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Complete Surfaces in E^3 with Constant Mean Curvature[1])

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§ 1. Introduction

In this paper we discuss complete surfaces in E^3 which have constant mean curvature H. Our main result states that a complete, immersed surface with $H \equiv c \neq 0$ on which the Gaussian curvature K does not change sign must be a sphere or a right circular cylinder.

It has long been conjectured that among compact surfaces only the sphere has constant H. This conjecture was proved by H. HOPF ([7], p. 241) for surfaces of genus zero. A proof for surfaces of arbitrary genus was given by A. D. ALEXANDROV ([2], or [8], Chapter 7) under the assumption that the surfaces have no self intersections, i.e., that they are embedded in E^3. While the theorem stated below is a step in the direction of characterizing all complete surfaces in E^3 with constant H, it does not give new information about the compact case.

On the other hand, our results do serve to verify for the very special case of surfaces with $H \equiv c \neq 0$ a conjecture suggested to the authors by J. MILNOR. The conjecture, motivated by the recent result of EFIMOV [5], applies to arbitrary complete surfaces in E^3 on which $K \not\equiv 0$ does not change sign. It states that such a surface must have an umbilic, or else have points at which both principal curvatures simultaneously assume values arbitrarily close to zero. A brief discussion of the conjecture is contained in § 4.

§ 2. The Main Results

The interesting variety of results about complete minimal surfaces on which $H \equiv 0$ suggests the study of complete surfaces in E^3 on which $H \equiv c \neq 0$. That the behavior of such surfaces can differ markedly from that of minimal surfaces is immediately seen. To cite one distinction, only the plane among complete minimal surfaces has a normal map which omits a set of positive capacity on the sphere ([11], p. 345). Yet, there are complete non-cylindrical surfaces in E^3 on which $H \equiv c \neq 0$ whose normal map omits a non-empty neighborhood on the sphere.

Such surfaces are obtainable by rotating a curve in parametric form around an axis. In addition to lines and circles, two classes of curves can be distinguished which satisfy the condition which insures that $H \equiv c \neq 0$ on the surface of revolution gener-

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ated. The first class consists of curves with no self intersections. Each of these when rotated yields a complete, doubly connected surface on which $H \equiv c \neq 0$, whose normal map image is an annular region on the sphere. The second class consists of curves with self intersections. Each of these when rotated yields a complete surface with $H \equiv c \neq 0$ which is a topologically immersed cylinder, and whose normal map covers the entire sphere. (The relevance of surfaces of revolution in this connection was pointed out to us by R. FINN. He suggested obtaining the two classes of complete surfaces indicated above by successive reflections of the non-parametric surfaces with $H \equiv c$ which he considers in [6], pp. 156–7.) In view of Lemma 4 below, we note that both classes of curves give rise to surfaces on which the quantity $H^2 - K$ is bounded away from zero.

It can be shown that the Gaussian curvature K changes sign on the surfaces of revolution just described, while, of course, $K \leq 0$ must hold any minimal surface. The following result indicates the strategic nature of this fact.

THEOREM. *A complete surface in E^3 with $H \equiv c$ on which K does not change sign is either a sphere, a minimal surface, or a right circular cylinder.*

The arguments needed to establish the Theorem are given in Section 3. In fact, the Theorem merely combines the statements of Propositions 1 and 2 below.

§ 3. Proof of the Theorem

In the discussion which follows, S denotes an oriented surface smoothly immersed in E^3. Where convenient, S also denotes the Riemann surface whose conformal structure is determined in the usual way, with $x + iy$ a conformal parameter if and only if x, y are isothermal coordinates.

LEMMA 1: *If $K \geq 0$ $(K \leq 0)$ over the domain D of isothermal coordinates x, y on S, then* $\log \lambda$ *with $\lambda = \lambda(x, y)$ defined by*

$$\mathrm{I} = \lambda(x, y)\{dx^2 + dy^2\}$$

is superharmonic (subharmonic) on D.

Proof. The theorem egregium equation gives

$$K = \frac{-1}{2\lambda} \Delta \log \lambda,$$

so that $\Delta \log \lambda \leq 0$ if $K \geq 0$ while $\Delta \log \lambda \geq 0$ if $K \leq 0$.

