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A generalization of the homology and homotopy suspension[1])

By T. Ganea

Introduction

Let $p: E \to B$ be a fibre map with fibre $F = p^{-1}(*)$, where $*$ is the base-point in B, let $E \cup CF$ result by erecting a cone over the subset F of E, and let $r: E \cup CF \to B$ extend p by mapping CF to the base-point. We may convert r into a homotopy equivalent fibre map, and our first result asserts that the fibre of r has the homotopy type of the join $F * \Omega B$. This yields a new proof of a theorem of Serre and enables us to generalize most of the classical results [25], [2] concerning the homology suspension; the latter occurs upon taking for p the standard fibration of the space of paths in B. Dually [6], let $d: A \to X$ be a cofibre (inclusion) map with cofibre B obtained by shrinking the subset A of X to a point, let $f: X \to B$ be the identification map, let F be the fibre of f, and let $e: A \to F$ lift d. In view of the above result, duality suggests that the homotopy type of the cofibre C_e of e is determined by those of B and ΣA. However, this turns out to be false and the main results of the third and fourth section only yield a description of C_e in low dimensions; specifically, with F_d standing for the fibre of d, there are maps $A \# F_d \to C_e \to \Omega(\Sigma A \flat B)$ which are $(m + n - 1 + \mathrm{Min}(m, n))$-connected in case A is $(n-1)$-connected and (X, A) is m-connected. This enables us to generalize for arbitrary cofibrations the well known EHP-sequence of G. W. Whitehead [24] which, in the classical case, arises upon taking for d the inclusion of A in the cone CA. The first homomorphism in the generalized sequence is induced by e, the second is related to a certain generalization of the Hopf invariant, and the third is given by a generalized Whitehead product.

The present generalizations can be used to study iterated fibrations or cofibrations. Starting, e. g., with a certain cofibration $A \to X \to B$, we obtain a second one $A \to F \to C_e$ which, in turn, yields a third, and so on; at each stage our results yield relations between a cofibration and the next. The last section of the paper studies this process with $A \to CA \to \Sigma A$ as original cofibration. There results a sequence of spaces and maps

$$A \to \ldots \to F_{k+1} \to F_k \to \ldots \to F_1 \to CA,$$

which is functorial in A and in which F_1 is equivalent to $\Omega\Sigma$. The sequence is used to solve some problems concerning the dual of Lusternik-Schnirelmann

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category, and spaces of finite cocategory appear as generalizations of H-spaces in a way similar to that in which nilpotent groups generalize the Abelian ones.

The above sequence gives rise to a spectral sequence, and many of the results in [26] and [11] can be dualized; in particular, the HOPF invariant of a cofibration described in § 4 readily yields the geometric interpretation of the first differential, as does the HOPF construction of a fibration in 1.4 for the dual case. However, we have not yet obtained all the relevant results (e. g. the dual of Lemma 2.2 in [11]), and omit details here.

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1. Extending fibrations

All spaces in this paper are provided with a base-point generally denoted by $*$, and all maps and homotopies are assumed to preserve base-points. A triple $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration if p has the covering homotopy property for any space and $F = p^{-1}(*)$; i is the inclusion map. Any map f can be converted into a homotopy equivalent fibre map p yielding the diagram

$$\begin{tikzcd} & & X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[d, equal] \\ \Omega Y \arrow[r, "\partial"] & F \arrow[ur, "j"] \arrow[r, "i"] & E \arrow[r, "p"] & Y \end{tikzcd} \tag{1}$$

in which

$$\left.\begin{aligned} E &= \{(x, \eta) \in X \times Y^I \mid f(x) = \eta(1)\},\ p(x, \eta) = \eta(0), \\ F &= p^{-1}(*) \subset X \times PY,\ i = \text{inclusion map}, \\ \partial(\omega) &= (*, \omega),\ j(x, \eta) = x,\ h(x) = (x, \eta_x). \end{aligned}\right\} \tag{2}$$

PY is the space of paths in Y emanating from $*$, ΩY is the loop space, $\eta_x(s) = f(x)$ for all $s \in I$, and h is a homotopy equivalence satisfying $p \circ h = f$ and $h \circ j \simeq i$. The triple $\Omega Y \to F \to X$ is the fibration induced by f from $\Omega Y \to PY \to Y$. We shall call F the fibre of f and sometimes denote it by F_f, noting that no real ambiguity occurs since the map $f^{-1}(*) \to F$ defined by h is a homotopy equivalence in case f is already a fibre map. Next, we may embed Y in the space $Y \cup_f CX$ obtained by attaching to Y the non-reduced cone over X by means of f. The subscript f will frequently be omitted; the points of CX are denoted by sx, the base-point is $1*$, and X is embedded in

CX by $x \to 1x$. The reduced cone C_0X may equally well be used, yielding the cofibre C_f of f. The identification map $\sigma: Y \cup CX \to \Sigma X$ shrinks the subset $Y \cup I*$ (resp. Y if we use the reduced cone) to the base-point and yields the reduced suspension of X, with points denoted by $\langle s, x\rangle$. The join $X*Y$ is taken as a quotient space of $X \times I \times Y$; its points are denoted by $(1-s)x + sy$ and the base-point is $\frac{1}{2}* + \frac{1}{2}*$.

Theorem 1.1. *Let* $\mathcal{F}: F \xrightarrow{i} E \xrightarrow{p} B$ *be a fibration in which* B *has the homotopy type of a* CW*-complex. Let* $r: E \cup CF \to B$ *extend* p *by mapping* CF *to the base-point and let* F_r *be the fibre of* r*. Then, there exists a weak homotopy equivalence* $w: F*\Omega B \to F_r$.

Proof. Since $r \mid E = p$ and $r(CF) = *$, by (2) one has

$$F_r = \{(a, \beta) \in E \times PB \mid p(a) = \beta(1)\} \cup (CF \times \Omega B).$$

We shall define w as the composite of three maps of which the first results by halving the join, the second is given by an extension $C\Omega B \to PB$ of the identity map of ΩB, and the third is suggested by the translation of fibres along paths in the base. Let $\lambda: \{(a, \beta) \in E \times B^I \mid p(a) = \beta(0)\} \to E^I$ be a lifting map for $\mathcal{F}$; λ assigns to any path β in B and any $a \in E$ lying over $\beta(0)$ a path in E over β starting at a [8]. For any path ξ, let ξ_s and $-\xi$ be given by $\xi_s(t) = \xi(st)$ and $-\xi(t) = \xi(1-t)$. Let w be the composite

$$F*\Omega B \xrightarrow{w_1} F \times C\Omega B \cup CF \times \Omega B \xrightarrow{w_2} F \times PB \cup CF \times \Omega B \xrightarrow{w_3} F_r$$

in which the last three spaces are subspaces of $CF \times C\Omega B$, $CF \times PB$, and $(E \cup CF) \times PB$ respectively, and

$$w_1((1-s)x + s\omega) = (\mathrm{Min}\,(1, 2-2s)x, \mathrm{Min}\,(2s, 1)\omega),$$

$$w_2(sx, t\omega) = (sx, \omega_t) \text{ for } (1-s)(1-t) = 0,$$

$$w_3(sx, \beta) = (s\lambda(x, \beta)(1), \beta).$$

Since its composite with the identification map $F \times I \times \Omega B \to F*\Omega B$ is continuous, so is w. Similarly, w_1 is continuous; it is also bijective and the composite of its inverse w_1^{-1} with any map of a compact Hausdorff space is continuous. This is enough for w_1 to be a weak homotopy equivalence. Next, since B has the homotopy type of a CW-complex the domain and range of the map $\varepsilon: (C\Omega B, \Omega B) \to (PB, \Omega B)$, given by $\varepsilon(t\omega) = \omega_t$, have the homotopy type of CW-pairs [16]; this, the free contractibility of $C\Omega B$ and PB, and the relation $\varepsilon(\omega) = \omega$, readily imply that ε is a homotopy equivalence of pairs.

Therefore, w_2 is a homotopy equivalence. In order to discuss w_3, recall [8] that there is a homotopy $H_t: E^I \to E^I$ with

$$H_0[\alpha] = \lambda(\alpha(0), p\circ\alpha), H_1 = 1, p\circ H_t[\alpha] = p\circ\alpha, \tag{3}$$

and define functions

$$F \times PB \cup CF \times \Omega B \xrightarrow{\varphi_t} F \times PB \cup CF \times \Omega B \xleftarrow{v} F_r \xrightarrow{\psi_t} F_r$$

by

$$\varphi_t(sx, \beta) = (sH_t[-\lambda(x, \beta)](1), \beta),$$
$$v(sa, \beta) = (s\lambda(a, -\beta)(1), \beta),$$
$$\psi_t(sa, \beta) = (sH_t[-\lambda(a, -\beta)](1), \beta).$$

Then $\varphi_0 = v\circ w_3$, $\varphi_1 = 1$, $\psi_0 = w_3\circ v$, $\psi_1 = 1$. However, $w_3, v, \varphi_t, \psi_t$ may fail to be continuous; nevertheless, by standard results on identification spaces, their composites with maps of compact HAUSDORFF spaces are continuous. Therefore, w_3 induces isomorphisms of homotopy groups and w is, as asserted, a weak homotopy equivalence.

Remark 1.2. We shall denote by j the composite of w with the projection $F_r \to E \cup CF$. Without altering the homotopy class of j we may replace $\lambda(x, \omega_{2s})(1)$ by $\lambda(x, \omega)(2s)$, and obtain

$$j((1-s)x + s\omega) = \begin{cases} \lambda(x, \omega)(2s) & \text{if } 0 \leq 2s \leq 1, \\ (2-2s)\lambda(x, \omega)(1) & \text{if } 1 \leq 2s \leq 2. \end{cases}$$

To interpret this result, notice that the homotopy $h_s: F \times \Omega B \to E$ given by $h_s(x, \omega) = \lambda(x, \omega)(s)$ connects $i\circ pr$ with $i\circ\varrho$, where $pr: F \times \Omega B \to F$ is the projection whereas the map $\varrho: F \times \Omega B \to F$, given by $\varrho(x, \omega) = \lambda(x, \omega)(1)$, expresses the operation of ΩB on F associated with the fibration $\mathcal{F}$[7]. Next, let $\varepsilon: E \cup CF \to E \cup C_0F$ shrink the segment $I*$ to a point, and let $j_0 = \varepsilon\circ j$, $r_0 = r\circ\varepsilon^{-1}$. We may obviously regard the triple

$$\mathcal{F}': F*\Omega B \xrightarrow{j} E \cup CF \xrightarrow{r} B \tag{4}$$

as a fibration, and the same remark applies to the triple

$$\mathcal{F}_0': F*\Omega B \xrightarrow{j_0} E \cup C_0F \xrightarrow{r_0} B \tag{5}$$

provided also F and E have the homotopy type of CW-complexes in which case ε is a homotopy equivalence. We shall, however, continue to write j and r even when using the reduced cone C_0F. The above results are closely connected with [5].

