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Degeneracy of Orbits of Actions of R^m on a Manifold[1])

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1. Introduction

Let V be a compact manifold (possibly with boundary) and let $\varphi : R^m \times V \to V$ be an action of Euclidean m-space (considered as an abelian group) on V. This means that $\varphi(0,p)=p$ and $\varphi(x+y,p)=\varphi(x,\varphi(y,p))$ for any x, y in R^m and p in V. If p is a point of V, the orbit of p, denoted by $O(p)=\{\varphi(x,p) : x \in R^m\}$ is a submanifold of V of dimension $d(p)$. If $I(p)=\{x \in R^m : \varphi(x,p)=p\}$ is the isotropy subgroup at p, $I(p)$ is closed in R^m and $O(p)$ is diffeomorphic to $R^m/I(p)$, which is the product of Euclidean space by a torus. The larger $I(p)$, the more degenerate is the orbit $O(p)$. Our purpose here is to investigate some conditions on V or φ which force an orbit to be degenerate.

Let $m(\varphi)$ denote the minimum value of $d(p)$. We say that φ is *locally free* if $m=m(\varphi)=d(p)$ and that φ is *foliating* if $d(p)=m(\varphi)$. The maximum possible values of $m(\varphi)$ for, respectively, locally free, foliating, and arbitrary actions are called the *rank*, *super rank*, and *total rank* of V and they are denoted by $R(V)$, $S(V)$, and $T(V)$. It is obvious that

$$0 \leqq R(V) \leqq S(V) \leqq T(V) \leqq n . \tag{1.1}$$

In this terminology, some of the known results in the subject are:

1. (H. HOPF [2]). The Euler characteristic of V is zero if and only if $R(V)>0$.
2. (LIMA [3]). $R(S^3)=1$.
3. (ROSENBERG [7]). $R(S^1 \times S^2)=1$.
4. (SACKSTEDER [8]). If φ is a foliating action with $m(\varphi)=n-1$, every non-dense orbit is properly imbedded in V.
5. (SACKSTEDER [8], cf. REEB [6], p. 110). Either there is a covering map $S^1 \times R^{n-1} \to V$, or every foliating action φ on V with $m(\varphi)=n-1$ has a compact orbit.
6. (LIMA and ROSENBERG [5]). Suppose that there is no covering map $S^1 \times R^{n-1} \to V$ and the fundamental group of V does not contain an $(n-1)$-fold direct sum of the integers. Then $R(V)<n-1$. (This contains 2. and 3. above).
7. (LIMA [4]). If $n=2$, $R(V)=S(V)=T(V)$.
8. (TRIVIAL). If $T(V)=n$, then $R(V)=S(V)=T(V)=n$ and V is the n-torus.

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Our results are most closely related to LIMA's Theorem 7. Theorem 1 asserts roughly that there is no need to consider actions of R^m with $m>n$ for purposes of investigating degeneracy of orbits because there is always an action of R^n with the same orbits as $\varphi : R^m \times V \to V$ in this case. Theorem 2 and its corollary give conditions under which it is possible to find a locally free action with the same orbits as a given foliating action. Theorems 3, 4, and 5 are our main results. Theorem 5 asserts that the conclusion of the Theorem of LIMA and ROSENBERG cited in 6. can be strengthened to: "Then $T(V)<n-1$." It therefore implies LIMA's Theorem cited in 7. (under different differentiability assumptions) and strengthens 2. and 3.

Various regularity assumptions are required for the proofs of our results and those cited above. These will be made explicit at the appropriate place. The most natural condition for our methods is described by the following definition. An action $\varphi : R^m \times V \to V$ is said to be a *piecewise C^k action* if V is the union of a countable set of compact submanifolds (with boundary) $V_1, V_2, \ldots$ such that each point of V is contained in only finitely many V_i's and the restriction of φ to $R^n \times V_i$ defines a C^k action on V_i for $i=1,2,\ldots$ The results cited in 4., 5., and 6. are valid for piecewise C^2 actions, although in some cases they were originally stated for C^2 actions. No essential changes in the proofs are required for the extension to the piecewise C^2 case.

