On the Isoperimetric Problem in a Riemann Space.

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To Professor Heinz Hopf on his 70th birthday

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Introduction

As well-known, the isoperimetric problem in an Euclidean space of two dimensions is to find the shortest simple closed curve enclosing a fixed area. The solution is a circle. The analogous problem in an Euclidean space of three dimensions is to find the simple closed surface with minimum area enclosing a fixed volume. Here again the classical answer is the sphere.

One knows (see, for instance, [1, 2]) that the closed surfaces with constant mean curvature are closely related to the isoperimetric problem, because of the following.

THEOREM. *Let S be a simple closed surface, then S has constant mean curvature H if and only if S is stationary with respect to the isoperimetric problem* ([1], p. 75).

In previous papers ([3, 4]), the author has investigated some properties of a closed orientable hypersurface with the first mean curvature $H_1 = \text{constant}$ in an $(m+1)$-dimensional Riemann space R^{m+1}.

It is the aim of the present paper to generalize the above Theorem to hypersurfaces in R^{m+1} and to investigate the connection with the isoperimetric problem in R^{m+1}. In §1 some integral formulas for a closed orientable hypersurface which is the boundary of a domain in R^{m+1} are derived; §2 gives a variational interpretation for these formulas and for a formula (I) of Minkowski type in R^{m+1} ([3], p. 288). In §3 the main theorem is proved.

§ 1. Some integral formulas

We consider a Riemann space $R^{m+1}(m+1 \geqq 3)$ of class $C^\nu(\nu \geqq 3)$ which admits a one-parameter continuous group G of transformations generated by an infinitesimal transformation

$$\bar{x}^i = x^i + \xi^i(x)\,\delta\tau \tag{1.1}$$

(where x^i are local coordinates in R^{m+1} and ξ^i are the components of a contravariant vector ξ). We suppose that the paths of these transformations cover R^{m+1} simply and that ξ is everywhere continuous and $\neq 0$. If ξ is a Killing vector, a homothetic Killing vector, a conformal Killing vector, etc. ([5], p. 32), then the group G is called isometric, homothetic, conformal, etc. respectively.

We now consider a domain D in R^{m+1} such that its boundary is a closed hypersurface V^m of class C^3 imbedded in R^{m+1}, locally given by

$$x^i = x^i(u^\alpha)\,; \tag{1.2}$$

here and henceforth, Latin indices run from 1 to $m+1$ and Greek indices from 1 to m.

Let us consider a differential form of degree m at a point P of the domain D, defined by

$$((\xi, \underbrace{dx, \ldots, dx}_{m})) = \sqrt{g}(\xi, dx, \ldots, dx) \tag{1.3}$$

where dx^k is a displacement in the domain D and g denotes the determinant of the metric tensor g_{ij} of R^{m+1}. Then the exterior differential of the differential form (1.3) divided by $m!$ becomes as follows

$$\frac{1}{m!}d((\xi, dx, \ldots, dx)) = -\tfrac{1}{2}g^{ij}\underset{\xi}{\mathscr{L}}g_{ij}\,dV \tag{1.4}$$

where $\mathscr{L}_\xi\, g_{ij}$ is the Lie derivative of the tensor g_{ij} with respect to the infinitesimal point transformation (1.1), and dV means the volume element of D.

Integrating both members of (1.4) over the whole domain D, and applying Stokes' theorem, we have

$$-\frac{1}{2}\int_D \cdots \int g^{ij}\underset{\xi}{\mathscr{L}}g_{ij}\,dV = \frac{1}{m!}\int_D \cdots \int d((\xi, dx, \ldots, dx)) = \frac{1}{m!}\int_{V^m} \cdots \int ((\xi, dx, \ldots, dx)) \tag{1.5}$$

V^m being the boundary of D. On the other hand, we can easily see the following relation $((\xi, dx, \ldots, dx)) = \xi^i n_i m!\,dA$, where dx^k means a displacement along the hypersurface V^m, i.e., $dx^k = (\partial x^k/\partial u^\alpha)du^\alpha$, and n_i is a unit normal covariant vector at a point P of the hypersurface V^m and dA denotes the area element of V^m. Thus we obtain the integral formula

$$-\tfrac{1}{2}\int_D \cdots \int g^{ij}\underset{\xi}{\mathscr{L}}g_{ij}\,dV = \int_{V^m} \cdots \int \xi^i n_i\,dA \qquad (\alpha).$$