The definition of j is valid even if B fails to have the homotopy type of a CW-complex, and j satisfies a naturality law expressed by

Proposition 1.3. *Suppose the rows in the commutative diagram on the left*

$$\begin{tikzcd} F \arrow[r] \arrow[d, "f"] & E \arrow[r] \arrow[d, "g"] & B \arrow[d, "h"] \\ F' \arrow[r] & E' \arrow[r] & B' \end{tikzcd} \qquad \begin{tikzcd} F * \Omega B \arrow[r, "j"] \arrow[d, "f * \Omega h"] & E \cup CF \arrow[d, "g \cup Cf"] \\ F' * \Omega B' \arrow[r, "j'"] & E' \cup CF' \end{tikzcd}$$

are fibrations. Then, with j and j' given by 1.2, *the diagram on the right homotopy-commutes; in particular, the homotopy class of j is unaffected by the choice of a lifting map.*

The proof uses homotopies satisfying (3); we omit the details.

The natural map $V : F * \Omega B \to \Sigma(F \times \Omega B)$ shrinks to a point the two ends of the join and the segment through the base-point. The Hopf construction corresponding to the operation $\varrho : F \times \Omega B \to F$ yields the composite $\Sigma\varrho \circ V$. Define $-1 : \Sigma X \to \Sigma X$ by $-1\,\langle s, x\rangle = \langle 1 - s, x\rangle$ and $-\sigma = (-1) \circ \sigma$. Then:

Theorem 1.4. *Homotopy-commutativity holds in the diagram*

$$\begin{tikzcd} F * \Omega B \arrow[rr, "j"] \arrow[d, equal] & & E \cup CF \arrow[d, "-\sigma"] \\ F * \Omega B \arrow[r, "V"] & \Sigma(F \times \Omega B) \arrow[r, "\Sigma\varrho"] & \Sigma F \end{tikzcd}$$

Proof. The result follows easily from 1.2 noting that

$$\sigma(E) = *,\ \sigma(sx) = \langle s, x\rangle,\ V((1-s)x + s\omega) = \langle s, (x, \omega)\rangle.$$

Let $\partial : \Omega B \to F$ be given by $\partial(\omega) = \varrho(*, -\omega)$, and let $S : F \to \Omega\Sigma F$ be the natural embedding defined by $S(x)(s) = \langle s, x\rangle$.

Proposition 1.5. *There is a map Γ such that the diagram*

$$\begin{tikzcd} \Omega(E \cup CF) \arrow[d, "\Omega\sigma"] & \Omega B \arrow[l, "\Gamma"'] \arrow[d, "\partial"] \\ \Omega\Sigma F & F \arrow[l, "S"'] \end{tikzcd}$$

homotopy-commutes and $\Omega r \circ \Gamma \simeq 1$; in particular, if B has the homotopy type of a CW-complex, there is a weak homotopy equivalence $\Omega(F * \Omega B) \times \Omega B \to \Omega(E \cup CF)$ and the homotopy sequence of $F * \Omega B \xrightarrow{j} E \cup CF \xrightarrow{r} B$ splits.

Proof. The map Γ given by

$$\Gamma(\omega)(s) = \begin{cases} (1-3s)* & \text{if} \quad 0 \leq 3s \leq 1, \\ (3s-1)\lambda(*,-\omega)(1) & \text{if} \quad 1 \leq 3s \leq 2, \\ \lambda(*,-\omega)(3-3s) & \text{if} \quad 2 \leq 3s \leq 3, \end{cases}$$

is easily seen to behave as asserted. We note that, by (3), its homotopy class is unaffected by the choice of a lifting map.

The main purpose of the next result is to introduce some maps needed later; η and η_k are the obvious inclusions whereas σ, σ', and σ_k are the obvious identification maps $(k = 0, 1, 2)$.

Proposition 1.6. *There are maps ψ, ψ', and a homotopy equivalence ζ yielding homotopy-commutativity in the diagram below, where r' extends r in the obvious way.*

$$\begin{tikzcd}
F \arrow[r, "i"] & E \arrow[r, "\eta"] \arrow[dr, "p"] & E \cup C_0F \arrow[r, "\sigma"] \arrow[d, "r"] & \Sigma F \arrow[r, "\Sigma i"] \arrow[d, "\psi"] & \Sigma E \arrow[d, "-1"] \\
(E \cup C_0F) \cup_j C_0(F * \Omega B) \arrow[rr, "r'"] \arrow[d, "\sigma'"] & & B \arrow[r, "\eta_0"] \arrow[d, "\eta_1"] & C_p \arrow[r, "\sigma_0"] \arrow[d, "\eta_2"] & \Sigma E \\
\Sigma(F * \Omega B) \arrow[rr, "\psi'"] \arrow[d, "\Sigma j"] & & C_r \arrow[r, "\zeta"] \arrow[d, "\sigma_1"] & C_\psi \arrow[d, "\sigma_2"] & \\
\Sigma(E \cup C_0F) \arrow[rr, "-1"] & & \Sigma(E \cup C_0F) \arrow[r, "\Sigma\sigma"] & \Sigma^2 F &
\end{tikzcd}$$

Proof. Let $\psi\langle s, x\rangle = (1-s)i(x)$ so that $\sigma_0 \circ \psi = -\Sigma i$. A homotopy h_t connecting $\eta_0 \circ r$ with $\psi \circ \sigma$ is easily found, and ζ is defined in terms of σ, η_0, h_t as in [19; 2.2]. It is a homotopy equivalence since, upon replacing B by the mapping cylinder B' of r, ζ is converted into the natural homeomorphism $B'/E \cup C_0F \to (B'/E)\,/\,(E \cup C_0F/E)$, where X/A is the space obtained by shrinking to a point the subset A of X. The map ψ' corresponds to (5) in the way ψ corresponds to $\mathcal{F}$; since $r \circ j$ is only null-homotopic, one has

$$\psi'\langle t, (1-s)x + s\omega\rangle = \begin{cases} \omega(2t \operatorname{Min}(2s, 1)) & \text{if} \quad 0 \leq 2t \leq 1, \\ (2-2t)j((1-s)x + s\omega) & \text{if} \quad 1 \leq 2t \leq 2. \end{cases} \tag{6}$$

2. Homology properties of extended fibrations

Throughout the paper we use reduced singular homology groups over the integers, and omit the tilde to simplify notations. We first show that a well known result of SERRE [21] readily follows from 1.1 and the relative HUREWICZ theorem in the form given by J. H. C. WHITEHEAD.

Proposition 2.1. *Let $\mathcal{F}: F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration. If B is $(m-1)$-connected and F is $(n-1)$-connected, then $p_* : H_q(E, F) \to H_q(B)$ is monomorphic for $q < m+n$ and epimorphic for $q \leq m+n (m \geq 1, n \geq 0)$.*

Proof. Suppose first that B has the homotopy type of a CW-complex. The connectivity assumptions imply that $F * \Omega B$ is $(m+n-1)$-connected, and the homotopy sequence of the fibration (4) reveals that the map r is $(m+n)$-connected. To obtain the result, we apply to r the HUREWICZ-WHITEHEAD theorem ($E \cup CF$ is certainly 0-connected and $\pi_1 = 0$ is not needed to pass from homotopy to homology) and then identify $H_q(E \cup CF)$ with $H_q(E, F)$. If B fails to have the homotopy type of a CW-complex, we replace the original fibration by the one it induces on the singular polytope of B.

The next result yields a useful exact sequence and gives information on the HOPF construction $\Sigma\varrho \circ V$ associated with $\mathcal{F}$. We use the notation of 1.6. Assuming F, E, B to have the homotopy type of CW-complexes, passing to homology in 1.6, and using 2.1, we see that

$$\psi \text{ is homology } (m+n)\text{-connected}, \tag{7}$$

$$\psi' \text{ is homology } (2m+n)\text{-connected}. \tag{8}$$

We identify $H_q(\Sigma X)$ with $H_{q-1}(X)$ in a natural way, write $H = \Sigma\varrho \circ V$ and $Z = \zeta \circ \psi' : \Sigma(F * \Omega B) \to C_\psi$, and prove

Theorem 2.2. *Under the assumptions of 2.1 and if F, E, B have the homotopy type of CW-complexes, the diagram*

$$\begin{tikzcd}
H_{N-1}(F*\Omega B) \arrow[r, "j_*"] \arrow[d, equal] & H_{N-1}(E \cup C_0F) \arrow[r, "r_*"] \arrow[d, "\sigma_*"] & H_{N-1}(B) \arrow[r, "T"] \arrow[d, "\eta_{0*}"] & H_{N-2}(F*\Omega B) \to \dots \arrow[d, equal] \\
H_{N-1}(F*\Omega B) \arrow[r, "-H_*"] & H_{N-1}(\Sigma F) \arrow[r, "\psi_*"] & H_{N-1}(C_p) \arrow[r, "Z_*^{-1} \circ \eta_{2*}"] & H_{N-2}(F*\Omega B) \to \dots
\end{tikzcd}$$

in which $N = 2m+n$ and T is the transgression, commutes and has exact rows.

Proof. Exactness of the top row follows from 1.1 and 2.1 upon replacing $H_q((E \cup C_0F) \cup_j C_0(F * \Omega B))$ by $H_q(B)$ for $q < N$ in the homology sequence of the cofibration $F * \Omega B \to E \cup C_0F \to (E \cup C_0F) \cup_j C_0(F * \Omega B)$. Thus, T coincides with $\sigma'_* \circ (r'_*)^{-1}$ followed by the identification $H_q(\Sigma(F * \Omega B)) \to H_{q-1}(F * \Omega B)$. Commutativity in the third and second square follows from 1.6, whereas in the first it follows from 1.4. To prove exactness in the bottom row, introduce the diagram

$$\begin{tikzcd}
 & H_q(F*\Omega B) \arrow[r, "\vartheta^{-1}"] & H_{q+1}(\Sigma(F*\Omega B)) \arrow[r, "(\Sigma H)_*"] \arrow[d, "Z_*"] & H_{q+1}(\Sigma^2 F) \arrow[d, "\vartheta"] & \\
\cdots \arrow[r] & H_{q+1}(C_p) \arrow[r, "\eta_{2*}"] & H_{q+1}(C_\psi) \arrow[r, "\partial"] \arrow[ur, "\sigma_{2*}"] & H_q(\Sigma F) \arrow[r, "\psi_*"] & H_q(C_p) \arrow[r] & \cdots
\end{tikzcd}$$

in which the bottom row is the homology sequence of the cofibration $\Sigma F \to C_p \to C_\psi$ and ϑ is the identification. By 1.6 one has $\sigma_2 \circ Z \simeq \Sigma\sigma \circ (-1) \circ \Sigma j$; on a double suspension one has $-1 \simeq \Sigma(-1)$ so that $\Sigma\sigma \circ (-1) = (-1) \circ \Sigma\sigma \simeq \Sigma(-\sigma)$ and, by 1.4, we obtain $\sigma_2 \circ Z \simeq \Sigma H$. The naturality of ϑ implies $\vartheta \circ (\Sigma H)_* \circ \vartheta^{-1} = H_*$ and, as is well known, $\vartheta \circ \sigma_{2*} = \partial$. Exactness in the bottom row of 2.2 now follows easily from that of the bottom row in the preceding diagram.