2. Preliminaries

Here we collect some results which will be needed later. Most of them are well-known and easy to prove. Let $\varphi : G \times V \to V$ be a piecewise C^2 action of a connected Lie group G on a compact n-manifold V. Let A denote the Lie algebra of G. It is easy to see that to each element x of A there corresponds a vector field φ'_x on V. For let $\{g(t) : 0 \leqq t \leqq 1\}$ be a C^1 curve in G such that $g(0)$ is the identity and $g'(0)$ corresponds to x. For any point p of V consider the curve $h(t)=\varphi(g(t), p)$. Then $\varphi'_x(p)$ is by definition the tangent vector at p determined by $h'(0)$. Each element g of G induces a map g^* which sends a tangent vector at p to one at $\varphi(g,p)$. When G is abelian it is easily seen that the vector fields φ'_x are invariant under the maps g^*. Therefore we have:

LEMMA 2.1. *Let $\varphi : R^m \times V \to V$ be a piecewise C^2 action. Let x be in R^m (considered as the Lie algebra of R^m) and let φ'_x be the corresponding R^m invariant vector field. Then, if $\varphi'_x(p)=0$, $\varphi'_x(q)=0$ for all q in $O(p)$.*

Under the conditions of Lemma 2.1, the subspace $B(p)=\{x \in R^m : \varphi'_x(p)=0\}$ will be called the *isotropy subspace* at p. Note that $B(p) \subset I(p)$ and Lemma 2.1 asserts that $B(p)=B(q)$ if q is in $O(p)$.

In all of the lemmas below φ is a piecewise $C^k (k \geqq 2)$ action of R^m on V and in the first $p_1, p_2, \ldots$ is a sequence in V such that $p=\lim p_i$.

LEMMA 2.2. *The isotropy subgroups and subspaces are upper semi-continuous in the sense that* $B(p) \supset \limsup B(p_i)$ *and* $I(p) \supset \limsup I(p_i)$.

The proof of this lemma is straightforward. The following one is well-known.

LEMMA 2.3. *Let x and y be in R^m. Then the vector fields φ'_x and φ'_y commute, that is* $[\varphi'_x, \varphi'_y]=0$. *Conversely, if $X_1, \ldots, X_m$ are piecewise C^k vector fields on V which commute, there is a piecewise C^k action $\varphi: R^m \times V \to V$ and a basis $e_1, \ldots, e_m$ of R^m such that $X_i = \varphi'_x$ if $x = e_i$. (Here, if V is a manifold with boundary $X_1, \ldots, X_m$ are required to be tangent to the boundary.)*

3. Actions with the Same Orbits

It is possible for two distinct actions on V to have the same orbits. In this case the actions are equivalent for most of the purposes of this paper. In this and the following section we investigate this phenomenon. Let $\varphi: R^m \times V \to V$ be a piecewise C^k action on a connected (not necessarily compact) n-manifold, where $k \geqq 2$.

THEOREM 1. *If $m > n$, almost all $(m-1)$-dimensional subspaces K of R^m are such that the orbits of φ restricted to K are the same as the orbits of φ. Therefore, there is a piecewise C^k action $\psi: R^n \times V \to V$ with the same orbits as φ.*

Proof. Define $B(p)$ as in section 2 and note that $B(p)$ has dimension $m-d(p)$. We say that a subspace F of R^m is *effective at p* if R^m is spanned by $F \cup B(p)$, and otherwise F is called *ineffective* at p. F is effective at p if and only if $O(p)=\{\varphi(x,p) : x \in F\}$. Moreover, if F is of dimension $m-1$, F is ineffective if and only if F contains $B(p)$. Lemma 2.1 shows that if F is effective at p, F is effective at every point in $O(p)$.

