Symmetry of Linking Coefficients.

Symmetry of Linking Coefficients

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Introduction. Consider a 3-link in the unit sphere S^{n+1}: namely three spheres S^{p_1}, S^{p_2}, S^{p_3} differentiably and disjointly embedded in S^{n+1}. Suppose $n - 1 > \max(p_1, p_2, p_3)$. Let i, j, k be any permutation of p_1, p_2, p_3. We know by ALEXANDER duality that $S^{n+1} - S^i$ has the same homotopy type as S^{n-i} and that $S^{n+1} - (S^i \cup S^j)$ has the same $(n-1)$-type as the wedge $S^{n-i} \vee S^{n-j}$. Hence S^k represents an element $\lambda^k \in \pi_k(S^{n-i} \vee S^{n-j})$.

HILTON, in [3], gave a direct sum decomposition for this group, namely

$$\pi_k(S^{n-i} \vee S^{n-j}) = \pi_k(S^{n-i}) + \pi_k(S^{n-j}) + \pi_k(S^{2n-i-j-1}) + \dots$$

The first two components λ_i^k and λ_j^k of λ^k in this decomposition are the linking elements of S^k with S^i and S^j respectively. It is known that λ_j^i and λ_i^j are equal, up to sign, after stable suspension (see § 5 of [4]).

We shall be concerned by the component λ_{ij}^k of λ^k in the third factor $\pi_k(S^{2n-i-j-1})$; this component is by definition the HILTON-HOPF invariant of λ^k. We shall prove the following symmetry relations. (They where suggested by the particular case $p_1 = p_2 = p_3 = 2d - 1$, $n + 1 = 3d$ studied by one of the authors [2] and were proved in that case by roundabout means.) E^i denotes the i-th fold suspension homomorphism, defined as in § 1.4.

Theorem. *For any 3-link S^{p_1}, S^{p_2}, S^{p_3} in S^{n+1}, the linking elements $\lambda_{ij}^k \in \pi_k(S^{2n-i-j-1})$, which are the HILTON-HOPF invariants of the elements $\lambda^k \in \pi_k(S^{n-i} \vee S^{n-j})$ represented by S^k embedded in the complement of $S^i \cup S^j$, satisfy the symmetry relations*

$$(-1)^{i+ij+nk} E^{n+2-i} \lambda_{jk}^i = (-1)^{j+jk+ni} E^{n+2-j} \lambda_{ki}^j .$$

On the way (in § 2), we give a geometric definition of the HILTON-HOPF invariant, which is very close to the original definition of HOPF.

1. Terminology. By a *manifold* M, we shall mean a differentiable compact manifold of class C^∞, possibly with boundary ∂M. A *submanifold* V of M will be a compact submanifold of class C^∞ of M; unless there is explicit statement of the contrary, the boundary ∂V of V will be contained in the boundary ∂M of M, and V will cut ∂M transversally along ∂V.

1.1. A *framed submanifold* $(V, \mathfrak{F})$ of M will be a submanifold V of M together with a framing $\mathfrak{F}$ (trivialization) of class C^∞ of its normal bundle. It is awkward to write $(V, \mathfrak{F})$ for the framed manifold, and we shall just write V with the particular framing understood. In particular, when V is just a point x of an oriented manifold M, x will often be considered as a framed submanifold with a frame giving the orientation of M.

It is clear that the boundary of a framed submanifold V of M is a framed submanifold ∂V of ∂M.

If V_1 and V_2 are two framed submanifolds of M and if they cut each other transversally (i. e. if $x \in V_1 \cap V_2$, the tangent space of M at x is the sum of the tangent spaces of V_1 and V_2 at x), the intersection $V_1 \cap V_2$ is again a framed submanifold; its framing is given by the direct sum in this order of the restrictions to $V_1 \cap V_2$ of the framings of V_1 and V_2.

Let V be a framed submanifold of M and let f be a differentiable map of a manifold M' into M which is transverse regular on V (see § 4 of [6]). Then $f^{-1}(V)$ is a framed submanifold of M'; its framing is the inverse image by f of the framing of V.

Two framed submanifolds V_0 and V_1 of M are *cobordant* if there exists a framed submanifold V of $I \times M$ such that $\partial V = (0 \times V_0) \cup (1 \times V_1)$. This is an equivalence relation.

The PONTRJAGIN-THOM construction (see [5], [6] or page 346 of [4]) associates to each framed submanifold (V, F) of M of codimension q a map of M into the q-sphere S^q. It induces a bijective correspondence between cobordism classes of framed submanifolds of codimension q in M and homotopy classes of maps of M in S^q.