Let now X and Y be arbitrary spaces. We shall need the sequence

$$\Omega X * \Omega Y \xrightarrow{W} X b Y \xrightarrow{L} X \vee Y \xrightarrow{J} X \times Y \xrightarrow{Q} X \# Y \tag{9}$$

where $X \vee Y$ is the subspace $(X \times *) \cup (* \times Y)$ of $X \times Y$, J is the inclusion map, $X \# Y$ results from $X \times Y$ by shrinking $X \vee Y$ to a point, and Q is the identification map. $X b Y$ is the fibre of J; by (2), it may readily be identified with $PX \times \Omega Y \cup \Omega X \times PY$ and, then, $L(\xi, \eta) = (\xi(1), \eta(1))$. The map W is given by

$$W((1-s)\xi + s\eta) = (\xi_{\mathrm{Min}(1, 2-2s)}, \eta_{\mathrm{Min}(2s, 1)})$$

and arguments similar to those referring to w_1 and w_2 in the proof of 1.1 reveal that W is a weak homotopy equivalence if X and Y have the homotopy type of CW-complexes.

In the next result C_i stands for $E \cup C_0 F$, $\triangle$ is the diagonal map, and

$$v\,\langle t, (1-s)x + s\omega\rangle = \langle t, x\rangle \# \omega(s).$$

Theorem 2.3. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration and suppose F, E, B have the homotopy type of CW-complexes. Then, homotopy-commutativity holds in the diagram*

$$\begin{tikzcd}
F*\Omega B \arrow[r, "j"] \arrow[d, "\psi^\bullet * 1"] & C_i \arrow[r, "r"] \arrow[d, "\tau"] & B \arrow[r, "\eta_1"] \arrow[d, "\triangle"] & C_r \arrow[dd, "\vartheta"] & \Sigma(F*\Omega B) \arrow[l, "\psi'"'] \arrow[dd, "v"] \\
\Omega C_p * \Omega B \arrow[d, "W"] & \Sigma F \vee C_i \arrow[d, "\psi \vee r"] & B \times B \arrow[d, "\eta_0 \times 1"] & & \\
C_p b B \arrow[r, "L"] & C_p \vee B \arrow[r, "J"] & C_p \times B \arrow[r, "Q"] & C_p \# B & \Sigma F \# B \arrow[l, "\psi \# 1"']
\end{tikzcd}$$

where $\psi^\bullet$ is adjoint to ψ, τ expresses the cooperation of ΣF on C_i, and ϑ is induced by the second square. Furthermore, $\psi^\bullet * 1$ is $(m + n - 1 + \mathrm{Min}(m, n))$-

connected and ϑ, ψ', $\psi \mathbin{\#} 1$, v are N-connected if B is $(m-1)$-connected and F is $(n-1)$-connected $(N = 2m + n,\ m \geq 1,\ n \geq 1)$.

Proof. The map τ is given [7] by

$$\tau(a) = (*, a) \text{ and } \tau(sx) = \begin{cases} (\langle 2s, x\rangle, *) & \text{if } 0 \leq 2s \leq 1, \\ (*, (2s-1)x) & \text{if } 1 \leq 2s \leq 2, \end{cases}$$

$\psi^{\cdot}$ stands for the composite $F \xrightarrow{S} \Omega\Sigma F \xrightarrow{\Omega\psi} \Omega C_p$, and it is easily seen that

$$\vartheta(b) = \eta_0(b) \mathbin{\#} b,\ \vartheta(ta) = \operatorname{Max}(0, 2t-1)a \mathbin{\#} p(a),\ \vartheta(tsx) = *.$$

Homotopy-commutativity is easily checked in the first three squares. Letting $v'\langle t, (1-s)x + s\omega\rangle = \operatorname{Min}(1, 2-2t)i(x) \mathbin{\#} \omega(s)$ yields a map $v' : \Sigma(F * \Omega B) \to C_p \mathbin{\#} B$ which, by 1.6 and (6), is easily seen to satisfy $v' \simeq (\psi \mathbin{\#} 1) \circ v$ and $v' \simeq \vartheta \circ \psi'$. The connectivities of $\psi^{\cdot} * 1$, $\psi \mathbin{\#} 1$, and ψ' are easily computed using (7) and (8), and noting that their domains and ranges are 1-connected. Expressing v as the composite

$$\Sigma(F * \Omega B) \to \Sigma(\Sigma(F \mathbin{\#} \Omega B)) \to \Sigma(F \mathbin{\#} \Sigma\Omega B) \to \Sigma(F \mathbin{\#} B) \to \Sigma F \mathbin{\#} B,$$

we easily find its connectivity which, by commutativity in the last square, yields that of ϑ.

Remark 2.4. Consider the fibration $\Omega B \to PB \to B$ and the diagram

$$\begin{tikzcd}
\Omega B * \Omega B \arrow[r, "j"] \arrow[d, equal] & PB \cup C\Omega B \arrow[r, "r"] \arrow[d, "-\sigma"] & B \arrow[d, equal] \\
\Omega B * \Omega B \arrow[r, "-\sigma \circ j"] & \Sigma\Omega B \arrow[r, "R"] & B
\end{tikzcd}$$

where $R\langle s, \omega\rangle = \omega(s)$. The right square homotopy-commutes and, if B has the homotopy type of a CW-complex, σ and η_0 are homotopy equivalences. Therefore, by 1.2, the bottom row is equivalent to a fibration, a result first proved in [2]. The map ϱ is now given by loop multiplication and C_p has the homotopy type of B; 2.2 yields the exact sequence associated with the homology suspension, and 1.4 and 2.3 yield the classical interpretation [25], [2] of the homomorphisms in the sequence.

Remark 2.5. The connectivity of a join is given in [15; Lemma 2.3]; an inductive form of the proof, involving (9) but bypassing the Künneth formula, is also available, Similarly, the connectivity of a join or reduced product of maps can be computed by means of the Künneth formula or analysing the cofibres as in [19; Satz 21].

3. Lifting cofibrations

Let $A \xrightarrow{d} X \xrightarrow{f} B$ be a cofibration, i. e. a triple in which A is a subspace of X, B results from X by shrinking A to a point, and the pair (X, A) has the homotopy extension property; d is the inclusion and f the identification map. Introduce the diagram

$$\begin{tikzcd}
B & E \arrow[l, "p"'] & F \arrow[l, "i"'] & \Omega B \arrow[l, "\partial"'] & \Omega E \arrow[l, "\Omega p"'] \\
B \arrow[u, equal] & X \arrow[l, "f"] \arrow[u, "h"] & A \arrow[l, "d"] \arrow[u, "e"] & F_d \arrow[l, "\varepsilon_0"] \arrow[u, "-\varphi"] & \Omega X \arrow[l, "\partial_0"] \arrow[u, "-\Omega h"]
\end{tikzcd} \tag{10}$$

in which the first square on the left and the top row result by converting f as in (1) into a homotopy equivalent fibre map. The fibration $F \xrightarrow{i} E \xrightarrow{p} B$ will be denoted by $\mathcal{F}$. Since d is an inclusion, its fibre F_d can be identified to $\{\xi \in PX \mid \xi(1) \in A\}$ and the projection ε_0 is given by $\varepsilon_0(\xi) = \xi(1)$; also, $\partial_0(\omega) = \omega$. We define $e(a) = (d(a), *)$ and $\varphi(\xi) = f \circ \xi$, denoting loop multiplication and inversion by $+$ and $-$. The diagram is essentially dual to the upper part of that in 1.6 and e lifts d to F.

Lemma 3.1. *The third square in* (10) *homotopy-commutes and the other squares commute. If A is $(n-1)$-connected and (X, A) is m-connected $(m \geq 1, n \geq 2)$, then e and φ are $(m+n-1)$-connected provided A, X, B have the homotopy type of CW-complexes.*

Proof. The first part is easily checked; the second follows by the 5 lemma from the BLAKERS-MASSEY theorem [4; Th. II] (which can be derived from 2.1 as in [17]) recalling that h is a homotopy equivalence.

We seek a suitable approximation to the fibre F_e and to the cofibre $C_e = F \cup C_0 A$ of e. In the diagram below, ε_1 is the projection, k the inclusion, and ∇ the folding map.

Theorem 3.2. *Let $A \xrightarrow{d} X \xrightarrow{f} B$ be a cofibration in which A, X, B have the homotopy type of CW-complexes. Introduce the diagram*

$$\begin{tikzcd}
C_e & F \arrow[l, "k"'] & A \arrow[l, "e"'] & F_e \arrow[l, "\varepsilon_1"'] \\
 & F \times \Omega B \arrow[u, "\varrho"] & A \vee A \arrow[u, "\nabla"] & \\
A \# F_d \arrow[uu, "\mu"] & A \times F_d \arrow[l, "Q"] \arrow[u, "e \times \varphi"] & A \vee F_d \arrow[l, "J"] \arrow[u, "1 \vee \varepsilon_0"] & A \flat F_d \arrow[l, "L"] \arrow[uu, "\nu"']
\end{tikzcd} \tag{11}$$

where ϱ expresses the operation of ΩB on F associated with $\mathcal{F}$. Then, the middle square homotopy-commutes and there result maps μ and ν yielding homotopy-commutativity in the other squares. Furthermore, if A is $(n-1)$-connected $(n \geq 2)$ and (X, A) is m-connected, then μ is $(N-1)$-connected for $m \geq 1$ and ν is $(N-2)$-connected for $m \geq 2$, where $N = m + n + \mathrm{Min}\,(m, n)$.

Proof. We may assume (cf. [7; Prop. 3.10] whose conventions differ slightly from ours) that $\varrho((x, \beta), \omega) = (x, -\omega + \beta)$, where we first traverse $-\omega$ and then β. Therefore, and by the first part of 3.1, the middle square homotopy-commutes; select a definite homotopy $h_t \colon A \vee F_d \to F$ connecting $e \circ \nabla \circ (1 \vee \varepsilon_0)$ to $\varrho \circ (e \times \varphi) \circ J$. By means of h_t we may construct as in [19; 2.2] a map μ' such that the diagram

$$\begin{tikzcd}
 & C_e & F \arrow[l, "k"'] \\
A \# F_d & C_J \arrow[l, "Q_2"'] \arrow[u, "\mu'"] & A \times F_d \arrow[l, "Q_1"'] \arrow[u, "\varrho\circ(e\times\varphi)"']
\end{tikzcd}$$

commutes. Since A and, as follows from [16], also F_d have the homotopy type of CW-complexes, the identification map Q_2 is a homotopy equivalence [19; Satz 16] with a map Q' as homotopy inverse. We define $\mu = \mu' \circ Q'$ and obtain homotopy-commutativity in the left square of (11) since $Q = Q_2 \circ Q_1$. Using h_t again, a map ν yielding commutativity in the right square of (11) is easily found.