Let the group G be conformal, that is, ξ^i satisfy the equation

$$\underset{\xi}{\mathscr{L}}g_{ij} \equiv \xi_{i;\,j} + \xi_{j;\,i} = 2\phi(x)g_{ij}$$

(cf. [5], p. 32), where the symbol "; " always means the covariant derivative, then (α) becomes

$$-(m+1)\int_D \cdots \int \phi\,dV = \int_{V^m} \cdots \int \xi^i n_i\,dA \qquad (\alpha)_c.$$

Let G be homothetic, that is, $\phi \equiv C =$ constant, then

$$-(m+1)CV = \int_{V^m} \cdots \int \xi^i n_i \, dA \qquad (\alpha)_h$$

V being the total volume of D. Especially, if our space R^{m+1} is an Euclidean space E^{m+1} and if we take a point of D as origin of the euclidean coordinates x^i and attach to each point x the vector ξ^i with the components $\xi^i = x^i$ (i.e., the position vector of x), then the transformations (1.1) are homothetic, that is, $C=1$, thus the formula $(\alpha)_h$ becomes the following well-known formula

$$(m+1)V = -\int_{V^m} \cdots \int x^i n_i \, dA .$$

In the case $m+1=3$, we have $3V = -\int \ldots \int_{V^2} x^i n_i dA$ (cf. [2], p. 18).

Furthermore in the Riemann space R^{m+1}, let G be isometric, that is, $C=0$, then we have

$$\int_{V^m} \cdots \int \xi^i n_i \, dA = 0 \qquad (\alpha)_i .$$

By making use of the formula $(\alpha)_c$ and the formula $(\mathrm{I})_c$ of the previous paper ([4], p. 3), we have the following

THEOREM 1.1. *If D is a domain in R^{m+1} admitting a conformal Killing vector ξ (i.e., $\xi_{i;j} + \xi_{j;i} = 2\phi g_{ij}$) and if its boundary V^m is a closed hypersurface with $H_1 =$ constant, then it follows that*

$$(m+1)H_1 \int_D \cdots \int \phi \, dV = \int_{V^m} \cdots \int \phi \, dA \qquad (1.6)$$

where H_1 means the first mean curvature of V^m.

Proof. Multiplying the formula $(\alpha)_c$ by H_1 (=const.), we obtain

$$-(m+1)H_1 \int_D \cdots \int \phi \, dV = H_1 \int_{V^m} \cdots \int \xi^i n_i \, dA .$$

By making use of the formula $(\mathrm{I})_c$ of the previous paper $H_1 \int \ldots \int_{V^m} \xi^i n_i dA = -\int \ldots \int_{V^m} \phi \, dA$ (cf. [4], p. 3), we see that $(m+1)H_1 \int \ldots \int_D \phi \, dV = \int \ldots \int_{V^m} \phi \, dA$.

COROLLARY. *If D is a domain in R^{m+1} admitting a homothetic Killing vector ξ (i.e., $\xi_{i;j} + \xi_{j;i} = 2\,C\,g_{ij}$) and if its boundary V^m is a closed hypersurface with $H_1 =$ const., then we have*

$$V = \frac{1}{m+1} \cdot \frac{A}{H_1} \qquad (1.7)$$

where A is the total area of V^m.

Proof. Substituting $\phi=C$ (=const.) into both members of (1.6), we obtain easily (1.7).

Especially, if our space R^{m+1} is an Euclidean space E^{m+1} and if V^m is a hypersphere with radius γ, then the formula (1.7) becomes $V=\gamma\cdot A/m+1$.

§ 2. On variational problems of integral formulas

In this section, we shall discuss the preceding integral formulas and the integral formulas of the previous paper ([4], p. 3) from the point of view of the calculus of variations.