The theorem will be proved by showing that the $(m-1)$-dimensional subspaces which are ineffective at some p in V form a subset of the Grassman manifold $M_{m,m-1}$ of dimension not greater than $n-1 < m-1 = \dim M_{m,m-1}$. It suffices to prove this for all points q where $d(q)=d(p)=d$ in a small neighborhood of an arbitrary point p of V, since countably many such sets cover V.

First suppose that V has a Riemannian metric and let $G(q)$ be the non-negative definite quadratic form on R^m which assigns to a vector x in R^m the length of $\varphi'_x(q)$ in the Riemannian metric on V. Note that a vector is of length zero in this metric if and only if it is in $B(q)$. Let $C(q)$ denote the subspace of R^m spanned by the eigenvectors belonging to the $m-d$ smallest eigenvalue of $G(q)$ with respect to a fixed positive definite metric on R^m. Note that near q, $C(q)$ depends $(k-1)$ times differentiably on q. Moreover, $C(q)=B(q)$ if $d(q)=d$.

Let S be the $(d-1)$-dimensional unit sphere in a subspace H of R^m which is complementary to $C(p)$. Let T be an $(n-d)$-dimensional submanifold through p which intersects the orbits transversally and is small enough so that $C(q)$ varies differentiably and is transversal to H for every q in T. If (q, s) is in $T \times S$, let $F(q, s)$ denote the $(n-1)$-dimensional subspace of R^m which contains $C(q)$ and whose intersection with H is perpendicular to s in the positive definite metric on R^m. If $d(q)=d$, $B(q)=C(q)$, hence $F(q, s)$ is ineffective by the remark at the beginning of the proof. Conversely, every ineffective $(m-1)$-dimensional subspace at q is of this type. Therefore the set of $(m-1)$-subspaces which are ineffective for some q in T such that $d(q)=d$ is contained in the range of F. Since $\dim T \times S = n-d+d-1 = n-1 < m-1 = \dim M_{m,m-1}$, Sard's Theorem implies that almost all $(m-1)$-subspaces are effective at every q in T such that $d(q)=d$. Since $B(q)=C(q)$ is constant along orbits if $d(q)=d$, this shows that almost all $(m-1)$-subspaces are effective for every q near p such that $d(q)=d$. This proves Theorem 1.

4. Foliating and Locally Free Actions

If φ is a piecewise C^k action $(k \geqq 2)$ of R^m on a the n-manifold V, V/φ will denote the quotient space obtained by identifying points on the same orbit and $f: V \to V/\varphi$ the projection map. Now assume that φ is a foliating action with $d(p)=d$ and define $B(p)$ as in section 2. Let $A(p)$ be the orthogonal compliment of $B(p)$ relative to some metric on R^m, so that $A(p)$ is always a point of the Grassman manifold $M_{m,d}$. Moreover, $A(p)$ is constant on the orbits of φ so that there is a map $h: V/\varphi \to M_{m,d}$, which sends $O(p)$ to $A(p)$. Let $V_{m,d}$ be the Stiefel manifold of orthogonal d-frames in R^m and let $\pi: V_{m,d} \to M_{m,d}$ be the usual projection.

Theorem 2. *Under the conditions described above, there exists a piecewise C^{k-1} locally free action $\psi: R^d \times V \to V$ whose orbits agree with those of φ if there is a piecewise C^{k-1} map $g: V/\varphi \to V_{m,d}$ such that $h = \pi g$.*

Note: Here g is said to be a piecewise C^{k-1} map if gf is C^{k-1} on V_i, $i=1, 2, \dots$, where each V_i is a compact n-dimensional manifold, $V = \cup V_i$, and every point of V is in only finitely many V_i's.