In order to compute the connectivities, notice first that:

B is m-connected, F_d is $(m-1)$-connected, F is $(n-1)$-connected. (12)

Next, introduce the diagram

$$\begin{tikzcd}
C_\gamma \arrow[dddd, "l"'] & & & & E \cup C_0 F \arrow[llll, "g"'] \arrow[r, "r"] \arrow[dddd, "-\sigma"] & B \\
 & & A * F_d \arrow[r, "e*\varphi"] \arrow[dd, "V'"] & F * \Omega B \arrow[ur, "j"] \arrow[dd, "V"] & & \\
 & & & & & X \cup C_0 A \arrow[uul, "\gamma"'] \arrow[uu, "f'"'] \arrow[dd, "-\sigma''"] \\
 & \Sigma(A \# F_d) \arrow[dl, "\Sigma\mu"'] & \Sigma(A \times F_d) \arrow[l, "\Sigma Q"'] \arrow[r, "\Sigma(e\times\varphi)"] & \Sigma(F \times \Omega B) \arrow[dr, "\Sigma\varrho"] & & \\
\Sigma C_e & & & & \Sigma F \arrow[llll, "\Sigma k"'] & \Sigma A \arrow[l, "\Sigma e"']
\end{tikzcd} \tag{13}$$

in which j and r are given by 1.2, $\gamma \mid X = h$ and $\gamma \mid C_0 A = C_0 e$, f' extends f by mapping $C_0 A$ to the base-point, σ and σ'' shrink E and X to a point, V and V' are natural maps as in 1.4, g is the inclusion map, and l is the composite

$$(E \cup C_0 F) \cup C_0(X \cup C_0 A) \to \Sigma F \cup C_0 \Sigma A \to \Sigma(F \cup C_0 A)$$

in which the first map is induced by $-\sigma$ and $C_0(-\sigma'')$ whereas the second results upon identifying $C_0\Sigma A$ with $\Sigma C_0 A$ in the obvious way. It follows from 3.1 and (12) that

$$e * \varphi \text{ is } N\text{-connected.} \tag{14}$$

Also, by 2.2 and (12), the top sequence in the diagram

$$\begin{tikzcd}
\ldots \arrow[r] & H_q(F*\Omega B) \arrow[r, "j_*"] & H_q(E \cup C_0F) \arrow[r, "r_*"] \arrow[d, equal] & H_q(B) \arrow[r, "T"] & H_{q-1}(F*\Omega B) \arrow[r] & \ldots \\
\ldots & H_q(C_\gamma) \arrow[l] & H_q(E \cup C_0F) \arrow[l, "g_*"'] & H_q(X \cup C_0A) \arrow[l, "\gamma_*"'] \arrow[u, "f'_*"'] & H_{q+1}(C_\gamma) \arrow[l] & \ldots \arrow[l]
\end{tikzcd}$$

is defined and exact for $q \leq 2m + n + 1$. One has $r \circ \gamma = f'$ and f'_* is, obviously, isomorphic in all dimensions. Therefore, γ_* is monomorphic and r_* is epimorphic for all $q \geq 0$. By exactness, it follows that g_* is always epimorphic whereas j_* is monomorphic for $q \leq 2m + n$, and routine arguments now reveal that

$$g \circ j \text{ is } (2m + n + 1)\text{-connected.} \tag{15}$$

Also, h being a homotopy equivalence readily implies that

$$l_* \text{ is isomorphic in all dimensions.} \tag{16}$$

Since $l \circ g = \Sigma k \circ (-\sigma)$, by 1.4, by the naturality of V and V', and by homotopy-commutativity in the left square of (11), one has

$$l \circ g \circ j \circ (e * \varphi) \simeq \Sigma\mu \circ \Sigma Q \circ V'.$$

Since $\Sigma Q \circ V'$ is well known to be a weak homotopy equivalence, by (14), (15) and (16) we see that $\Sigma\mu$ is N-connected and the connectivity of μ follows upon noticing that $A \# F_d$ and C_e are 1-connected.

Finally, the connectivity of ν follows from that of μ noting that the map $A \vee F_d \to A$ is m-connected and applying the "relative J. H. C. WHITEHEAD theorem" given in [17; Th. 1.8 (I)]. The assumption $m \geq 2$ is needed in order that F_d be 1-connected, as required in [17; Th. 1.8]. Thus, 3.2 is completely proved.

We close this section by describing the behaviour of $e: A \to F$ under suspension.

Proposition 3.3. *There exists a homotopy equivalence α such that the composite*

$$\Sigma A \xleftarrow{\alpha} C_\varphi \xleftarrow{\varphi} \Sigma F \xleftarrow{\Sigma e} \Sigma A,$$

where ψ is as in 1.6, is homotopic to the identity; in particular, there is a homotopy equivalence $\Sigma F \to \Sigma C_e \vee \Sigma A$ and, if A, X, B have the homotopy type of CW-complexes, the homology sequence of $A \xrightarrow{e} F \xrightarrow{k} C_e$ splits.

Proof. By [19; Satz 2] the map f' in (13) has a homotopy inverse f'' and the left square in the diagram

$$\begin{array}{ccccc} E & \xrightarrow{p} & B & \xrightarrow{\eta_0} & C_p \\ \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle \alpha} \\ X & \xrightarrow{d'} & X \cup C_0 A & \xrightarrow{\sigma''} & \Sigma A \end{array}$$

where η_0 and d' are inclusions and h' is the projection, homotopy-commutes. There results a map α yielding commutativity in the right square. Since h' and f'' are homotopy equivalences so is α [19; Hilfssatz 7], and 3.3 is easily checked using explicit expressions for the maps involved.

Remark 3.4. For the cofibration $A \xrightarrow{d} C_0 A \xrightarrow{f} \Sigma A$, the projection $\varepsilon_0 : F_d \to A$ and the inclusion $\partial : \Omega B \to F$ in (10) are homotopy equivalences. Using ε_0 and $\partial \circ (-1)$ as identifications, e and φ are converted into the natural embedding $S : A \to \Omega \Sigma A$, and, in (11), $\varrho \circ (e \times \varphi)$ into $M \circ (S \times S)$, where M is the loop multiplication.

Remark 3.5. Theorem 1.1 does not dualize: the homotopy type of the cofibre C_e is not determined by those of B and ΣA. Thus, as pointed out by M. G. Barratt, if $A = S^p \vee S^q \vee S^{p+q}$ and $A' = S^p \times S^q$, where S^n is the n-sphere, then ΣA and $\Sigma A'$ have the same homotopy type whereas, if p and q are even, the cofibres $\Omega \Sigma A / A$ and $\Omega \Sigma A' / A'$ have non-isomorphic integral cohomology rings.

4. The Hopf invariant of a cofibration

The purpose of this section is to provide an alternative approximation to the cofibre C_e. We maintain the notations of the previous section.

For arbitrary spaces X and Y, consider the composite

$$M : \Omega(X \times Y) \to \Omega X \times \Omega Y \xrightarrow{\Omega i \times \Omega j} \Omega(X \vee Y) \times \Omega(X \vee Y) \to \Omega(X \vee Y)$$

in which the first map is the obvious homeomorphism, the last is given by loop multiplication, and $i : X \to X \vee Y$, $j : Y \to X \vee Y$ are the inclusions. With the notations introduced in (9), it is well known that, in the sequence

$$\Omega(X \flat Y) \underset{T}{\overset{\Omega L}{\rightleftarrows}} \Omega(X \vee Y) \underset{M}{\overset{\Omega J}{\rightleftarrows}} \Omega(X \times Y) \xrightarrow{\partial} X \flat Y,$$

one has $\Omega J \circ M \simeq 1$ so that $\partial \simeq 0$, and the properties of fibrations yield a unique homotopy class of maps T such that

$$\Omega L \circ T + M \circ \Omega J \simeq 1, \; T \circ \Omega L \simeq 1, \; T \circ (1 - M \circ \Omega J) \simeq T. \tag{17}$$

We also need the map

$$\Phi : X \flat Y \to \Omega(X \# Y) \text{ given by } \Phi(\xi, \eta)(t) = \xi(t) \# \eta(t).$$

Let now $\mathcal{C}: A \xrightarrow{d} X \xrightarrow{f} B$ be a cofibration and let ΣA cooperate on B through $\chi : B \to \Sigma A \vee B$ [7]. We define the "delicate" and "crude Hopf invariant of $\mathcal{C}$" as the composites

$$\mathcal{H}: \Omega B \xrightarrow{\Omega\chi} \Omega(\Sigma A \vee B) \xrightarrow{T} \Omega(\Sigma A \flat B),$$
$$\mathcal{H}' : \Omega B \xrightarrow{\mathcal{H}} \Omega(\Sigma A \flat B) \xrightarrow{\Omega\Phi} \Omega^2(\Sigma A \# B).$$

This is obviously consistent with previous generalizations of the Hopf invariant [12; § 3], and the map $\mathcal{J}$ below is related to the relative Hopf invariant introduced in [23]. Define

$$G : A \# F_d \to \Omega^2(\Sigma A \# B) \text{ by } G(a \# \xi)(s)(t) = \langle 1 - t, a\rangle \# f \circ \xi(s).$$

Theorem 4.1. *Let $\mathcal{C}: A \xrightarrow{d} X \xrightarrow{f} B$ be a cofibration in which A, X, B have the homotopy type of countable CW-complexes. Then, there exists a map $\mathcal{J}$ yielding homotopy-commutativity in the diagram*

$$\begin{tikzcd}
A \# F_d \arrow[r, "\mu"] \arrow[d, "G"] & C_e \arrow[d, "\mathcal{J}"] & F \arrow[l, "k"'] \\
\Omega^2(\Sigma A \# B) & \Omega(\Sigma A \flat B) \arrow[l, "\Omega\Phi"'] & \Omega B \arrow[l, "\mathcal{H}"'] \arrow[u, "\partial"']
\end{tikzcd}$$

Furthermore, $\mathcal{J}$ is $(m + n - 1 + \mathrm{Min}\,(m, n))$-connected if A is $(n-1)$-connected and (X, A) is m-connected $(m \geq 1, n \geq 2)$.