We now consider a variation of a geometrical object in R^{m+1}, defined by

$$\bar{x}^i = x^i + \xi^i(x)\varepsilon \tag{2.1}$$

where ε is a parameter near $\varepsilon=0$; then substituting (1.2) into (2.1), we get a family $\bar{x}^i=\bar{x}^i(u^\alpha, \varepsilon)$ of admissible hypersurfaces of the form

$$\bar{x}^i = x^i(u^\alpha) + \xi^i(x^j(u^\alpha))\varepsilon. \tag{2.2}$$

For each value of ε near $\varepsilon=0$, we thus obtain a domain $D(\varepsilon)$ with a boundary $V^m(\varepsilon)$, where $D(0)=D$, $V^m(0)=V^m$; let $V(\varepsilon)$ be the total volume of $D(\varepsilon)$. Now we have the following

THEOREM 2.1. *If* $(\delta V/\partial\varepsilon)_{\varepsilon=0}$ *is the first variation of the total volume of* $D(\varepsilon)$ along D *with respect to a direction* ξ^i, *then*

$$\left(\frac{\delta V}{\partial\varepsilon}\right)_{\varepsilon=0} = \frac{1}{2}\int_D\cdots\int g^{ij}\underset{\xi}{\mathscr{L}}g_{ij}\,dV. \tag{2.3}$$

Proof. Let $\bar{V}$ be the total volume of $D(\varepsilon)$, which is given by the integral form

$$\bar{V} = \int_{D(\varepsilon)}\cdots\int \sqrt{\bar{g}}(d\bar{x}, \ldots, d\bar{x})$$

where $\bar{g}=g(x,\varepsilon)$ and $d\bar{x}^i=dx^i+(\partial\xi^i/\partial x^l)dx\varepsilon$. For the first variation of $\bar{V}$ along D we have

$$\frac{\delta V}{\partial\varepsilon} = \int_{D(\varepsilon)}\cdots\int \frac{\partial}{\partial\varepsilon}\sqrt{\bar{g}}(d\bar{x}, \ldots, d\bar{x}) + \sqrt{\bar{g}}\frac{\partial}{\partial\varepsilon}(d\bar{x}, \ldots, d\bar{x}),$$

$$\left(\frac{\delta V}{\partial\varepsilon}\right)_{\varepsilon=0} = \frac{1}{2}\int_D\cdots\int\sqrt{g}\,g^{ij}\left(\frac{\partial g_{ij}}{\partial x^l}\xi^l + g_{lj}\frac{\partial\xi^l}{\partial x^i} + g_{li}\frac{\partial\xi^l}{\partial x^j}\right)(dx, \ldots, dx)$$

$$= \frac{1}{2}\int_D\cdots\int g^{ij}\underset{\xi}{\mathscr{L}}g_{ij}\,dV$$

because of $dV=\sqrt{g}\,(dx,\dots,dx)$ and $\mathscr{L}_\xi\, g_{ij}=(\partial g_{ij}/\partial x^l)\,\xi^l+g_{lj}(\partial\xi^l/\partial x^i)+g_{il}(\partial\xi^l/\partial x_j)$ (cf. [5], p. 4).

Therefore we evidently have the following

COROLLARY 2.1. The first variation of the total volume of $D(\varepsilon)$ along D, with respect to a direction ξ^i becomes as follows

$$\left(\frac{\delta V}{\partial\varepsilon}\right)_{\varepsilon=0}=(m+1)\int\cdots\int_D\phi\,dV,\quad \left(\frac{\delta V}{\partial\varepsilon}\right)_{\varepsilon=0}=(m+1)\,C\,V,\quad \text{or}\quad \left(\frac{\delta V}{\partial\varepsilon}\right)_{\varepsilon=0}=0$$

according to ξ^i being a conformal Killing vector ($\mathscr{L}_\xi\, g_{ij}=2\,\phi\, g_{ij}$), a homothetic Killing vector, or a Killing vector.

COROLLARY 2.2. The first variation of the total volume of $D(\varepsilon)$ along D, with respect to a direction ξ^i, is given by

$$\left(\frac{\delta V}{\partial\varepsilon}\right)_{\varepsilon=0}=-\int\cdots\int_{V^m}\xi^i n_i\,dA. \tag{2.4}$$

The proof easily follows from the integral formula (α) and (2.3).