Proof. For typographical convenience we modify the notation $\varphi'_x(p)$ established in Section 2 by writing $\varphi'(x, p)$ instead. Let $e_1, \dots, e_m$ be a basis for R^m and define vector fields $X_1, \dots, X_d$ by $X_i(p) = \varphi'(g_i(p), p)$, where $g_i(p)$ is the i'th vector in the frame $g(O(p))$. If $g_i(p) = \sum_{j=1}^m a_{ij}(p) e_j$, each a_{ij} will be constant along orbits. Then if $Y_j(p) = \varphi'(e_j, p)$, $X_i(p) = \sum_{j=1}^m a_{ij}(p)\, Y_j(p)$ and $Y_k a_{ij}$ (the directional derivative of a_{ij} in the direction Y_k will be zero for all $k=1, \dots, m$. Now Lemma 2.3 implies that $[Y_i, Y_j]=0$, hence $Y_k a_{ij}=0$ implies that $[X_i, X_k]=0$. The vector fields $X_1, \dots, X_d$ are linearly independent because $g_1(p), \dots, g_d(p)$ span $A(p)$. The conclusion now follows from Lemma 2.3.

Corollary 4.1. *Let $\varphi: R^m \times V \to V$ be as in Theorem 2 with $m = d+1$ and $2d > n$. Then there is a piecewise C^{k-1} locally free action $\psi: R^d \times V^* \to V^*$* where either $V^* = V$

or $c: V^* \to V$ is a two sheeted covering. In the first case, the orbits of and φ agree and in the second c maps each ψ orbit homeomorphically onto a φ orbit.

Proof. The assumption that $m=d+1$ implies that $M_{m,d}=P^d=$ projective d-space here. Therefore hf: $V \to P^d$. Let $b: S^d \to P^d$ be the two sheeted covering and construct $c: V^* \to V$ in such a way that there is a map $(hf)^*: V^* \to S^d$ such that $b(hf)^* = (hf)c$. The covering c can turn out to be either the identity map or a two-sheeted covering. The action φ can be lifted to an action φ^* on V^* and one has the following commutative diagram in which the existence of g^* will be proved later

$$\begin{tikzcd}
 & & SO(m) \arrow[r, "\alpha"] \arrow[d, "\pi^*"] & V_{m,d} \arrow[ddl, "\pi"] \\
V^* \arrow[r, "f^*"] \arrow[d, "c"'] & V^*/\varphi^* \arrow[r, "h^*"] \arrow[d] \arrow[ur, dashed, "g^*"] & S^d \arrow[d, "b"'] & \\
V \arrow[r, "f"] & V/\varphi \arrow[r, "h"] & P^d &
\end{tikzcd}$$

In the diagram, the map π^* sends an m-frame in R^m to the mth vector in it and α sends an m-frame to the $(m-1)$-frame obtained by removing the last vector. The piecewise C^{k-1} map g^* is constructed as follows: First observe that the image of $(hf)^*=$image $h^*f^*=$ image h^* is a zero set in S^d, because $(hf)^*$ is of rank not greater than $n-d<d$. Therefore the image is contractible in S^d and the existence of g^* follows from the fact that the fibration π^* is trivial above the complement of a point of S^d. Now the existence of ψ follows by an application of Theorem 2 in which V^*, φ^*, bh^*, and αg^* replace V, φ, h, and g, respectively. The remaining assertions of Corollary 4.1 are simple consequences of the commutativity of the above diagram.

An example: Let V be the Klein bottle which we represent as the square $|x| \leqq 1, |y| \leqq 1$ with the identifications $(x,-1)=(x,1)$ and $(1,y)=(-1,-y)$. Define the action $\varphi: R^2 \times V \to V$ by $\varphi((u,v),(x,y))=(x, y+(u \cos x\pi/2)+(v \sin x\pi/2))$. It is easy to check that the action is compatible with the identifications and has as orbits the sets $x=$ const., hence φ is a foliating action. However, there is no locally free action with the same orbits as φ. In this example, $m=n=2$ and $d=1$; however one can obtain examples where $2d>n$ and $m=d+1$ from this one, e.g. on $V \times S^1$. Such examples show that the introduction of the covering space V^* is essential in Corollary 4.1.