LEMMA 2: *Near any non-umbilic point on an S with $H \equiv c$ there are coordinates x, y in terms of which*

$$\begin{aligned} (\sqrt{H^2 - K})\, \mathrm{I} &= dx^2 + dy^2 \\ (\sqrt{H^2 - K})\, \mathrm{II} &= (H + \sqrt{H^2 - K})\, dx^2 + (H - \sqrt{H^2 - K})\, dy^2 . \end{aligned} \tag{1}$$

Proof. Using only isothermal coordinates anywhere on S, consider the expression

$$(L-N)-2iM \tag{2}$$

with L, M, N the coefficients of the second fundamental form II. It is well known ([7], p. 241) that $H\equiv c$ on S if and only if (2) is always analytic. Since umbilics can be characterized as the zeros of (2), it follows that an S with $H\equiv c$ is either totally umbilic or else has only isolated umbilics, if any. Since (2) varies with changing isothermal coordinates like the coefficient of a quadratic differential, it follows also that near any non-umbilic point on S, special isothermal coordinates can be found in terms of which $(L-N)-2iM=2$ ([3], p. 103), so that $(L-N)=2$, $M=0$. Because $\mathrm{I}=\lambda(dx^2+dy^2)$, we recognize x, y as lines of curvature coordinates. But then, letting $k_1\geq k_2$ denote the principal curvatures on S, we have $L=k_1\lambda$, $N=k_2\lambda$, with $2H=k_1+k_2$, $K=k_1k_2$, and simple arithmetic yields (1). We have also proved thereby the following statement.

LEMMA 3: *Away from umbilic points on an S with $H=c$, the metric $(\sqrt{H^2-K})\mathrm{I}$ is flat.*

LEMMA 4: *If $\varepsilon^2\leq(H^2-K)$ for a fixed $\varepsilon>0$ on a complete S with $H\equiv c$, then the universal covering surface Σ of S is conformally the plane, so that S is conformally the plane, the once punctured plane, or the torus.*

Proof. The conformal metric $\varepsilon\mathrm{I}$ is complete on S. But since $\varepsilon\mathrm{I}\leq(\sqrt{H^2-K})\,\mathrm{I}$, the flat conformal metric $(\sqrt{H^2-K})\,\mathrm{I}$ is also complete on S. Thus, Σ with the lifted metric $(\sqrt{H^2-K})\,\mathrm{I}$ is isometric to the plane ([10], p. 394).

Of special importance below is the fact that a complete surface with $H\equiv c\neq 0$ and $K\leq 0$ satisfies the hypotheses of Lemma 4 with $\varepsilon=|c|$. This implies in particular that such an S is parabolic, so that a subharmonic (superharmonic) function bounded from above (below) on S must be a constant ([1], p. 204).

LEMMA 5. *Let $K\geq 0$ on a complete S. Then either $K\equiv 0$ or else S is simply connected. And either S is conformally the sphere or else S is parabolic.*

Proof. These statements follow easily from well known facts. For, if $K\not\equiv 0$ while S is not simply connected, then over the universal covering surface Σ of S,

$$\iint_{\Sigma} K\,dA=\infty$$

which would contradict the theorem of COHN-VOSSEN ([4], p. 80) which implies instead that

$$\iint K\,dA\leq 2\pi. \tag{3}$$

Next, a compact S with $K \geq 0$ must have genus zero by the Gauss-Bonnet theorem, and all compact surfaces of genus zero are conformally equivalent to the sphere. Finally, if a complete S with $K \geq 0$ is not compact, the theorem of Blanc-Fiala-Huber states that S is parabolic ([9], p. 71).

PROPOSITION 1. *If $K \leq 0$ and $H \equiv c$ on a complete S, then S is either a minimal surface or a right circular cylinder.*

Proof. If $c = 0$, S is a minimal surface. If $c \neq 0$, Lemmas 1 and 2 with $K \leq 0$ imply that $\log (H^2 - K)$ is superharmonic and bounded from below by $\log (c^2) \neq -\infty$. Using Lemma 4, we conclude that S is parabolic, so that $(H^2 - K)$, and therefore K, must be a constant. It follows then from (1) that I is flat, making $K \equiv 0$. Because $c \neq 0$, equations (1) give the fundamental forms I and II of a right circular cylinder, and by the fundamental uniqueness theorem of surface theory, S must itself be (locally) a right circular cylinder. That S is globally a right circular cylinder follows easily from the fact that any surface on which $H \equiv c$ is real analytic ([8], p. 114), though other arguments would suffice.