1.2. Similarly we can consider pairs (V, W) of disjoint framed submanifolds in M. Two such pairs (V_0, M_0) and (V_1, M_1) are (framed) cobordant if there exists a pair (V, W) of disjoint framed submanifolds in $I \times M$ such that

$$\partial V = (0 \times V_0) \cup (1 \times V_1) \text{ and } \partial W = (0 \times W_0) \cup (1 \times W_1).$$

The analogue of the PONTRJAGIN-THOM construction will give a bijective correspondence between cobordism classes of pairs (V, W) of disjoint framed submanifolds of M of codimension (p, q) and the homotopy classes of maps of M in the wedge $S^p \vee S^q$. The construction is as follows. The framings of V and W identify disjoint tubular neighbourhoods T of V and T' of W with $V \times D^p$ and $W \times D^q$ respectively; by projection on the second factor, one gets a differentiable map of $T \cup T'$ on the disjoint union $D^p \cup D^q$; after identification of the boundary of $D^p \cup D^q$ to one point b, one obtains a map of $T \cup T'$ on

$S^p \vee S^q$ mapping the boundary of $T \cup T'$ on b; it is extended to the whole of M by mapping the complement of $T \cup T'$ to b. Conversely, given a map of M in $S^p \vee S^q$, it is homotopic to a map f which is differentiable on $f^{-1}(S^p \vee S^q - b)$; one gets a pair (V, W) of framed submanifolds of M by taking the inverse image by f of points $x \in S^p - b$ and $x' \in S^q - b$ on which f is transverse-regular.

1.3. In § 3, we shall have to consider a framed submanifold V of M whose boundary is not contained in the boundary of M. In such a case, the boundary ∂V of V will be the framed submanifold obtained in restricting to ∂V the framing of V and in adding as last vector the normal to ∂V in V pointing outside V. Notice that if $M = S^n$, then the framed submanifold ∂V represents the trivial element of $\pi_n(S^q)$, where $q - 1 =$ codimension of V.

1.4. Let V be a framed submanifold (without boundary) of an oriented disc D^p itself embedded in S^{p+r}; then V represents an element α of $\pi_p(S_q)$, where $q =$ codimension of V in D^p. If one completes the framing of V with the framing of D^p (which gives the normal orientation of D^p), one gets a framed submanifold in S^{p+r} which represents the r-fold suspension of α. Indeed, D^p is isotopic to a disc linearly embedded in S^{p+r} and we can apply 1.4 of [4].

1.5. Let $(M, \mathfrak{D})$ be a framed submanifold of S^p representing an element $\alpha \in \pi_p(S^q)$, We can identify a tubular neighbourhood T of M with $M \times D^q$ in such a way that M, as a framed submanifold, is identified with $f^{-1}(0)$, where f is the projection $M \times D^q \to D^q$ and 0 is the origin of the unit disk D^q. On the other hand, let N be a submanifold contained in the interior of D^q with a framing $\mathfrak{F}$ representing an element $\beta \in \pi_q(S^r)$. Then the framed submanifold $M \times N \subset M \times D^q = T \subset S^p$ with the framing $\mathfrak{D} \times \mathfrak{F}$ represents the composition $\beta \circ \alpha$.

1.6. Let M be a submanifold with a framing $\mathfrak{F}$ in the interior of D^q representing an element $\alpha \in \pi_p(S^i)$. Similarly, let $N \subset D^q$ be a submanifold with a framing $\mathfrak{G}$ representing an element β of $\pi_q(S^j)$. Then the framed submanifold $M \times N$ with the framing $\mathfrak{F} \times \mathfrak{G}$ represents the element

$$(-1)^{iq} E^i \beta \circ E^q \alpha = E^j \alpha \circ (-1)^{pq} E^p \beta,$$

where E^s denotes the s-fold suspension hormomorphism. This follows from 1.4 and 1.5.

1.7. We embed $I \times S^n$ in S^{n+1} by the formula

$$\eta : (t, x_0, \ldots, x_n) \to (\vartheta, \alpha x_0, \ldots, \alpha x_n)$$

where $\vartheta = 1/4\,(t - 1/2)$ and $\alpha = (1 - \vartheta^2)^{1/2}$.

By this embedding of degree $+1$, $I \times S^n$ will often be implicitely considered as a subspace of S^{n+1}.

1.8. We shall adopt the original definition of J. H. C. WHITEHEAD for his product (see for example [3]).

1.9. Suppose that S^p, S^q are two oriented spheres differentiably and disjointly embedded in S^{n+1} with $n - 1 > \max(p, q)$. Then $S^{n+1} - S^p$ has the same homotopy type as S^{n-p}. We fix a homotopy equivalence using a map $j : S^{n-p} \to S^{n+1} - S^p$ such that the linking number of $j(S^{n-p})$ with S^p is $+1$.

