformly distributed in G according to the above definition. Taking for H the trivial subgroup consisting of the identity, we see that $\{x_\nu\}$ is uniformly distributed in G according to the standard definition. Conversely, if $\{x_\nu\}$ is uniformly distributed in G according to the standard definition, let H be a closed normal subgroup of compact index, and let $f: G/H \to C$ be a continuous function. It must be shown that

$$\frac{1}{N}\sum_{\nu=1}^{N} f(\varphi_H(x_\nu)) \to \int_{G/H} f dm' \quad as \quad N \to \infty,$$

where m' is HAAR measure on G/H. But $f \circ \varphi_H$ is continuous on G, so that

$$\frac{1}{N}\sum_{\nu=1}^{N} f(\varphi_H(x_\nu)) \to \int_{G} f o \varphi_H dm \quad as \quad N \to \infty,$$

and the result follows, since

$$\int_{G/H} f dm' = \int_G f \circ \varphi_H dm,$$

by the uniqueness of HAAR measure on G/H.

Proposition. *If* $G = Z$, *then the above definition coincides with* NIVEN'*s definition.*

Proof. It is easily seen that the subgroups H of Z that are of compact index are just the groups $nZ = \{0, \pm n, \pm 2n, \ldots\}$ for $n = 1, 2, 3, \ldots$. The corresponding quotient groups are Z_n, the cyclic group with n elements. But it is easy to see that $\{x_\nu\}$ is uniformly distributed mod (n) if and only if the residue classes of the x_ν are uniformly distributed as elements of Z_n, since the HAAR measure on Z_n assigns mass $1/n$ to each of its elements.

From this proposition, the next results follow easily. They were partially formulated and proved by NIVEN [7], and completed by UCHIYAMA [11].

Theorem (NIVEN/UCHIYAMA). *In order that the sequence* $\{x_\nu\}$ *of integers be uniformly distributed* mod (n), *it is necessary and sufficient that*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{\nu=1}^{N} e\left(\frac{j}{n} x_\nu\right) = 0$$

for each $j = 1, 2, \ldots, n-1$, *where*

$$e(x) = \exp(2\pi i x).$$

Corollary. *In order that the sequence* $\{x_\nu\}$ *of integers be uniformly distributed, it is necessary and sufficient that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\nu=1}^{N} e(r x_\nu) = 0$$

for each rational number r *that is not an integer.*

Proof. The characters γ on Z_n are easily seen to be of the form

$$(x, \gamma) = e\left(x \frac{j}{n}\right) \qquad j = 0, 1, 2, \ldots, n-1 .$$

A direct application of the WEYL criterion now gives the proof.

Remark. It is now clear what the definition of uniform distribution on the GAUSSian integers $Z^2 = Z \oplus Z$, or on Z^n in general, should be. Regarding Z^n as a lattice in EUCLIDean n-space R^n, the condition that the sequence $\{x_\nu\}$ be uniformly distributed is just the requirement that the x_ν fall with proper limiting frequency in each n-dimensional sublattice of Z^n, since these are easily seen to be the subgroups of compact index. It is now possible to extend part of a result of NIVEN [7] to the case of higher dimensions, giving a result bearing some resemblance to KRONECKER's theorem. The proof is entirely analogous to NIVEN's proof, and we omit it. We have not sought a complete characterization of the n-tuples $(\vartheta_1, \vartheta_2, \ldots, \vartheta_n)$ that work. In case $n = 1$, NIVEN showed that the numbers ϑ of the form $\vartheta = 1/k$, $k = \pm 1, \pm 2, \ldots$ work, as well as the irrational numbers ϑ, and these are all. It seems likely that some of the other results of NIVEN and of M. UCHIYAMA and S. UCHIYAMA [10] will also generalize.

Theorem. *Suppose that the* n-tuple $(\vartheta_1, \vartheta_2, \ldots, \vartheta_n)$ *of real numbers is such that the set* $1, \vartheta_1, \vartheta_2, \ldots, \vartheta_n$ *is linearly independent over the rationals. Then the sequence* $\{x_\nu\}$ *in* Z^n *given by*

$$x_\nu = ([\nu\vartheta_1], [\nu\vartheta_2], \ldots, [\nu\vartheta_n])$$

is uniformly distributed in Z^n, *where* $[y]$ *denotes the integral part of* y.