We consider next a closed orientable hypersurface V^m of class C^3 imbedded in R^{m+1}, locally given by (1.2). then we obtain a family $\bar{x}^i=x^i(u^\alpha,\varepsilon)$ of admissible hypersurfaces of the form (2.2). For each value of ε near $\varepsilon=0$, we have a hypersurface $V^m(\varepsilon)$, where $V^m(0)=V^m$, and we have a value $A(\varepsilon)$ of the total area of $V^m(\varepsilon)$. Then we shall prove the following theorem.

THEOREM 2.2. *Let $(\delta A/\partial\varepsilon)_{\varepsilon=0}$ be the first variation of the total area of $V^m(\varepsilon)$ along V^m, with respect to a direction ξ^i, then*

$$\left(\frac{\delta A}{\partial\varepsilon}\right)_{\varepsilon=0}=\frac{1}{2}\int\cdots\int_{V^m}\mathscr{L}_\xi\, g_{ij}\frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^j}{\partial u^\beta}g^{\alpha\beta}\,dA. \tag{2.5}$$

Proof. As well-known, the total area of $V^m(\varepsilon)$ is given by the form

$$A(\varepsilon)=\int\cdots\int_{V^m(\varepsilon)}\sqrt{\tilde{g}(\varepsilon)}\,(du,\,\dots,\,du)$$

where $\tilde{g}(\varepsilon)$ means the determinant of the metric tensor $g_{\alpha\beta}(\varepsilon)$ of the hypersurface $V^m(\varepsilon)$ (i.e., $g_{\alpha\beta}(\varepsilon)=g_{ij}(\bar{x})\,(\partial\bar{x}^i/\partial u^\alpha)\,(\partial\bar{x}^j/\partial u^\beta)$).

Differentiating the above integral form with respect to ε, we have

$$\frac{\delta A}{\partial\varepsilon}=\int\cdots\int_{V^m(\varepsilon)}\frac{\partial}{\partial\varepsilon}\sqrt{\tilde{g}(\varepsilon)}\,(du,\,\dots,\,du)$$

where u^α and ε are independent parameters.

On making use of the following results

$$\frac{\partial}{\partial\varepsilon}\sqrt{\tilde g}=\frac{1}{2\sqrt{\tilde g}}\left\{\frac{\partial\tilde g}{\partial\bar x^k}\xi^k+\frac{\partial\tilde g}{\partial(\partial\bar x^k/\partial u^\alpha)}\frac{\partial\xi^k}{\partial u^\alpha}\right\},$$

$$\frac{\partial\tilde g}{\partial\bar x^k}=\frac{\partial\bar g_{ij}}{\partial\bar x^k}\frac{\partial\bar x^i}{\partial u^\alpha}\frac{\partial\bar x^j}{\partial u^\beta}g^{\alpha\beta}(\varepsilon)\tilde g,\qquad \frac{\partial\tilde g}{\partial(\partial\bar x^k/\partial u^\alpha)}=2g_{kj}(\bar x)\frac{\partial\bar x^j}{\partial u^\beta}g^{\alpha\beta}(\varepsilon)\tilde g,$$

$$\frac{\partial\tilde g}{\partial(\partial\bar x^k/\partial u^\alpha)}\frac{\partial\xi^k}{\partial u^\alpha}=\left(g_{kj}(\bar x)\frac{\partial\xi^k}{\partial\bar x^i}+g_{ki}(\bar x)\frac{\partial\xi^k}{\partial\bar x^j}\right)\frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^j}{\partial u^\beta}g^{\alpha\beta}(\varepsilon)\tilde g,$$

we obtain

$$\left(\frac{\partial\sqrt{\tilde g}}{\partial\varepsilon}\right)_{\varepsilon=0}=\tfrac{1}{2}\underset{\xi}{\mathscr{L}}g_{ij}\frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^j}{\partial u^\beta}g^{\alpha\beta}d\sqrt{\tilde g}A.$$

Consequently for the first variation of the total area of $V^m(\varepsilon)$ along V^m, we can see

$$\left(\frac{\delta A}{\partial\varepsilon}\right)_{\varepsilon=0}=\tfrac{1}{2}\int_{V^m}\cdots\int\underset{\xi}{\mathscr{L}}g_{ij}\frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^j}{\partial u^\beta}g^{\alpha\beta}dA.$$