PROPOSITION 2. *If $K \geq 0$ and $H \equiv c$ on a complete S, then S is either a sphere, a plane, or a right circular cylinder.*

Proof. If $K \equiv 0$, S is either a plane or a right circular cylinder. If $K \not\equiv 0$ and S is compact, then S is conformally the sphere by Lemma 5, and thus must be a sphere ([7], p. 234). If $K \not\equiv 0$ and S is not compact, then Lemma 5 states that S is parabolic, while $H \equiv c \neq 0$ must hold. Here Lemmas 1 and 2 with $K \geq 0$ imply that $\log (H^2 - K)$ is subharmonic and bounded from above by $\log (c^2) \neq +\infty$. Since the value $-\infty$ of $\log (H^2 - K)$ at the possible isolated umbilics on S causes no trouble ([1], p. 135), we conclude once again that K is constant, making I flat and $K \equiv 0$, a contradiction here.

§ 4. The Conjecture

It is well known that a complete surface on which $K \geq c > 0$, must be compact. The Gauss-Bonnet theorem then implies that such a surface has genus zero, and thus cannot be free of umbilics. Recalling the recent EFIMOV result [5], which states that $K \leq c < 0$ cannot hold on a complete surface in E^3, we conclude that no complete, umbilic free surface exists in E^3 on which K is bounded away from zero. The following conjecture would generalize this statement, and would yield Proposition 1 (and Proposition 2 if stated for an umbilic free S) as a simple consequence.

MILNOR CONJECTURE. If the expression

$$k_1^2 + k_2^2 \tag{4}$$

is bounded away from zero on a complete, umbilic free surface in E^3, then K either must change sign or else must vanish identically.

Put another way, the conjecture states that the curvature diagram of a complete, umbilic free surface in E^3 on which $K \not\equiv 0$ does not change sign, cannot avoid a neighborhood of the origin in the k_1, k_2-plane. Thus it may be thought of as part of the larger problem of characterizing the curvature diagrams of all complete surfaces in E^3.

It is easy to check that the conjecture coincides with the assertion of Efimov's theorem for quasiminimal surfaces. On a quasiminimal surface, $K \leq 0$ and, except where $k_1 = k_2 = 0$, the ratio k_1/k_2 remains bounded away from both zero and $-\infty$. Thus K is bounded away from zero on a quasiminimal surface if and only if the expression (4) is bounded away from zero. A proof of the Efimov theorem for quasiminimal surfaces can be found in [10].

Our Propositions 1 and 2 check the conjecture for the much more special case of surfaces with $H \equiv c \neq 0$. Some of our procedures shed light on the $K \geq 0$ case of the conjecture as well. For, by Lemma 5, any umbilic free surface on which $K \geq 0$ while $K \not\equiv 0$ must be conformally the plane. Using global isothermal coordinates, Lemma 1 would then provide a superharmonic function, $\log \lambda$. It would remain to show that if (4) is bounded away from zero, then $\log \lambda$ is bounded from below. The consequence $\lambda \equiv c$ would imply $K \equiv 0$, a contradiction.

Further remarks and references

(added June 18, 1966).

1. For an extensive discussion of surfaces of revolution of constant mean curvature, together with references and "practical" applications, see the book "On Growth and Form" by D'ARCY W. THOMPSON, Cambridge University Press, 1942, p. 368.

2. Statements equivalent to lemma 3 above had been obtained long ago by various authors, including RICCI. (See G. RICCI-CURBASTRO, *Opere*, Vol. 1. p. 411.) In the form stated here, it may be found in K. M. BELOV, "On surfaces of constant mean curvature," Sibirskii Mat. J. *5* (1964), 748.

3. A result which fits into the circle of ideas discussed in this paper is given in the recent paper of S.-S. CHERN, "On the curvatures of a piece of hypersurface in euclidean space," Abh. Math. Sem. Univ. Hamburg *29* (1965), 82. It is the following: a hypersurface $z = F(x_1, \ldots, x_m)$ of constant mean curvature, defined for all values of $x_1, \ldots, x_m$, is necessarily a minimal hypersurface.

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ADDED IN PROOF (Dec. 19, 1966): A recent paper by J. A. WOLF (Proceedings Amer. Math. Soc. *17* (1966) 1103–1111) contains a number of theorems on surfaces of constant mean curvature. His first theorem (p. 1104) coincides with lemma 2 of the present paper.