5. The Main Theorems

Here, $\varphi: R^m \times V \to V$ is a piecewise C^k action $(k \geqq 2)$ on a compact n-manifold V.

THEOREM 3. *Suppose that every orbit of φ is of dimension n or $n-1$. Then there is a piecewise C^k foliating action $\psi: R^n \times V \to V$ such that all orbits are of dimension $n-1$.*

In view of Corollary 4.1, Theorem 3 and 8 of Section 1 imply the following theorem.

THEOREM 4. *Under the conditions of Theorem 3 there is a piecewise C^{k-1} locally free action of R^{n-1} on V^*, where either $V^* = V$ or V^* is a 2-fold covering of V. Therefore if $T(V) \geqq n-1$, $T(V) = S(V) = R(V^*)$, where T, S are defined with respect to piecewise C^k actions and R is defined with respect to piecewise C^{k-1} actions.*

This theorem, together with the theorem of LIMA and ROSENBERG [5] (cf. 6) Section 1 above) implies

THEOREM 5. *Suppose that the compact connected n-manifold V is such that there is no covering map $S^1 \times R^{n-1} \to V$ and the fundamental group of V does not contain an $(n-1)$-fold direct sum of the integers. Then $T(V) < n-1$.*

Special cases of the conclusion of Theorem 5 are $T(S^n) < n-1$ and $T(S^1 \times S^n) < n$.

6. Lemmas

LEMMA 6.1. *Let V_0 be a (not necessarily compact) manifold with a foliated structure of co-dimension one, cf.* [1, 6]. *Let F be a compact leaf of the foliation. Then there is a tubular neighborhood B of F such that any component J of $B - F$ satisfies either* (i) *every leaf which intersects J contains F in its closure, or* (ii) *there are compact leaves in J arbitrarily close to F.* This lemma is essentially the same as a theorem of REEB ([6], p. 139). We therefore omit the proof.

The rest of the lemmas employ the hypotheses of Theorem 3. Moreover, it will always be assumed that $m = n$, which does not represent a loss of generality in view of Theorem 1. The symbol $\cong$ will mean "diffeomorphic to" and T^{n-1} will denote the $n-1$ torus.

LEMMA 6.2. *Let p be any point of V. Then the closure of $O(p)$ contains a compact orbit unless φ is already a foliating action.*

Proof. Assume that φ is not a foliating action. It can be assumed that $d(p) = n-1$, because if $d(p) = n$, p can be replaced by any point in the boundary of $O(p)$. Let $R^{n-1} \subset R^n$ be a subspace which is effective at p (cf. Section 3). Then there is an open neighborhood V_0 of $O(p)$ such that φ restricted to $R^{n-1} \times V_0$ defines a locally free action on V_0 by Lemma 2.1. Applying Theorem 8 of [8] and [6, p. 103] to this action, one sees that $O(p)$ contains a compact leaf in its closure.

LEMMA 6.3. *Let p be a point of V such that there is a sequence $p_1, p_2, \ldots$ of points of V with $O(p_i) \cong T^{n-1}$ and $p = \lim p_i$. Then $O(p) \cong T^{n-1}$.*

Proof. Since $d(p_i) = n-1$, Lemma 2.2 implies that $d(p) = n-1$ because of

$m(\varphi)=n-1$. Let q be a point in the closure of $O(p)$ such that $O(q)\cong T^{n-1}$. Such a q (possibly $q=p$) exists by Lemma 6.2. Let $R^{n-1}\subset R^n$ be effective at q, and let V_0 be a neighborhood of $O(q)$ such that the restriction of φ to $R^{n-1}\times V_0$ defines a locally free action on V_0. Apply Lemma 6.1 to the foliation defined by this action, taking $F=O(q)$. There must be points of $O(p)$ in some component J of $B-O(q)$ arbitrarily close to $O(q)$ if $O(p)$ is not compact. Thus the alternative (ii) of the conclusion of Lemma 6.1 could not hold in this case because the compact orbits separate J. Therefore (i) holds. However, this is also impossible because $O(p_i)$ must intersect J for large i, but such $O(p_i)$, being compact, cannot contain $O(q)$ in their closure. This shows that $O(p)$ must be compact. This proves Lemma 6.3.