Proof. There is [7; p. 11] a homotopy $h_s : X \to \Sigma A \vee B$ with

$$h_0(x) = (*, f(x)), \; h_1 = \chi \circ f, \; h_s \circ d(a) = (\langle s, a\rangle, *).$$

We also need the maps

$$\sigma : B \xrightarrow{\chi} \Sigma A \vee B \xrightarrow{pr} \Sigma A \text{ and } \vartheta : B \xrightarrow{\chi} \Sigma A \vee B \xrightarrow{pr} B,$$

and recall [7; Th. 3.1′] that there is a homotopy

$$\vartheta_s : B \to B \text{ satisfying } \vartheta_0 = \vartheta, \vartheta_1 = 1.$$

To define $\mathcal{T}$, introduce the diagram

$$\begin{array}{ccccc} A & \xrightarrow{e} & F & \xrightarrow{k} & C_e \\ \downarrow l & & \downarrow D & & \downarrow \mathcal{T} \\ \Omega(\Sigma A \times B) & \xrightarrow{M} & \Omega(\Sigma A \vee B) & \xrightarrow{T} & \Omega(\Sigma A \flat B) \end{array}$$

in which $l(a)(s) = (\langle 1 - s, a\rangle, *)$ and

$$D(x, \beta)(s) = \begin{cases} \chi \circ \beta(3s) & \text{if } 0 \leq 3s \leq 1, \\ (1 \vee \vartheta_{2-3s}) \circ h_{2-3s}(x) & \text{if } 1 \leq 3s \leq 2, \\ (*, \vartheta \circ \beta(3 - 3s)) & \text{if } 2 \leq 3s \leq 3. \end{cases}$$

The left square is obviously homotopy-commutative, by (17) one has $T \circ M \simeq 0$, and there results a map $\mathcal{T}$ yielding homotopy-commutativity in the right square. By (17), the map

$$\mathcal{D} = D - M \circ \Omega J \circ D : F \to \Omega(\Sigma A \vee B)$$

satisfies $\mathcal{D} = (1 - M \circ \Omega J) \circ D \simeq \Omega L \circ T \circ D \simeq \Omega L \circ \mathcal{T} \circ k$, so that

$$T \circ \mathcal{D} \simeq \mathcal{T} \circ k \tag{18}$$

and also $\mathcal{D}^{\cdot} \simeq L \circ \mathcal{T}^{\cdot} \circ \Sigma k$, where $\mathcal{D}^{\cdot}$ and $\mathcal{T}^{\cdot}$ are adjoint to $\mathcal{D}$ and $\mathcal{T}$. Passing to loop spaces and then composing with T, we obtain

$$T \circ \Omega \mathcal{D}^{\cdot} \simeq \Omega \mathcal{T}^{\cdot} \circ \Omega \Sigma k. \tag{19}$$

To prove homotopy-commutativity in the left square of 4.1, define a map $H : \Sigma(A \times F_d) \to \Sigma A \flat B$ by

$$H\langle s, (a, \xi)\rangle(t) = \begin{cases} (\langle u, a\rangle, \vartheta \circ f \circ \xi \circ w) & \text{if } 1 \leq 5s \leq 4,\ 1 \leq 2t \leq 2, \\ (\sigma \circ f \circ \xi \circ v, \vartheta \circ f \circ \xi \circ w) & \text{otherwise,} \end{cases}$$

where the real functions u, v, w are given by

$$u = u(s, t) = 2 - 2t + (2t - 1)\,\mathrm{Max}\,(2 - 5s, 0, 5s - 3),$$
$$v = v(s, t) = \mathrm{Max}\,(1 - 2t, 0) + \mathrm{Min}\,(2t, 1)\,\mathrm{Max}\,(1 - 5s, 0, 5s - 4),$$
$$w = w(s, t) = \mathrm{Max}(1 - 2t, 0) + \mathrm{Min}\,(2t, 1)\,\mathrm{Max}\,(1 - 5s, 0, \mathrm{Min}\,(5s - 2, 1)).$$

The definition of H is suggested by a null-homotopy of $J \circ \mathcal{D}^{\cdot}$ in $\Sigma A \times B$ and, when dealing with H, we shall tacitly use the fact that $(\sigma(b), \vartheta(b)) = \chi(b) \in \Sigma A \vee B$ for any $b \in B$. As is easily seen, $L \circ H$ is homotopic to the composite

$$\Sigma(A \times F_d) \xrightarrow{\Sigma(e \times \varphi)} \Sigma(F \times \Omega B) \xrightarrow{\Sigma \varrho} \Sigma F \xrightarrow{\mathcal{D}^{\cdot}} \Sigma A \vee B$$

so that, passing to loop spaces, composing with T, and then applying (19), we obtain

$$\Omega H \simeq T \circ \Omega L \circ \Omega H \simeq T \circ \Omega \mathcal{D}^{\cdot} \circ \Omega \Sigma(\varrho \circ (e \times \varphi)) \simeq \Omega \mathcal{T}^{\cdot} \circ \Omega \Sigma k \circ \Omega \Sigma(\varrho \circ (e \times \varphi)).$$

Therefore, inspection of (13) reveals that $\Omega H \simeq \Omega Z$, where

$$Z : \Sigma(A \times F_d) \xrightarrow{\Sigma Q} \Sigma(A \# F_d) \xrightarrow{\Sigma \mu} \Sigma C_e \xrightarrow{\mathcal{T}^{\cdot}} \Sigma A \flat B.$$

Define $\Phi_u : A * F_d \to \Omega(\Sigma A \# B)$ by

$$\Phi_u(y)(t) = \begin{cases} \sigma \circ f \circ \xi(1 - 2t) \# \vartheta \circ f \circ \xi(1 - 2t + 2tsu) & \text{if } 0 \leq 2t \leq 1, \\ \langle 2 - 2t, a \rangle \# \vartheta_{1-u} \circ f \circ \xi(s) & \text{if } 1 \leq 2t \leq 2, \end{cases}$$

where $y = (1 - s)a + s\xi$. Then, it is easily seen that Φ_1 and Φ_0 are respectively homotopic to the composites

$$A * F_d \xrightarrow{V'} \Sigma(A \times F_d) \xrightarrow{H} \Sigma A \flat B \xrightarrow{\Phi} \Omega(\Sigma A \# B)$$

and

$$A * F_d \xrightarrow{V'} \Sigma(A \times F_d) \xrightarrow{\Sigma Q} \Sigma(A \# F_d) \xrightarrow{G^{\cdot}} \Omega(\Sigma A \# B),$$

where $G^{\cdot}$ is adjoint to G and V' is the natural map. Passing to loop spaces and replacing, as we may, ΩH by ΩZ we obtain

$$\Omega \Phi \circ \Omega \mathcal{T}^{\cdot} \circ \Omega \Sigma \mu \circ \Omega \Sigma Q \circ \Omega V' \simeq \Omega G^{\cdot} \circ \Omega \Sigma Q \circ \Omega V'.$$

Since A, X, B have the homotopy type of countable CW-complexes, $\Sigma Q \circ V'$ is a homotopy equivalence so that the left square homotopy-commutes in the diagram

$$\begin{tikzcd}
\Omega^2(\Sigma A \# B) & \Omega \Sigma(A \# F_d) \arrow[l, "\Omega G^{\cdot}"'] \arrow[d, "\Omega \Sigma \mu"] & A \# F_d \arrow[l, "S"'] \arrow[d, "\mu"] \\
\Omega(\Sigma A \flat B) \arrow[u, "\Omega \Phi"] & \Omega \Sigma C_e \arrow[l, "\Omega \mathcal{T}^{\cdot}"'] & C_e \arrow[l, "S"']
\end{tikzcd}$$

To obtain homotopy-commutativity in the left square of 4.1, it only remains to notice that, with S standing for the natural embedding, the right square also homotopy-commutes whereas $G = \Omega G^{\cdot} \circ S$ and $\mathcal{J} = \Omega \mathcal{J}^{\cdot} \circ S$.

To prove homotopy-commutativity in the right square of 4.1, notice that

$$M \circ \Omega J \circ \Omega \chi \circ \omega(s) = \begin{cases} (\sigma \circ \omega(2s), *) & \text{if } 0 \leq 2s \leq 1, \\ (*, \vartheta \circ \omega(2s-1)) & \text{if } 1 \leq 2s \leq 2, \end{cases}$$

whereas, since $\partial(\omega) = (*, \omega)$, $\Omega J \circ D \circ \partial$ is homotopic to the map $\Omega B \to \Omega(\Sigma A \times B)$ given by $\omega \to (\sigma \circ \omega, *)$. This readily implies $(1 - M \circ \Omega J) \circ \Omega \chi \simeq \mathcal{D} \circ \partial$ which, by (17), (18) and the definition of $\mathcal{H}$, yields the desired result.

Finally, G may be expressed as the composite

$$A \# F_d \to \Omega^2 \Sigma^2 (A \# F_d) \to \Omega^2 (\Sigma A \# \Sigma F_d) \xrightarrow{\Omega^2((-1) \# \varphi^{\cdot})} \Omega^2 (\Sigma A \# B)$$

in which $\varphi^{\cdot}$ is adjoint to φ, and it is easily seen that G is $(m + n + \mathrm{Min}(m, n-1))$-connected. Also, by 3.1 applied to the cofibration $\Sigma A \vee B \to \Sigma A \times B \to \Sigma A \# B$, it follows that Φ is $(m + n + 1 + \mathrm{Min}(m, n))$-connected. By commutativity in the left square of 4.1, the connectivity of $\mathcal{J}$ now follows from that of μ as given in 3.2.

Remark 4.2. It is well known that $\Omega(\Sigma A \flat B)$ is homotopically equivalent to the "cojoin" of ΣA and B, i. e. the space $P(\Sigma A \vee B; \Sigma A, B)$ of all paths in $\Sigma A \vee B$ which start in ΣA and end in B. In this sense, the right square in 4.1 can be regarded as dual to the diagram obtained upon replacing $F * \Omega B$ by the actual fibre F_r of r in the top row of 1.4; the left vertical in 1.4 should then be replaced by the weak homotopy equivalence w of 1.1 which appears as dual to the $(m + n - 1 + \mathrm{Min}(m, n))$-connected map $\mathcal{J}$ of 4.1. This duality becomes actually more striking if the results of 4.1 are expressed in terms of the cojoin. For traditional reasons however, we prefer to use $\Omega(\Sigma A \flat B)$ and the present generalization of the Hopf invariant.