2. Construction of the HILTON-HOPF invariant

2.1. Let α be an element of $\pi_n(S^{n-p} \vee S^{n-q})$; let $f : S^n \to S^{n-p} \vee S^{n-q}$ be a representative which is differentiable on $f^{-1}(S^{n-p} \vee S^{n-q} - b)$. Taking the inverse image of two regular values, $x \in S^{n-p} - b$, $y \in S^{n-q} - b$ (as above) we get a pair (M^p, M^q) of disjoint framed submanifolds of S^n. Let V^{p+1} be a framed submanifold in $I \times S^n$ with boundary $M^p = 0 \times M^p$ in $0 \times S^n$ and N^p in $1 \times S^n$; similarly let V^{q+1} be a framed submanifold in $I \times S^n$ with boundary $M^q = 0 \times M^q$ in $0 \times S^n$ and N^q in $1 \times S^n$. We suppose in addition that N^p and N^q are separated in $1 \times S^n$ by an equator and that V^{p+1} meets V^{q+1} transversally. Such V^{p+1} and V^{q+1} always exist; for instance, one can get them by moving M^p and M^q, as t varies from 0 to 1, by an isotopy to push them finally into opposite hemispheres of S^n. Then $W = V^{p+1} \cap V^{q+1}$ is a framed closed submanifold of $I \times S^n \subset S^{n+1}$ and we may apply the PONTRJAGIN-TOM construction to get an element

$$\tau(M^p, M^q) \in \pi_{n+1}(S^{2n-p-q}).$$

In some sense, this element measures how much M^p and M^q are linked in S^n.

Lemma 2.2. *The element $\tau(M^p, M^q)$ depends only on the cobordism class of the pair (M^p, M^q) and yields a homomorphism h' of $\pi_n(S^{n-p} \vee S^{n-q})$ into $\pi_{n+1}(S^{2n-p-q})$.*

Proof. To prove the first assertion amounts to showing that if M^p and M^q are separated by an equator to begin with, then $\tau(M^p, M^q) = 0$. Indeed, suppose $\tilde{M}^p, \tilde{M}^q$ is a pair of submanifolds of S^n cobordant (by the pair Q^{p+1}, Q^{q+1} say) to M^p, M^q and that $\tilde{V}^{p+1}, \tilde{V}^{q+1} \subset I \times S^n$ are two candidates for use in the construction of $\tau(\tilde{M}^p, \tilde{M}^q)$. Similarly let $V^{p+1}, V^{q+1} \subset I \times S^n$ be candidates for $\tau(M^p, M^q)$. We thus have three pairs of submanifolds of $I \times S^n$. Paste these together (with the first pair in the middle) across the faces where they agree. We arrive at the situation mentioned in the first line.

By a rotation, arrange that some separating equator of N^p and N^q lies vertically above a separating equator for M^p, M^q; and that M^p and N^p lie on the same side of these equators. Let $V^{p+1} \cap V^{q+1} = W$. Place $I \times S^n$ in $I \times I \times S^n$ as $0 \times I \times S^n$, and pull V^{p+1}, V^{q+1} apart in $I \times I \times S^n$ so that, if one regards the last parameter as time, M^p, M^q, N^p and N^q remain fixed throughout and at the end V^{p+1}, V^{q+1} are separated by an equator in S^{n+2}. This presents W as the boundary of a framed manifold. Hence $\tau(M^p, M^q) = 0$. The last assertion follows from the additive property of the PONTRJAGIN-THOM construction.

We now compare this homomorphism $h' : \pi_n(S^i \vee S^j) \to \pi_{n+1}(S^{i+j})$ with the homomorphism $h : \pi_n(S^i \vee S^j) \to \pi_n(S^{i+j-1})$ given by the HILTON-HOPF invariant.

Proposition 2.3. If $\alpha \in \pi_r(S^i \vee S^j)$ then $h'(\alpha) = (-1)^{r+i+j} Eh(\alpha)$.

Proof. Let ι_1, ι_2 denote the classes of the inclusion of S^i in $S^i \vee S^j$, and of S^j in $S^i \vee S^j$, respectively. Suppose that $\alpha \in \pi_r(S^i \vee S^j)$, and that ι_ω is a basic WHITEHEAD product in ι_1, ι_2 with m entries of ι_1 and n entries of ι_2. By HILTON's decomposition (see 6.1 of [3]) there exist elements

$$\alpha_\omega \in \pi_r(S^{m(i-1)+n(j-1)+1})$$

such that

$$\alpha = \iota_1 \circ \alpha_1 + \iota_2 \circ \alpha_2 + \sum_\omega \iota_\omega \circ \alpha_\omega, \tag{2.4}$$

where ω runs over the basic WHITEHEAD products of weight ≥ 2. We shall prove the proposition by evaluating h' on each component of this decomposition. (We regard HILTON's invariant as being defined with respect to the product