Now we define W to be the set of points p of V such that $p=\lim p_i$, where $O(p_i)\cong T^{n-1}$ and $p\neq p_i$. W is clearly closed, hence compact, and if p is in W, $O(p)\cong T^{n-1}$, by Lemma 6.3.

LEMMA 6.4. *There exist a finite number of compact n-dimensional submanifolds of V, $K_1, \ldots, K_r$ such that*

(i) $W\subset \text{interior} \cup K_i$

(ii) *The restriction of φ to $R^m\times K_i$ defines an action on K_i;*

(iii) *If $i\neq j$, $K_i\cap K_j$ is either empty or consists of at most two compact orbits;*

(iv) *K_i is a fiber space over T^{n-1} with fiber I. Moreover, there is a map $\pi_i\colon T^{n-1}\to T^{n-1}$ defined by $\pi_i(t^1, \ldots, t^{n-1})=(t^1(1+\Delta_i^1),\ldots,t^{n-1}(1+\Delta_i^{n-1}))$, (where $(t^1,\ldots,t^{n-1})$ are real numbers mod 1 which can be taken as coordinates of T^{n-1} and $\Delta_i^j=0$ or 1 such that the pullback of K_i by π_i is $T^{n-1}\times I$, ($I=[0,1]$).*

Proof. Let p be a point of W with $p=\lim p_i$, etc. as above. Let $R^{n-1}\subset R^n$ be effective at p, hence define a locally free action on a neighborhood V_0 of $O(p)$, by restricting φ to $R^{n-1}\times V_0$.

In view of Lemma 6.3, Lemma 6.1 can be applied to $F=O(p)$. Let J be a component of $B-O(p)$ such that $O(p_i)$ intersects J for infinitely many i. Then for some large i, $O(p_i)$ must separate J. Let K be the closure of the component of $J-O(p)$ which contains points near $O(p)$. It is clear that K satisfies (ii) above with K_i replaced by K. To see that (iv) can be satisfied, let $\Delta=(\Delta^1, \ldots, \Delta^n)$ be an orientation cocycle for the normal bundle of $O(p)$ in V. That is, if $O(p)=T^{n-1}$ has coordinates $(t^1, \ldots, t^{n-1})$ as in (iv), Δ^j is defined to be one or zero according as the orientation of the normal fiber is reversed or remains the same along the path $t^i=$ const. $i\neq j$ and $0\leqq t^j\leqq 1$. It is clear then that the map $\pi\colon T^{n-1}\to T^{n-1}=O(p)$ defined by $\pi(t^1, \ldots, t^{n-1})=(t^1(1+\Delta^1), \ldots, t^{n-1}(1+\Delta^{n-1})$, has the required properties. This shows that K satisfies (iv). A finite number of such K cover W, hence (i) can be satisfied. Obvious modification of a set of K's which cover W can be made to assure that (iii) holds. This proves Lemma 6.4.

LEMMA 6.5. *Let $K=K_i$ be one of the sets with the properties described in the* *conclusion of Lemma 6.4. Suppose that a C^k action ψ of R^n is defined on the boundary of K such that each component of the boundary is an orbit of ψ. Then ψ can be extended to a C^k action on all of K.*

Proof. Let $\pi=\pi_i\colon T^{n-1}\to T^{n-1}$ be as in Lemma 6.4. Then the action ψ lifts to an action ψ_0 on the boundary of $T^{n-1}\times I$ by the pullback map. Each component of the boundary will be an orbit.