For the final result of this section, we need a third map closely related to $\mathcal{H}$. Let $\mathcal{F}'_0$ result as in (5) from the fibration $\mathcal{F}$ obtained in (10) by converting f into a fibre map. Introduce the composite

$$\gamma' : B \xrightarrow{f''} X \cup C_0 A \xrightarrow{\gamma} E \cup C_0 F$$

where γ and f'' are defined in (13) and in the proof of 3.3. One has

$$r \circ \gamma' = f' \circ f'' \simeq 1 \text{ and } \Omega r \circ \Gamma \simeq 1, \tag{20}$$

where Γ is defined in 1.5. Therefore, if B has the homotopy type of a CW-complex, 1.2 yields a map

$$\mathscr{H}'' : \Omega B \to \Omega(F * \Omega B) \text{ such that } \Omega\gamma' - \Gamma \simeq \Omega j \circ \mathscr{H}''; \tag{21}$$

by 1.3 and the remark concluding the proof of 1.5, the homotopy class of $\mathscr{H}''$ is uniquely determined. We now write $\Psi = \alpha \circ \psi$ and $\Upsilon = \alpha \circ \eta_0$, where α is given by 3.3, then pass to loop spaces in 2.3, and obtain the diagram

$$\begin{tikzcd}
\Omega(F * \Omega B) \arrow[r, "\Omega j"] \arrow[d, "\Omega(\Psi^{\cdot} * 1)"] & \Omega C_i \arrow[rr, shift left, "\Omega r"] \arrow[d, "\Omega\tau"] & & \Omega B \arrow[ll, shift left, "{\Omega\gamma', \Gamma}"] \arrow[d, "\Omega\Delta"] \arrow[ddll, "\Omega\chi"'] \\
\Omega(\Omega\Sigma A * \Omega B) \arrow[d, "\Omega W"] & \Omega(\Sigma F \vee C_i) \arrow[d, shift left, "\Omega(\Psi \vee r)"] & & \Omega(B \times B) \arrow[d, "\Omega(\Upsilon \times 1)"] \\
\Omega(\Sigma A \flat B) \arrow[r, shift left, "\Omega L"] & \Omega(\Sigma A \vee B) \arrow[l, shift left, "T"] \arrow[u, shift left, "\Omega(\Sigma e \vee \gamma')"] \arrow[rr, shift left, "\Omega J"] & & \Omega(\Sigma A \times B) \arrow[ll, shift left, "M"]
\end{tikzcd}$$

where $\Psi^{\cdot}$ is adjoint to Ψ and $C_i = E \cup C_0 F$.

Proposition 4.3. *Under the assumptions of 3.2, one has* $\mathscr{H} \simeq \Omega W \circ \Omega(\Psi^{\cdot} * 1) \circ \mathscr{H}''$ *and* $\Omega W \circ \Omega(\Psi^{\cdot} * 1)$ *is* $(m + n + \operatorname{Min}(m, n - 1))$*-connected.*

Proof. Commutativity in the right square in the proof of 3.3 yields $\Upsilon = \sigma'' \circ f''$ so that, by well known properties of cooperation,

$$J \circ \chi \simeq (\Upsilon \times 1) \circ \Delta.$$

Commutativity in the second square of (10) and the naturality of χ imply $(\Sigma e \vee \gamma') \circ \chi \simeq \tau \circ \gamma'$ so that, by 3.3 and (20),

$$\chi \simeq (\Psi \vee r) \circ \tau \circ \gamma'.$$

Finally, using 1.5 and the definition of ψ given in 1.6, it is easy to see that

$$M \circ \Omega(\Upsilon \times 1) \circ \Omega\Delta \simeq \Omega(\Psi \vee r) \circ \Omega\tau \circ \Gamma.$$

By (17) and (21), the three preceding relations yield

$$T \circ \Omega\chi \simeq T \circ (1 - M \circ \Omega J) \circ \Omega\chi \simeq T \circ \Omega(\Psi \vee r) \circ \Omega\tau \circ \Omega j \circ \mathscr{H}'',$$

and the first result follows from the definition of $\mathscr{H}$, homotopy-commutativity in the left square of 2.3 and hence of the preceding diagram, and (17). The connectivity follows from 2.3.

5. The generalized EHP sequence

Recall first that the generalized Whitehead product of two maps $f:\Sigma X \to Z$ and $g:\Sigma Y \to Z$ is a map $[f, g]$ such that the composite

$$\Sigma(X \times Y) \xrightarrow{\Sigma Q} \Sigma(X \# Y) \xrightarrow{[f,g]} Z$$

represents the commutator $(f' + g') + (-f' - g')$ of

$$f': \Sigma(X \times Y) \xrightarrow{\Sigma p_1} \Sigma X \xrightarrow{f} Z \text{ and } g': \Sigma(X \times Y) \xrightarrow{\Sigma p_2} \Sigma Y \xrightarrow{g} Z$$

in the group $\pi(\Sigma(X \times Y), Z)$. As in [3; 6.9], the construction of $[f, g]$ is valid if X and Y have non-degenerate base-points and then, by [19; Folgerung, p. 333], the homotopy class of $[f, g]$ is uniquely determined by those of f and g. Define $R:\Sigma\Omega Z \to Z$ by $R\langle s, \omega\rangle = \omega(s)$, and let $f: X \to Z$, $g: Y \to Z$ be arbitrary maps.

Lemma 5.1. *If X and Y have the homotopy type of countable CW-complexes, there exists a homotopy equivalence ϑ yielding homotopy-commutativity in the diagram*

$$\begin{tikzcd} Z \arrow[d, equal] & Z \vee Z \arrow[l, "\nabla"'] & X \vee Y \arrow[l, "f \vee g"'] & X \flat Y \arrow[l, "L"'] \\ Z & & & \Sigma(\Omega X \# \Omega Y) \arrow[lll, "{[R \circ \Sigma\Omega f, R \circ \Sigma\Omega g]}"] \arrow[u, "\vartheta"'] \end{tikzcd}$$

Proof. Since ΩX and ΩY also have the homotopy type of countable CW-complexes [16], the weak homotopy equivalence

$$\Sigma Q \circ V: \Omega X * \Omega Y \to \Sigma(\Omega X \times \Omega Y) \to \Sigma(\Omega X \# \Omega Y)$$

has a homotopy inverse Λ. Define $\vartheta = W \circ \Lambda$, where the homotopy equivalence $W: \Omega X * \Omega Y \to X \flat Y$ is as in (9). One has

$$h_0 \simeq \nabla \circ (f \vee g) \circ L \circ W \text{ and } h_1 = [R \circ \Sigma\Omega f, R \circ \Sigma\Omega g] \circ \Sigma Q \circ V,$$

provided the values of $h_t((1-s)\xi + s\eta)$ on the quarters of $0 \le s \le 1$ are

$$f \circ \xi(1 - t + 4st), g \circ \eta(4s - 1), f \circ \xi(3 - 4s), g \circ \eta(1 - t + (4 - 4s)t).$$

The result now follows easily.

With the notation of 3.2, let F_k be the fibre of k, let e' lift e to F_k, and let ε_0 be as in (10). Define

$$\partial'(\omega) = (*, \omega) \text{ and } \mathcal{R}\langle s, \alpha \# \delta\rangle(t) = \alpha(1 - s) \# \delta(1 - t).$$

Theorem 5.2. *Let* $\mathcal{C}: A \xrightarrow{d} X \xrightarrow{f} B$ *be a cofibration in which* A, X, B *have the homotopy type of countable* CW*-complexes. Then, homotopy-commutativity holds in the diagram*

$$\begin{tikzcd} A \arrow[r, "e'"] \arrow[d, equal] & F_k & \Omega C_e \arrow[l, "\partial'"'] & \Omega(A \# F_d) \arrow[l, "\Omega\mu"'] \\ A & & & \Sigma(\Omega A \# \Omega F_d) \arrow[lll, "{[R, R \circ \Sigma\Omega\varepsilon_0]}"'] \arrow[u, "\mathcal{R}"'] \end{tikzcd}$$

Furthermore, e' *is* $(m + 2n - 2)$*-connected and* $\mathcal{R}$ *is* $(m + n - 2 + \mathrm{Min}(m, n))$*-connected if* A *is* $(n - 1)$*-connected and* (X, A) *is* m*-connected* $(m \geq 1, n \geq 2)$.

Proof. With the notation of 3.2 one has

$$\varepsilon_1 \circ \nu \simeq \nabla \circ (1 \vee \varepsilon_0) \circ L. \tag{22}$$

Replacing in (10) the original cofibration by $A \to F \to C_e$ and then by $A \vee F_d \to A \times F_d \to A \# F_d$, we obtain maps

$$\varphi' : F_e \to \Omega C_e \text{ and } \Phi' : A \flat F_d \to \Omega(A \# F_d)$$

which, by naturality and commutativity in 3.2, satisfy

$$\Omega\mu \circ \Phi' \simeq \varphi' \circ \nu. \tag{23}$$

As in 3.1, one has $\partial' \circ (-\varphi') \simeq e' \circ \varepsilon_1$ so that, by (23) and (22),

$$\partial' \circ \Omega\mu \circ (-\Phi') \simeq e' \circ \nabla \circ (1 \vee \varepsilon_0) \circ L : A \flat F_d \to F_k. \tag{24}$$

The map Φ' is given by $\Phi'(\alpha, \delta)(t) = \alpha(t) \# \delta(t)$ and, letting

$$H_u((1 - s)\alpha + s\delta)(t) = \alpha((t + u - tu)\,\mathrm{Min}(1, 2 - 2s)) \# \delta(t\,\mathrm{Min}(2s, 1)),$$

we obtain $H_0 = \Phi' \circ W$ and $H_1 \simeq U$ in the diagram

$$\begin{tikzcd} \Omega A * \Omega F_d \arrow[rr, "W"] \arrow[dr, "U"'] & & A \flat F_d \arrow[dl, "\Phi'"] \\ & \Omega(A \# F_d) & \end{tikzcd}$$

where W is as in (9) and $U((1 - s)\alpha + s\delta)(t) = \alpha(1 - s) \# \delta(t)$. Thus defined, U coincides with the composite

$$\Omega A * \Omega F_d \xrightarrow{V} \Sigma(\Omega A \times \Omega F_d) \xrightarrow{\Sigma Q} \Sigma(\Omega A \# \Omega F_d) \xrightarrow{\mathcal{R}} \Omega(A \# F_d) \xrightarrow{-1} \Omega(A \# F_d)$$

and, by the definitions of Λ and ϑ in the proof of 5.1, we obtain

$$\mathcal{R} \simeq (-U) \circ \Lambda \simeq (-\Phi') \circ W \circ \Lambda = (-\Phi') \circ \vartheta.$$

Therefore, by (24) and 5.1, we have

$$\partial' \circ \Omega\mu \circ \mathcal{R} \simeq \partial' \circ \Omega\mu \circ (-\Phi') \circ \vartheta \simeq e' \circ \nabla \circ (1 \vee \varepsilon_0) \circ L \circ \vartheta \simeq e' \circ [R, R \circ \Sigma\Omega\varepsilon_0].$$

Finally, the connectivity of e' follows from 3.1 and that of $\mathcal{R}$ is easily computed.