We now turn to generalizations of cyclic groups.

Definition (VAN DANTZIG). *An Abelian group* G *is called monothetic if there is an element* x *in* G *such that the sequence* $\{nx\}, n = 1, 2, 3, \ldots,$ *is everywhere dense in* G.

Definition. *An* ABELian *group* G *is called monogenic if there is an element* x *in* G *such that the sequence* $\{nx\}, n = 1, 2, 3, \ldots$ *is uniformly distributed in* G.

A result of ECKMANN [2], using the WEYL criterion, shows in effect that every monothetic group is monogenic, while it is clear that every compact monogenic group is monothetic, since a uniformly distributed sequence in a compact group must be everywhere dense. Not every monogenic group is monothetic, however, as the example $G = R_d$ shows, where R_d is the real numbers in the discrete topology. Indeed, it is easily seen that every sequence in R_d is uniformly distributed. It is known [2], [3] that each monothetic group, except for Z, is compact, and also that each compact connected ABELian group is monothetic.

Proposition. *Let* G *be an* ABELian *group, and let* Γ *be its dual. In order that* G *be monogenic, it is necessary, but not sufficient, that each discrete subgroup of* Γ *be isomorphic to a subgroup of* T, *the circle group.*

Proof. The condition is necessary, since if Γ_1 is a discrete subgroup of Γ, then Γ_1 is the dual of G/H, where

$$H = \Gamma_1^{\perp} = \{x \in G : (x, \gamma) = 1 \text{ for each } \gamma \in \Gamma_1\},$$

and H is of compact index since G/H is compact, being the dual of a discrete group. From [2], [3] we have that a compact group is monothetic if and only if its dual group is a subgroup of T. Furthermore, a compact group is monothetic if and only if it is monogenic. This proves the first assertion. For the second assertion, take $G = R$. It is easy to see that R is not monogenic. But the dual group of R is again R, and each discrete subgroup of R is of the form $aZ = \{0, \pm a, \pm 2a, \ldots\}$. If $a \neq 0$, then aZ is isomorphic to the subgroup of T generated by $\exp 2\pi i\vartheta$ for any fixed irrational ϑ, while the case $a = 0$ is trivial.

We conclude with some remarks. One might say that an ABELian group G is unithetic, say, in case each discrete subgroup of the dual of G is isomorphic to a subgroup of the circle group. But this notion does not seem too promising. Also, the definition has been proposed (in conversation) that a sequence $\{x_\nu\}$ be called, say, equidistributed in the ABELian group G provided that $\{x_\nu\}$ is uniformly distributed in the BOHR compactification of G. This definition seems unpromising, especially since it differs in the case $G = Z$ from NIVEN's definition.

In another direction, the sequence $\{\nu^{1/2}\}$ is uniformly distributed in R. This is equivalent to the assertion, whose simple proof we omit, that $\{t\nu^{1/2}\}$ is uniformly distributed mod (1) for each real $t, t \neq 0$.

Addet in proof:

It has recently come to our attention that J. CIGLER, in his paper Folgen normierter Masse auf kompakten Gruppen, Zeitschrift für Wahrscheinlichkeitstheorie *1*, 3–13 (1962), has a definition of uniform distribution on R in terms of almost periodic functions, that can be shown to be equivalent to ours. In a talk, of which there is a mimeographed version, at the Mathematical Center of Amsterdam in January 1964, entitled Einige Fragen der Theorie der Gleichverteilung, CIGLER outlined some further ideas related to the contents of this paper.

Two of our colleagues at the University of Illinois, J. J. ROTMAN and C. T. RAJAGOPALAN, have recently obtained a characterization of the discrete monogenic groups as being certain distinguished subgroups of direct products of finite groups and groups of p-adic integers. Finally, we have recently obtained some results that extend the results of the present paper to apply to sequences of measures (actually to so-called densities, which are more general than measures) on locally compact groups, and expect to publish them in due course.

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(Received July 2, 1964)