$[\iota_1, \iota_2]$.) Consider the Whitehead product $[\alpha, \beta]$ where $\alpha \in \pi_p(S^i \vee S^j)$, $\beta \in \pi_q(S^i \vee S^j)$. Let $f: (D^p, S^{p-1}) \to (S^i \vee S^j, a)$ and $g: (D^q, S^{q-1}) \to (S^i \vee S^j, a)$ be representatives for α and β which are differentiable, except at the inverse image of the base-point a. Then the following map, h, of $\partial(D^p \times D^q) = D^p \times S^{q-1} \cup S^{p-1} \times D^q$ into $S^i \vee S^j$ defined by

$$h(u, v) = \begin{cases} f(u); & u \in D^p,\ v \in S^{q-1} \\ g(v); & u \in S^{p-1},\ v \in D^q, \end{cases}$$

is a representative for $[\alpha, \beta]$ which is differentiable except at $h^{-1}(a)$. (Here $D^p \times D^q$ has the product orientation and $S^{p-1} \times D^q \cup D^p \times S^{q-1}$ is oriented as the boundary.) Suppose $x \in S^i - a$, $y \in S^j - a$ and that $f^{-1}(x) = M_1$, $f^{-1}(y) = M_2$, $g^{-1}(x) = N_1$, $g^{-1}(y) = N_2$; so that α is represented by the pair (M_1, M_2) of disjoint framed submanifolds of D^p and β is represented by the pair $(N_1, N_2) \subset D^q$. Then $[\alpha, \beta]$ is represented by the pair

$$(M_1 \times S^{q-1} \cup S^{p-1} \times N_1,\ M_2 \times S^{q-1} \cup S^{p-1} \times N_2) \tag{2.5}$$

of framed submanifolds of $\partial(D^p \times D^q)$.

(i) First we calculate the value of h' on the Whitehead product $[\iota_1, \iota_2] \in \pi_{i+j-1}(S^i \vee S^j)$. Let $\psi_1: (D^i, S^{i-1}) \to (S^i \vee S^j, a)$ denote the composition of a relative diffeomorphism of degree $+1$, $\bar{\psi}_1: (D^i, S^{i-1}) \to (S^i, a)$, and the natural inclusion of S^i in $S^i \vee S^j$. (Define $\psi_2: (D^j, S^{j-1}) \to (S^i \vee S^j, a)$ similarly.) Then ψ_1 represents ι_1, and $\psi_1^{-1}(x) =$ a point, x' say, and $\psi_1^{-1}(y)$ is void. Similarly $\psi_2^{-1}(y) =$ a point, y' say, and $\psi_2^{-1}(x) = \varnothing$. Hence, by 2.5, $[\iota_1, \iota_2]$ is represented by the pair $(x' \times S^{j-1}, S^{i-1} \times y')$ in $\partial(D^i \times D^j)$. To compute $h'([\iota_1, \iota_2])$ we may use the framed submanifolds $U_1 = x' \times D^j$ and $U_2 = D^i \times y'$. $U_1 \cap U_2 = x' \times y'$, a point, and so $h'([\iota_1, \iota_2]) = \pm 1$, depending on the orientation of the field at $x' \times y'$. Now as x' has a frame $\mathfrak{F}$ which gives the positive orientation of D^i, $x' \times S^{j-1}$ and $U_1 = x' \times D^j$ have framings which, at the point $x' \times y'$ determine the positive orientation of $D^i \times y'$. Similarly for $S^{i-1} \times y'$ and $U_2 = D^i \times y'$; where $\mathfrak{G}$ is the frame of y'. Hence the framing of $x' \times y'$ is, by convention, $\mathfrak{F} \times \mathfrak{G}$ which determines the positive orientation of $D^i \times D^j$. Hence $h'([\iota_1, \iota_2]) = -1$.

(ii) We now show that $h'(\iota_\omega) = 0$ if ι_ω is any basic Whitehead product other then $[\iota_1, \iota_2]$. Clearly $h'(\iota_1) = 0 = h'(\iota_2)$, so we may concern ourselves with Whitehead products of weight greater than 2. If $[\alpha, \beta]$ is such a product, then either $\alpha_1 = 0 = \alpha_2$ or $\beta_1 = 0 = \beta_2$. We may suppose the former, and we shall show more generally that if $\alpha \in \pi_p(S^i \vee S^j)$, $\beta \in \pi_q(S^i \vee S^j)$ and

$\alpha_1 = 0 = \alpha_2$ then $h'([\alpha, \beta]) = 0$. Let (M_1, M_2) be a pair of framed submanifolds of D^p representing α: similarly, let $(N_1, N_2) \subset D^q$ represent β. Since $\alpha_1 = 0 = \alpha_2$, $M_i = \partial V_i = V_i \cap D^p \times 0$ where V_i is a framed submanifold of $D^p \times [0, \varepsilon]$ and $V_i \cap D^p \times \varepsilon = \varnothing$, $i = 1, 2$. And we may arrange that V_1 and V_2 intersect transversally in W. Let $U_i = \xi(V_i \times S^{q-1}) \subset \subset D^p \times D^q (i = 1, 2)$ where ξ is the embedding of $D^p \times [0, \varepsilon] \times S^{q-1}$ in $D^p \times D^q$ defined by $\xi(x, t, y) = (x, (1-t)y)$. Now N_1, N_2 are closed manifolds and lie in the interior of D^q. Hence we may suppose that ε is so small that

$$\xi(D^p \times [0, \varepsilon] \times S^{q-1}) \cap D^p \times N_i = \varnothing \qquad (i = 1, 2).$$

We may then use the submanifolds $Q_i = U_i \cup D^p \times N_i (i = 1, 2)$ of $D^p \times D^q$ to construct $h'([\alpha, \beta])$. Clearly $X = Q_1 \cap Q_2 = U_1 \cap U_2 = \xi(W \times S^{q-1})$. But in $D^p \times D^q \times I$, X bounds a framed submanifold diffeomorphic to $W \times D^q$. Hence $h'(\alpha, \beta]) = 0$.