We conclude this section with a result which is, to a certain extent, dual to 2.2. Introduce the diagram

$$\begin{tikzcd}
 & & F & \Omega B \arrow[l, "\partial"'] \\
 & F_k & A \arrow[u, "e"] \arrow[l, "e'"'] & F_d \arrow[u, "-\varphi"'] \arrow[l, "\varepsilon_0"'] \\
\Omega^2(\Sigma A \flat B) & \Omega C_e \arrow[u, "\partial'"] \arrow[l, "\Omega\mathcal{T}"'] & F_e \arrow[u, "\varepsilon_1"'] \arrow[l, "-\varphi'"'] & F_{(-\varphi)} \arrow[u, "\varepsilon_2"'] \arrow[l, "\zeta"'] \\
 & \Omega F \arrow[u, "\Omega k"] & \Omega F \arrow[u, "\partial_1"'] \arrow[l, "-1"'] & \Omega^2 B \arrow[u, "\partial_2"'] \arrow[l, "\Omega\partial"']
\end{tikzcd} \tag{25}$$

in which ∂_1, ∂_2, ε_2 have obvious meanings whereas all other maps, except ζ, have been defined in connection with (10), 3.2, 4.1, 5.2; ζ is induced by the top square and, since h in (10) is a homotopy equivalence, an argument dual to that in 1.6 reveals that ζ is, in turn, a homotopy equivalence. The diagram homotopy-commutes. By 4.1 and then by 3.1 applied with $A \to F \to C_e$ as original cofibration, it follows that

$$Z = \Omega\mathcal{T} \circ (-\varphi') \circ \zeta \text{ is } (m + n - 2 + \operatorname{Min}(m, n))\text{-connected}$$

if A is $(n-1)$-connected and (X, A) is m-connected. We identify $\pi_q(\Omega Y)$ with $\pi_{q+1}(Y)$ in a natural way, denote by $\mathbf{E}$, $\mathbf{L}$ and $\mathbf{P}$ the homomorphisms induced by e and by the top rows in 4.1 and 5.2 respectively, and prove

Theorem 5.3. *Let $A \xrightarrow{d} X \xrightarrow{f} B$ be a cofibration in which A, X, B have the homotopy type of countable CW-complexes. If A is $(n-1)$-connected and (X, A) is m-connected $(m \geq 1, n \geq 2)$, then the diagram*

$$\begin{tikzcd}
\cdots \leftarrow \pi_{N-3}(A) & \pi_{N-2}(A \# F_d) \arrow[l, "\mathbf{P}"'] & \pi_{N-2}(F) \arrow[l, "\mathbf{L}"'] & \pi_{N-2}(A) \arrow[l, "\mathbf{E}"'] \\
\cdots \leftarrow \pi_{N-3}(F_d) \arrow[u, "\varepsilon_0 *"] & \pi_{N-2}(\Omega(\Sigma A \flat B)) \arrow[l, "\varepsilon_{2*} \circ Z_*^{-1}"'] \arrow[u, "G_*^{-1} \circ (\Omega\Phi)_*"] & \pi_{N-2}(\Omega B) \arrow[l, "\mathcal{H}_*"'] \arrow[u, "\partial_*"] & \pi_{N-2}(F_d) \arrow[l, "-\varphi_*"'] \arrow[u, "\varepsilon_0 *"]
\end{tikzcd}$$

where $N = m + n +$ Min (m, n), commutes and has exact rows.

Proof. According to 4.1 we may replace $G_*^{-1} \circ (\Omega\Phi)_*$ by $\mu_*^{-1} \circ \mathcal{T}_*^{-1}$. The top sequence results upon using the maps e' and μ of 5.2 and 3.2 in order to replace $\pi_q(F_k)$ and $\pi_q(C_e)$ for $q \leq N-2$ by $\pi_q(A)$ and $\pi_q(A \# F_d)$ in the homotopy sequence of the fibration $F_k \to F \to C_e$. Commutativity in the first square (from the right) follows from 3.1 and in the second from 4.1; to prove it in the third, it suffices to notice that, in (25), one has $e' \circ \varepsilon_0 \circ \varepsilon_2 \simeq \partial' \circ (-\varphi') \circ \zeta$. To prove exactness in the bottom row, introduce the diagram

$$\begin{tikzcd}
\pi_{q+1}(\Omega(\Sigma A \flat B)) & \pi_q(\Omega^2(\Sigma A \flat B)) \arrow[l, "\vartheta^{-1}"'] & \pi_q(\Omega^2 B) \arrow[l, "(\Omega\mathcal{H})_*"'] \arrow[dl, "\partial_{2*}"] & \\
\cdots \leftarrow \pi_q(F_d) & \pi_q(F_{(-\varphi)}) \arrow[l, "\varepsilon_{2*}"'] \arrow[u, "Z_*"] & \pi_{q+1}(\Omega B) \arrow[l, "\triangle"'] \arrow[u, "\vartheta"'] & \pi_{q+1}(F_d) \leftarrow \cdots \arrow[l, "-\varphi_*"']
\end{tikzcd}$$

where the bottom row is the homotopy sequence of the fibration $F_{(-\varphi)} \to F_d \to \Omega B$ and ϑ is the identification. Inspection of (25) reveals that $Z \circ \partial_2 \simeq \Omega\mathcal{T} \circ \Omega k \circ (-1) \circ \Omega\partial$ so that, by 4.1, $Z \circ \partial_2 \simeq -\Omega\mathcal{H}$. The naturality of ϑ implies $\vartheta^{-1} \circ (\Omega\mathcal{H})_* \circ \vartheta = \mathcal{H}_*$ and, as is well known, $\partial_{2*} \circ \vartheta = \triangle$. Exactness in the bottom row of 5.3 now follows from that of the bottom row in the preceding diagram noting that $\vartheta^{-1} \circ Z_*$ is isomorphic for $q \leq N-3$.

Remark 5.4. For the cofibration $A \to C_0 A \to \Sigma A$ one has $m = n$ and $N = 3n$ if A is $(n-1)$-connected $(n \geq 2)$. As noticed in 3.4, e can be identified to the natural embedding $A \to \Omega\Sigma A$ whereas ∂ and ε_0 are homotopy equivalences. Hence, replacing F_d by A and ε_0 by the identity map, and writing $\mathbf{H}$ for the composite $G_*^{-1} \circ (\Omega\Phi)_* \circ \mathcal{H}_* \circ \partial_*^{-1}$, we obtain the exact sequence

$$\pi_{3n-2}(A) \xrightarrow{\mathbf{E}} \pi_{3n-2}(\Omega\Sigma A) \xrightarrow{\mathbf{H}} \pi_{3n-2}(A \# A) \xrightarrow{\mathbf{P}} \pi_{3n-3}(A) \to \cdots$$

where, according to 5.2, **P** coincides with $[R, R]_* \circ \mathcal{R}_*^{-1} \circ \vartheta$; as before, ϑ is the identification $\pi_{q+1}(Y) \to \pi_q(\Omega Y)$. This is, essentially, the well known EHP-sequence of G. W. Whitehead [24] in the slightly more general form given by Barcus [1]. Obviously, it could be rewritten for generalized homotopy groups.

6. Nilpotency and cocategory

Let A be any space. Define a sequence of cofibrations

$$\mathcal{C}_k \colon A \xrightarrow{e_k} F_k \xrightarrow{f_k} B_k \qquad (k \geq 0)$$

as follows. $\mathcal{C}_0$ is the standard cofibration $A \to C_0 A \to \Sigma A$. Assuming $\mathcal{C}_k$ to be defined, let F'_{k+1} be the fibre of f_k and let $e'_{k+1} \colon A \to F'_{k+1}$ lift e_k as in (10). Define F_{k+1} as the reduced mapping cylinder of e'_{k+1}, let e_{k+1} be the obvious inclusion map, and let B_{k+1} and f_{k+1} result by shrinking the subset A of F_{k+1} to a point. We also need the fibre D_k of e_k, with projection $\varepsilon_k \colon D_k \to A$. The results of the preceding sections refer to F'_{k+1} and e'_{k+1}; obviously, they apply equally well to F_{k+1} and e_{k+1}, and will be used when passing from $\mathcal{C}_k$ to $\mathcal{C}_{k+1}$.

Definition 6.1. *The cocategory of A,* cocat A, *is the least integer $k \geq 0$ for which there is a map $r \colon F_k \to A$ such that $r \circ e_k \simeq 1$; if no such integer exists,* cocat $A = \infty$.

Remark 6.2. Interpreting F_k as a functor and e_k as a natural transformation, we see that the above definitions yield a left structure in the sense of [18] on the category of based topological spaces. A previous definition [9; 2.1] of the dual of Lusternik-Schnirelmann category may be restated as follows: cocat $A = 0$ if and only if A is contractible, and cocat $A \leq k+1$ if and only if there exists a fibration $F \to E \to B$ such that F dominates A and cocat $E \leq k$. Its equivalence with 6.1 is easily proved using the next result, in which cocategory is as in 6.1.

Lemma 6.3. *If $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration, then* cocat $F \leq$ cocat $E + 1$.

Proof. Suppose cocat $E = k$ and introduce the diagram

$$\begin{tikzcd}
F'_{k+1}(F) \arrow[rr, "j"] \arrow[d, shift left, "s'"] & & F_k(F) \arrow[rr, "f_k"] \arrow[d, "F_k(i)"] & & B_k(F) \arrow[d, "g"] \\
F \arrow[u, shift left, "e'_{k+1}"] \arrow[urr, "e_k"] \arrow[r, "i"] & E \arrow[r, "e_k"] & F_k(E) \arrow[r, "r"] & E \arrow[r, "p"] & B
\end{tikzcd}$$

in which j is the projection and r is given by 6.1. The pair $(F_k(E), E)$ has the homotopy extension property so we may assume that $r \circ e_k = 1$. Hence, by the naturality of e_k, $p \circ r \circ F_k(i) \circ e_k = *$ and there results g satisfying $p \circ r \circ F_k(i) = g \circ f_k$. Therefore, $p \circ r \circ F_k(i) \circ j \simeq 0$ and there results s' with $i \circ s' \simeq r \circ F_k(i) \circ j$ so that, since $j \circ e'_{k+1} = e_k$, $i \circ s' \circ e'_{k+1} \simeq i$. Let $\varrho : F \times \Omega B \to F$ express the operation associated with the given fibration. By [7; Th. 4.2] there is a map $u : F \to \Omega B$ such that

$$\varrho \circ ((s' \circ e'_{k+1}) \times u) \circ \triangle \simeq 1 : F \to F, \tag{26}$$

where $\triangle : F \to F \times F$ is the diagonal map. It follows from 3.3 that there is a map $v : F'_{k+1}(F) \to \Omega B$ with $v \circ e'_{k+1} \simeq u$. Define

$$s : F'_{k+1}(F) \xrightarrow{\triangle} F'_{k+1}(F) \times F'_{k+1}(F) \xrightarrow{s' \times v} F \times \Omega B \xrightarrow{\varrho} F .$$

Then, by (26), one has $s \circ e'_{k+1} \simeq 1$, i. e. $\operatorname{cocat} F \leq k + 1$.