Let $T^{n-1}\times I$ have coordinates $t=(t^1, \ldots, t^n)$ where t^j is defined mod 1 and $0\leqq t^n\leqq 1$. Let R^n have coordinates $x=(x^1, \ldots, x^n)$ and define the action $\gamma\colon R^n\times T^{n-1}\times I\to T^{n-1}\times I$ by $\gamma^j(x,t)=x^j+t^j$ mod 1 if $j<n$ and $\gamma^n(x,t)=t^n$. Then there are linear maps $L_i\colon R^n\to R^n$ $(i=0, 1)$ such that on $T^{n-1}\times\{i\}$, $\psi_0(x,t)=\gamma(L_ix, t)$. It is possible to choose L_0 and L_1 in such a way that there is a C^k homotopy $L(t)$ $(0\leqq t\leqq 1)$ such that $L(i)=L_i (i=0,1)$ and for every t, $L(t)$ is a linear map of R^n whose range is of dimension $(n-1)$ and transversal to the x^n axis. (This assertion amounts to saying that the Stiefel manifold $V_{n,n-1}$ is arcwise connected.) Therefore one can define $\psi_0(x,t)=\gamma(L(t^n)x, t)$. This defines an extension of ψ_0 to an action on $T^{n-1}\times I$. The bundle map $b\colon T^{n-1}\times I\to K$ defines the desired action on K by the formula $\psi(x,p)=b(\psi_0(x, b^{-1}(p)))$, where b^{-1} is any local inverse of b defined near p. This definition easily seen to be independent of the choice of b^{-1} and to give an action with the desired properties.

7. Proof of Theorem 3

Let K_i be as in Lemma 6.4 and let M be the closure of $V-K_1\cup\cdots\cup K_r$. M is a manifold with boundary. From definition of W and $W\subset$ interior $\cup K_i$ it follows that M contains, at most, finitely many compact orbits of the action φ. Let $R^{n-1}\subset R^n$ be a subspace which is effective on each of these orbits. Then R^{n-1} is also effective at every point p of M such that $d(p)=n-1$. This follows from Lemmas 6.2, and 2.2 and the fact that M is closed and invariant under φ. Let $P\colon R^n\to R^{n-1}$ be the orthogonal projection with respect to some metric on R^n. Then $\psi(x,p)=\varphi(Px,p)$ defines a foliating action in M with all of its orbits of dimension $n-1$. In fact, the $(n-1)$-dimensional orbits of φ will be orbits of ψ also.

It remains to define ψ on $K_1, \ldots, K_r$. This is done by induction. Suppose that ψ is defined on $M\cup K_1\cup\cdots\cup K_i$ $0\leqq i<r$ in such a way that ψ defines a piecewise C^k foliating action on $M\cup K_1\cup\cdots\cup K_i$ with orbits of dimension $n-1$, then Lemma 6.5 enables one to extend the action to $M\cup K_1\cup\cdots\cup K_{i+1}$, retaining the desired properties. In the application of the Lemma, ψ can be defined arbitrarily on any components of the boundary of K_{i+1} which are not contained in $M\cup K_1\cup\cdots\cup K_i$. This completes the proof of Theorem 3.

8. Comments

Theorem 3 can be proved in a somewhat different way, which is longer than the one given here, but which has the advantage of revealing more about the orbits of φ. The alternate proof shows that all n-dimensional orbits of φ have either one or two components in their boundary. If one of the components consists of a single leaf the orbit is diffeomorphic to $R^1 \times T^{n-1}$ and its closure is either $T^{n-1} \times I$ or all of V. The leaf in the component is, of course, compact.

No examples of compact manifolds V are known where $R(V)$, $S(V)$, and $T(V)$ are unequal. It would be interesting to known if they can differ. There is some reason to hope that these numbers are related to the multiplicity of -1 as a root of the Poincaré polynomial of V, over some field. If, for example, one defines $Z(V)$ to be this multiplicity over the rationals, an interesting question is whether $R(V) \leqq Z(V)$ always holds. $R(V) \geqq Z(V)$ holds if V is a Lie group. Finally, if $V = V_1 \times V_2$ is a product, $R(V) \geqq R(V_1) + R(V_2)$ and similarly for S and T, but no examples are known where equality fails to hold.

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