(iii) Finally we show that if $\varphi \in \pi_r(S^i \vee S^j)$ and $\gamma \in \pi_p(S^r)$, then

$$h'(\varphi \circ \gamma) = (-1)^{p+r} h'(\varphi) \circ E\gamma + (-1)^{r(r+j)} E^j \varphi_1 \circ E^r \varphi_2 \circ h'(\gamma),$$

where φ_1 is the component of φ in $\pi_r(S^i)$ and φ_2 is the component of φ in $\pi_r(S^j)$. (Here $h'(\gamma)$ denotes $h'(\varDelta \circ \gamma)$, where $\varDelta : S^r \to S^r \vee S^r$ is the canonical pinching map which shrinks the equator to one point.)

This formula, together with (i), (ii), and 2.4 will prove 2.3.

Let $M_1, M_2 \subset S^r$ be two disjoint framed submanifolds of S^r which represent φ and let $P_1, P_2 \subset I \times S^r$ be two framed submanifolds, constructed as in 2.1, of which the intersection P represents $h'(\varphi)$. The framed submanifolds $M'_k = P_k \cap (1 \times S^r)$, $k = 1, 2$, of $1 \times S^r$ are contained in two disjoint discs D^r_1 and $D^r_2 \subset 1 \times S^r$ which we may take as small as we please. Moreover, if $i, j > 1$, as we suppose, we may further arrange that $P_k \cap I \times a = \varnothing$ $(k = 1, 2)$ and that $1 \times a \notin D^r_1 \cup D^r_2$, where $a \in S^r$ is the base-point.

Let $g' : S^p \to S^r$ be a map representing γ and obtained by applying the Pontrjagin-Thom construction to a framed submanifold $N \subset S^p$. Define $\bar{g} : I \times S^p \to I \times S^r$ by $\bar{g}(t, x) = (t, g'(x))$. Then $\bar{g}$ is transverse-regular to D^r_1, D^r_2, M_1, M_2. Approximate $\bar{g}$ by g, where g agrees with $\bar{g}$ in a neighbourhood of the boundary and is transversal to P_1, P_2. The framed submanifold $g^{-1}(P)$ represents $(-1)^{p+r} h'(\varphi) \circ E\gamma$.

Now $g_k = g \mid k \times S^p = k \times g'$, $k = 0, 1$, and $g_0^{-1}(M_1)$ and $g_0^{-1}(M_2)$ represent $\varphi \circ \gamma$. To construct $h'(\varphi \circ \gamma)$, we proceed in two steps. First we consider the framed submanifolds $g^{-1}(P_1)$ and $g^{-1}(P_2)$ in $I \times S^p$ of which the intersection is $g^{-1}(P)$: if $g_1^{-1}(M'_1)$ and $g_1^{-1}(M'_2)$ were separated by an equator in S^p, then $g^{-1}(P)$ would represent $h'(\varphi \circ \gamma)$. But this will not be the case in general if $p \geqslant 2r - 1$. Indeed if x_1 and x_2 are points of M_1' and M_2', the framed submanifolds $N_1 = g_1^{-1}(x_1)$ and $N_2 = g_1^{-1}(x_2)$ may be linked in S^p.

Let Q_1 and Q_2 be two framed submanifolds in $[1, 2] \times S^p$ such that $\partial Q_i = N_i \cup N_i'$ where $N_i \subset 1 \times S^p$, $N_i' \subset 2 \times S^p (i = 1, 2)$ and N_1' and N_2' are separated by an equator in $2 \times S^p$. Then $Q = Q_1 \cap Q_2$ represents $h'(\gamma)$. Now using the framings of Q_1 and Q_2, we can construct tubular neighbourhoods $T_1 \approx Q_1 \times D_1^r$ and $T_2 \approx Q_2 \times D_2^r$ of Q_1 and Q_2 in $[1, 2] \times S^p$ such that:

(a) $T_i \cap (1 \times S^p) = N_i \times D_i^r$ and the natural projection $N_i \times D_i^r \to D_i^r$ is just the restriction of g_1 to $N_1 \times D_i^r$, $i = 1, 2$.

(b) $T_1 \cap (2 \times S^p)$ and $T_2 \cap (2 \times S^p)$ are separated by an equator in $2 \times S^p$.