Next, let $\varphi : \Omega A \# \Omega A \to \Omega A$ denote the adjoint to the WHITEHEAD product $[R, R]$ defined in § 5. Let

$$\varphi_0 = 1 \text{ and } \varphi_{k+1} : (\Omega A)^{(k+2)} \xrightarrow{1 \# \varphi_k} \Omega A \# \Omega A \xrightarrow{\varphi} \Omega A ,$$

where $X^{(k)}$ is the k-fold reduced product inductively defined by $X^{(1)} = X$ and $X^{(k+1)} = X \# X^{(k)}$. Define nil A as the least integer $k \geq 0$ such that $\varphi_k \simeq 0$; if no such integer exists, nil $A = \infty$. The construction of φ is valid if A, hence ΩA, has a non-degenerate base-point and the preceding definition is then equivalent to that introduced in [3]. As a motivation, recall [3; § 2] that

$$\operatorname{nil} A = \sup \operatorname{nil} \pi(\Sigma X, A) = \sup \operatorname{nil} \pi(X, \Omega A),$$

where nil G denotes the nilpotency class of the abstract group G, and X ranges over all based topological spaces.

Lemma 6.4. *If A is a countable CW-complex, then for every $k \geq 0$ there is a map λ_k such that φ_k is homotopic to the composite*

$$(\Omega A)^{(k+1)} \xrightarrow{\lambda_k} \Omega D_k \xrightarrow{\Omega \varepsilon_k} \Omega A .$$

If A is $(n - 1)$-connected $(n \geq 2)$, then λ_k is $(k + 2)(n - 1)$-connected.

Proof. We may obviously assume that $D_0 = A$ and $\varepsilon_0 = 1$, so that 6.4 holds true for $k = 0$ with $\lambda_0 = 1$. Suppose 6.4 is true for some $k \geq 0$ and introduce the diagram

$$\begin{tikzcd}
\Omega D_{k+1} \arrow[dr, "\Omega\varepsilon_{k+1}"'] & \Omega(A\, b\, D_k) \arrow[l, "\Omega\nu_{k+1}"'] \arrow[d, "f"] & \Omega\Sigma(\Omega A \# \Omega D_k) \arrow[l, "\Omega\vartheta"'] \arrow[d, "g"] & \Omega A \# \Omega D_k \arrow[l, "S"'] \arrow[d, "h"] & (\Omega A)^{(k+2)} \arrow[l, "1 \# \lambda_k"'] \arrow[dl, "1 \# \varphi_k"] \\
& \Omega A & \Omega\Sigma(\Omega A \# \Omega A) \arrow[l, "{\Omega[R,R]}"'] & \Omega A \# \Omega A \arrow[l, "S'"'] &
\end{tikzcd}$$

where $f = \Omega(\nabla \circ (1 \vee \varepsilon_k) \circ L)$, $g = \Omega\Sigma(1 \# \Omega\varepsilon_k)$, and $h = 1 \# \Omega\varepsilon_k$; S and S' are the natural embeddings, ν_{k+1} is given by 3.2 and ϑ by 5.1. The left triangle homotopy-commutes by 3.2. Next, by 5.1 and the naturality of the generalized WHITEHEAD product, one has

$$\nabla \circ (1 \vee \varepsilon_k) \circ L \circ \vartheta \simeq [R, R \circ \Sigma\Omega\varepsilon_k] = [R, R] \circ \Sigma(1 \# \Omega\varepsilon_k).$$

Commutativity in the second square is obvious. Finally, homotopy-commutativity in the right triangle is granted by the induction hypothesis. Obviously, $\Omega[R, R] \circ S' = \varphi$, and the first result follows upon defining

$$\lambda_{k+1} = \Omega\nu_{k+1} \circ \Omega\vartheta \circ S \circ (1 \# \lambda_k).$$

Next, it is easily seen that

$$D_k \text{ is } (k+1)(n-1)\text{-connected}.$$

Also, by 3.2, ν_{k+1} is $((k+3)(n-1)+1)$-connected, and the connectivity of λ_{k+1} follows easily from that of λ_k recalling that ϑ is a homotopy equivalence.

From 6.4 and 6.1 it is easy to derive the following two known results [9; Th. 2.12], [10; Th. 1.4]:

Proposition 6.5. nil $A \leq$ cocat A.

Proposition 6.6. cocat $A \leq k$ *if* A *is an* $(n-1)$*-connected* CW*-complex such that* $\pi_q(A) = 0$ *for* $q > (k+1)(n-1)$, $(n \geq 2,\ k \geq 0)$.

Let W-long A denote the least integer $k \geq 0$ for which any $(k+1)$-fold WHITEHEAD product $[\alpha_1, \ldots, [\alpha_k, \alpha_{k+1}] \ldots]$, with $\alpha_i \in \pi_{q_i}(A)$, $q_i \geq 1$, vanishes. We prove

Theorem 6.7. *Let A be an $(n-1)$-connected countable CW-complex $(n \geq 1)$ and let $k \geq 0$. If $\pi_q(A) = 0$ for $q > (k+1)(n-1)+n$, then* $\operatorname{cocat} A \leq k$ *if and only if* $\operatorname{nil} A \leq k$. *If $\pi_q(A) = 0$ for $q > (k+1)(n-1)+1$, then* $\operatorname{cocat} A \leq k$ *if and only if* W-long $A \leq k$.

Proof. If $n = 1$, we have [9; Th. 2.15], without assuming countability, $\operatorname{nil} \pi_1(A) = \operatorname{nil} A = \operatorname{cocat} A$. Let $n \geq 2$. Let $R: \Sigma\Omega D_k \to D_k$ satisfy $R\langle s, \omega\rangle = \omega(s)$. If $\varphi_k \simeq 0$, then, by 6.4 and the naturality of R, $\varepsilon_k \circ R \circ \Sigma\lambda_k \simeq 0$. Hence, there is a map $s: H \to A$ such that $s \circ \eta \simeq 1$, where $\eta: A \to H$ is the inclusion map and $H = A \cup C_0\Sigma(\Omega A)^{(k+1)}$ results upon attaching the cone by means of $\varepsilon_k \circ R \circ \Sigma\lambda_k$. The map $\Phi = 1 \cup C_0(R \circ \Sigma\lambda_k): H \to A \cup C_0 D_k$, where $C_0 D_k$ is attached by means of ε_k, and the extension $r: A \cup C_0 D_k \to F_k$ of e_k, given by 1.1, obviously satisfy $r \circ \Phi \circ \eta = e_k$. It follows from 1.1 and 6.4 that the composite $r \circ \Phi$ is $((k+2)(n-1)+2)$-connected and an obstruction argument yields a map $t: F_k \to A$ satisfying $t \circ r \circ \Phi \simeq s$. Hence, $t \circ e_k \simeq 1$ and the first result is proved. Next, $(\Omega A)^{(k+1)}$ is $((k+1)(n-1)-1)$-connected and its $(k+1)(n-1)$-dimensional homotopy group can be identified to the $(k+1)$-fold tensor product in the left bottom corner of the diagram

$$\begin{array}{ccc}
\pi_n(A) \otimes \cdots \otimes \pi_n(A) & \xrightarrow{W} & \pi_{(k+1)(n-1)+1}(A) \\
\downarrow & & \downarrow \\
\pi_{n-1}(\Omega A) \otimes \cdots \otimes \pi_{n-1}(\Omega A) & \xrightarrow{\Phi} & \pi_{(k+1)(n-1)}(\Omega A)
\end{array}$$

The top row is given by the $(k+1)$-fold WHITEHEAD product, Φ is the homomorphism induced by φ_k, and the verticals are given by a natural isomorphism $\pi_{q+1}(A) \to \pi_q(\Omega A)$. It follows from a result by SAMELSON [20] that the diagram commutes up to a sign, so that $\Phi = 0$ if $W = 0$. Since $\pi_q(\Omega A) = 0$ for $q > (k+1)(n-1)$, an obstruction argument now implies $\varphi_k \simeq 0$ and the second result follows from the first.

Remark 6.8. It follows from [14] that $\operatorname{cocat} A \leq 1$ if and only if A is an H-space, and 6.5, 6.6, 6.7 generalize well known results on H-spaces; the first part of 6.7 generalizes a theorem by SUGAWARA [22], and the second dualizes an unpublished result by I. BERSTEIN on LUSTERNIK-SCHNIRELMANN category.

As a final result, we express $\pi_1(F_k(A))$ in terms of $\pi_1(A)$. Recall that the lower central series of a group π consists of the commutator subgroups $\pi_{(n)}$ of π, given by $\pi_{(0)} = \pi$ and $\pi_{(n+1)} = [\pi, \pi_{(n)}]$.

Theorem 6.9. *Let A be a connected CW-complex with fundamental group π. Then, for every $k \geq 0$, $e_k : A \to F_k(A)$ induces an epimorphism of fundamental groups under which $\pi_1(F_k(A))$ is isomorphic to $\pi/\pi_{(k)}$.*

Proof. Suppose first that $m = n = 1$ in 3.1. Then, F_d is 0-connected, 2.1 implies that $A \cup CF_d \to X$ is homology 2-connected, and a 5 lemma argument reveals that the same holds for $\Sigma F_d \to B$. Since ΣF_d and B are 1-connected, applying the Hurewicz-Whitehead theorem and then passing to loop spaces we see that $\Omega \Sigma F_d \to \Omega B$ is 1-connected. Since $F_d \to \Omega \Sigma F_d$ is 1-connected, it follows that φ is 1-connected, and a 5 lemma argument reveals that also e is 1-connected. An obvious induction argument now reveals that e_k is 1-connected for all $k \geq 0$. Obviously, cocat $F_k(A) \leq k$ so that, by 6.5, nil $\pi_1(F_k(A)) \leq k$; therefore, the kernel E of the epimorphism induced by e_k contains $\pi_{(k)}$. To prove the converse, let Y be a connected aspherical CW-complex with fundamental group $\pi/\pi_{(k)}$, and let $g : A \to Y$ induce the canonical homomorphism $\pi \to \pi/\pi_{(k)}$. One has nil $\pi/\pi_{(k)} \leq k$ so that, by [9; Th. 2.15], cocat $Y \leq k$ and there results a map $r : F_k(Y) \to Y$ such that $r \circ e_k(Y) = 1$, hence $r \circ F_k(g) \circ e_k = g$. This obviously implies $E \subset \pi_{(k)}$, and 6.9 is proved. The crux of the matter is Lemma 5.4 in [13] which implies 2.15 in [9].

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