(c) $T = T_1 \cap T_2$ is diffeomorphic to $Q \times D_1^r \times D_2^r$, where under this diffeomorphism $T_1 \cap Q_2$ maps into $Q \times D_1^r \times 0$ and $T_2 \cap Q_1$ onto $Q \times 0 \times D_2^r$.

That (a) can be satisfied follows from our choice of representive, g, for γ: to see that (c) is possible is a little more difficult. It may be proved using the tubular neighbourhood theorem of J. MILNOR. From (a) it follows that $g_1^{-1}(M_i') = N_i \times M_i' \subset N_i \times D_i^r \subset 1 \times S^p$, $i = 1, 2$. The element $h'(\varphi \circ \gamma)$ will be represented by the union of $g^{-1}(P)$ and the framed submanifold

$$(Q_1 \times M_1') \cap (Q_2 \times M_2') = Q \times M_1' \times M_2' \text{ by (c)}.$$

Let $\mathfrak{F}_i$ be the framing of M_i' in $1 \times S^r$, let $\mathfrak{Q}_i$ be the framing of Q_i in $[1,2] \times S^p$; and write $\mathfrak{Q}_i = \mathfrak{R}_i \times \mathfrak{F}_i : i = 1, 2$. Then Q with the framing $\mathfrak{Q}_1 \times \mathfrak{Q}_2$ represents $h'(\gamma)$. And the representative map goes from S^{p+1}, with orientation determined by that of the subspace $[1, 2] \times S^p$, into S^{2r} with orientation that determined by the field $\mathfrak{Q}_1 \times \mathfrak{Q}_2$. By 1.4, the submanifold $M_2' \subset S^{2r}$, where M_2' has framing $\mathfrak{F}_2 \times \mathfrak{Q}_1$ and S^{2r} has orientation given by $\mathfrak{Q}_2 \times \mathfrak{Q}_1$, will represent $E^r \varphi_2$. Hence when S^{2r} has orientation given by $\mathfrak{Q}_1 \times \mathfrak{Q}_2$, M_2' with the framing $\mathfrak{Q}_1 \times \mathfrak{F}_2$ will represent $(-1)^{r(r+j)} E^r \varphi_2$. Again by 1.4, if S^{r+j} has orientation given by $\mathfrak{Q}_1 \times \mathfrak{F}_2$ then M_1' with framing $\mathfrak{F}_1 \times \mathfrak{F}_2$ will represent $E^j \varphi_1$. Thus the framed submanifold $Q \times M_1' \times M_2'$ in $[1, 2] \times S^p$ represents (by 1.5, 1.6)

$$(-1)^{r(r+j)} E^j \varphi_1 \circ E^r \varphi_2 \circ h'(\gamma).$$

The result now follows by the additivity of the PONTRJAGIN-THOM construction.

2.6. We can arrange that proposition 2.3 is much neater by using throughout either the homotopy convention or the homology convention, instead of using them both, each one in its own context. If one chooses the homology orientation convention, that is, if $M = \partial V$, the orientation of $V =$ the outward normal $+$ the orientation of M, then to be consistent, one must redefine suspension by placing the suspension parameter first, that is, as the first coordinate. Then if $\tilde{E}$ denotes this suspension homomorphism and if $\alpha \in \pi_p(S^r)$, $\tilde{E}\alpha = (-1)^{p+r}E\alpha$. Clearly, then, $h' = -\tilde{E}h$.

If, alternatively, one adopts the homotopy orientation convention throughout, that is, the orientation of $V =$ the orientation of $M +$ the outward normal, then one must, in order to be consistent, write $S^r \times I$ instead of $I \times S^r$, and one must change the convention for WHITEHEAD products in the way that W. D. BARCUS and M. G. BARRATT do in their paper '*On the homotopy classification of extensions of a fixed map*' (Trans. A. M. S. 88, 1958, pp. 57–74). In this case, if $\alpha \in \pi_p(X)$ and $\beta \in \pi_q(X)$ and if $[\alpha, \beta]'$ is the product defined with respect to the homotopy convention, $[\alpha, \beta]' = (-1)^{p+q-1}[\alpha, \beta]$. And with h and h' redefined according to this convention, proposition 2.3 again reads $h' = -Eh$.

3. Proof of the theorem. A sphere S^p differentiably embedded in S^{n+1} is h-cobordant to zero (see [1]) if S^p bounds in the $(n+2)$-disk D^{n+2} a contractible submanifold D^{p+1} (homotopy $(p+1)$-disk). A 2-link formed by two disjointly embedded spheres S^p and S^q in S^{n+1} is h-cobordant to zero (cf. [2]) if S^p and S^q bound in D^{n+2} two disjoint contractible submanifolds D^{p+1} and D^{q+1}.

In that case, let T_p, T_q be tubular neighbourhoods of D^{p+1}, D^{q+1} in D^{n+2} which touch at one point $a \in \partial D^{n+2} = S^{n+1}$. Let $\dot{T}_p$, $\dot{T}_q$ denote the sphere bundles over D^{p+1}, D^{q+1} which are the boundaries of T_p and T_q. As bundles they are trivialized by the framings. To have a definite homotopy-equivalence between $D^{n+2} - (D^{p+1} \cup D^{q+1})$ and $S^{n-p} \vee S^{n-q}$ we must choose definite framings. We choose one which, for each disc, agrees with the convention of 1.9. Let S^{n-p}, S^{n-q} be the fibres of $\dot{T}_p$, $\dot{T}_q$ which contain a. Map $T_p - D^{p+1}$ onto $\dot{T}_p$ by collapsing radially; now use the framing to map the whole of $\dot{T}_p$ onto the fibre S^{n-p}. Call this map $\bar{\varphi}_p$ and let $\bar{\varphi}_q : T_q - D^{q+1} \to S^{n-q}$ be similarly defined.

By POINCARÉ duality, the inclusion of the wedge formed by the fibres $S^{n-p} \vee S^{n-q}$ in $D^{n+2} - (D^{p+1} \cup D^{q+1})$ is an homotopy equivalence. Hence there is no obstruction to extending $\bar{\varphi}_p \cup \bar{\varphi}_q$ to a map $\bar{\varphi} : D^{n+2} - (D^{p+1} \cup D^{q+1}) \to$ $\to S^{n-p} \vee S^{n-q}$. Moreover, as $\bar{\varphi}_p$ and $\bar{\varphi}_q$ are differentiable, we may suppose $\bar{\varphi}$ to be differentiable (except on a). Let φ be the restriction of $\bar{\varphi}$ to $S^{n+1} -$ $- (S^p \cup S^q)$. It is a $(n-1)$-homotopy equivalence. If $x \in S^{n-p} - a$, and $y \in S^{n-q} - a$ are regular values for φ, $\varphi^{-1}(x)$ and $\varphi^{-1}(y)$ are disjoint open framed submanifolds and

$$V_q^{p+1} = \varphi^{-1}(x) \cup S^p \text{ and } V_p^{q+1} = \varphi^{-1}(y) \cup S^q$$

are disjoint compact framed submanifolds in S^{n+1} with boundaries S^p and S^q (see 1.3). We have thus proved the following lemma.

Lemma 3.1. *Let S^p, S^q be two disjoint differentiable spheres in S^{n+1} such that S^p, S^q is h-cobordant to zero. There exist disjoint bounded framed submanifolds V_q^{p+1}, V_p^{q+1} in S^{n+1} such that $\partial V_q^{p+1} = S^p$, $\partial V_p^{q+1} = S^q$.*

The PONTRJAGIN-THOM construction applied to the pair $V_q^{p+1} - S^p$, $V_p^{q+1} - S^q$ yields a map of $S^{n+1} - (S^p \cup S^q)$ into $S^{n-p} \vee S^{n-q}$ which is an $(n-1)$-homotopy equivalence.

3.2. Let $S^p \subset S^{n+1}$ be a sphere h-cobordant to zero in S^{n+1} and let D_0^{p+1} and D_1^{p+1} be two $(p+1)$-disks in D^{n+2} whose boundary is S^p. Using D_0^{p+1} (resp. D_1^{p+1}) we can construct as above a framed submanifold V_0^{p+1} (resp. V_1^{p+1}) whose boundary is S^p. Suppose now that D_0^{p+1} and D_1^{p+1} are h-cobordant, i. e. there exists in $I \times D^{n+2}$ an homotopy disk D^{p+2} whose boundary is the union B of $0 \times D_0^{p+1}$, $1 \times D_1^{p+1}$, and $I \times S^p$. Then there exists a framed submanifold V^{p+2} in $I \times S^{n+1}$ whose boundary is the union of $0 \times V_0^{p+1}$, $1 \times V_1^{p+1}$ and $I \times S^p$.

If D_0^{p+1} and D_1^{p+1} are not h-cobordant, a modification of D_1^{p+1} in an arbitrary small neighbourhood of one of its points will make D_1^{p+1} h-cobordant to D_0^{p+1}. Indeed it is sufficient to replace (D^{n+2}, D_1^{p+1}) by its connected sum with the pair $-(\partial(I \times D^{n+2}), B)$.

3.3. Now let $L = (S^{p_1}, S^{p_2}, S^{p_3})$ be a 3-link in S^{n+1} with $n-1 > \max(p_1, p_2, p_3)$ as always. Let i, j, k be a permutation of p_1, p_2, p_3. Denote by L_i the 3-link obtained in dropping the component S^i in L and replacing it by the boundary of an $(i+1)$-disk which does not intersect the two other components S^j and S^k. The inverse $-L_i$ of L_i is the symmetrical of L_i with respect to reflection in an equator of S^{n+1} (see [2]).

Let Λ be the 3-link which is the sum of L, $-L_{p_1}$, $-L_{p_2}$, and $-L_{p_3}$. The linking elements λ_{ij}^k of L and Λ are the same because they vanish for each L_i; moreover each 2-sublink of Λ is h-cobordant to zero. Hence it is sufficient to prove the theorem when each 2-sublink of L is h-cobordant to zero. From now on we assume this.

According to lemma 3.1, for any permutation (i, j, k) of (p_1, p_2, p_3), one can construct framed submanifolds V_k^{j+1} and V_j^{k+1} in S^{n+1} such that

$$\partial V_k^{j+1} = S^j \text{ and } V_k^{j+1} \cap V_j^{k+1} = \varnothing .$$

Let W_{jk}^i be a framed submanifold of $I \times S^{n+1}$ such that $\partial W_{jk}^i = I \times S^i \cup 0 \times V_j^{i+1} \cup 1 \times V_k^{i+1}$. The existence of such W_{jk}^i is assured by 3.2. (Having defined W^i_{jk} for a positive permutation (i, j, k) of (p_1, p_2, p_3), we could define W^i_{kj} to be the inverse image of W^i_{jk} in $I \times S^{n+1}$ under the orientation-reversing homeomorphism $(t, x) \to (1 - t, x), x \in S^{n+1}, t \in I$.) Denote $V^{k+1}_j \cap S^i$ by M^i_k.

Lemma 3.4. $\tau(M^i_j, M^i_k) = (-1)^{i+j+k} E \lambda^i_{jk}$.

Proof. Consider the following pair of manifolds in $I \times S^{n+1}$.

$$Q = W^i_{jk} \cap (S^i \times I), \ I \times M^i_k.$$

It is clear that Q and $I \times M^i_k$ are two manifolds which qualify for use in the definition of $\tau(M^i_j, M^i_k)$, since $\partial Q = M^i_j$. So $Q \cap (I \times M^i_k)$ is a closed framed submanifold of $I \times S^i \subset I \times S^{n+1}$ which represents $\tau(M^i_j, M^i_k)$. Now if

$$\varphi : S^{n+1} - (S^i \cup S^k) \to S^{n-j} \vee S^{n-k}$$

is the map of 3.1, $\varphi \mid S^i : S^i \to S^{n-j} \vee S^{n-k}$ is a representative for λ^i. By definition $V^{j+1}_k = \varphi^{-1}(x)$ and $V^{k+1}_j = \varphi^{-1}(y)$ for some regular values $x \in S^{n-j} - b$, $y \in S^{n-k} - b$. So

$$(\varphi \mid S^i)^{-1}(x) = M^i_j, \ (\varphi \mid S^i)^{-1}(y) = M^i_k.$$

Hence by lemma 2.3, $\tau(M^i_j, M^i_k) = (-1)^{i+j+k} E \lambda^i_{jk}$.

We wish to prove symmetry. First notice that by 2.1 we could have used the pairs $[I \times M^i_j, W^k_{ji} \cap (I \times S^i)]$ or the pair $[W^j_{ki} \cap (I \times S^i), W^k_{ji} \cap (I \times S^i)]$ instead of $[Q, I \times M^i_k]$ to define $\tau(M^i_j, M^i_k)$.

Let $T = W^k_{ij} \cap (I \times V^{i+1}_k) \cap (I \times V^{j+1}_k)$. It is a framed submanifold of $I \times S^{n+1}$, under the conventions of 1.1 and $\partial T = A \cup B$ where

$$A = W^k_{ji} \cap I \times S^i \quad \cap I \times V^{j+1}_k = W^k_{ji} \cap I \times M^i_j$$

$$B = W^k_{ji} \cap I \times V^{i+1}_k \cap I \times S^j \quad = W^k_{ji} \cap I \times M^j_i$$

and this time we break 1.1 and suppose that A, B are framed according to convention 1.3. If we write A_1, B_1 for the manifolds A, B reframed according to the convention of 1.1, and if $\nu(M^p) \in \pi_q(S^{q-p})$ denotes the element obtained by applying the Pontrjagin-Thom construction to the framed submanifold $M^p \subset S^q$, then

$$\nu(A_1) = (-1)^{n+k} \nu(A), \ \nu(B_1) = (-1)^{k+i} \nu(B).$$

Moreover, because $A \cup B = \partial T$, $\nu(A) = \nu(B)$; and by lemma 3.4,

$$\nu(A_1) = (-1)^{(n+j)(n+i+1)} E^{n-i+1} \tau(M_k^i, M_j^i) = (-1)^{(n+j)(n+i+1)+i+j+k} E^{n-i+2} \lambda_{kj}^i,$$

$$\nu(B_1) = E^{n-j+1} \tau(M_k^j, M_i^j) = (-1)^{i+j+k} E^{n-j+2} \lambda_{ki}^j .$$

But $E \lambda_{jk}^i = (-1)^{(n+j)(n+k)} E \lambda_{jk}^i$; hence

$$E^{n-i+2} \lambda_{jk}^i = (-1)^{(n+j)(i+k)+i+j} E^{n-j+2} \lambda_{ki}^j .$$

The theorem is proved.

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