COROLLARY 2.3. The first variation of the total area of $V^m(\varepsilon)$ along V^m, with respect to a direction ξ^i, becomes as follows

$$\left(\frac{\delta A}{\partial\varepsilon}\right)_{\varepsilon=0}=m\int_{V^m}\cdots\int\phi\,dA,\quad \left(\frac{\delta A}{\partial\varepsilon}\right)_{\varepsilon=0}=mCA\quad\text{or}\quad\left(\frac{\delta A}{\partial\varepsilon}\right)_{\varepsilon=0}=0$$

according to ξ^i being a conformal Killing vector, a homothetic Killing vector or a Killing vector.

From Theorem 2.2 and the formula (I) of the previous paper (cf. [4], p. 3), we can see easily the following

COROLLARY 2.4. The first variation of the total area $V^m(\varepsilon)$ along V^m with respect to a direction ξ^i, has the form

$$\frac{1}{m}\left(\frac{\delta A}{\partial\varepsilon}\right)_{\varepsilon=0}=-\int_{V^m}\cdots\int H_1\xi^i n_i\,dA.\tag{2.6}$$

If our space R^{m+1} is an Euclidean space E^{m+1} and if we take to each point x the vector $\xi^i(x)$ with the components $\xi^i=x^i$ (i.e., the position vector of x), then the vector ξ^i is a homothetic Killing vector with $C=1$, and $\xi^i n_i$ is the support function p for $x\in V^m$. In this case, the formula (2.6) becomes

$$\int_{V^m}\cdots\int H_1\,p\,dA+A=0,$$

this being nothing but the formula of Minkowski type of V^m in E^{m+1} given by C. C. HSIUNG (cf. [6], p. 286). Therefore we can see the formula (2.6):

$$\int_{V^m}\cdots\int H_1 \xi^i n_i dA + \frac{1}{m}\left(\frac{\delta A}{\partial \varepsilon}\right)_{\varepsilon=0} = 0$$

as a generalization of the formula of Minkowski type.

Remark 1. Although the vector field $\xi^i(x)$ is not defined on the whole Riemann space but defined on a certain domain including both D and V^m, all the preceding theorems are valid.

Remark 2. In case an arbitrary vector η^i is defined on the hypersurface V^m given by (1.2), we can find also the following formulas

$$\left(\frac{\delta V}{\partial \varepsilon}\right)_{\varepsilon=0} = -\int_{V^m}\cdots\int \eta^i(u^\alpha) n_i dA \tag{2.7}$$

and

$$\left(\frac{\delta A}{\partial \varepsilon}\right)_{\varepsilon=0} = -m\int_{V^m}\cdots\int H_1 \eta^i(u^\alpha) n_i dA \tag{2.8}$$

for the first variation of the total volume of $D(\varepsilon)$ along D and the first variation of the total area of $V^m(\varepsilon)$ along V^m, by means of a family $\bar{x}^i = x^i(u^\alpha, \varepsilon)$ of the hypersurfaces of the form

$$\bar{x}^i(u^\alpha, \varepsilon) = x^i(u^\alpha) + \eta^i(u^\alpha)\varepsilon.$$

§ 3. The isoperimetric problems

In this section, we shall prove the following theorems closely related to what may be called an isoperimetric problem in R^{m+1}.

If $(\delta A/\partial\varepsilon)_{\varepsilon=0}=0$ for all variations with respect to a direction such that $(\delta V/\partial\varepsilon)_{\varepsilon=0}=0$, then the hypersurface V^m is called a pseudo-stationary hypersurface.

THEOREM 3.1. *Let V^m be a closed orientable hypersurface in R^{m+1}. Then the first mean curvature of V^m is constant if and only if V^m is a pseudo-stationary hypersurface.*

Proof. Suppose H_1 is constant; if $(\delta V/\partial\varepsilon)_{\varepsilon=0}=0$, then we get from (2.7)

$$\left(\frac{\delta V}{\partial \varepsilon}\right)_{\varepsilon=0} = -\int_{V^m}\cdots\int \eta^i n_i dA = 0$$

and hence from (2.8)

$$\left(\frac{\delta A}{\partial \varepsilon}\right)_{\varepsilon=0} = -m\int_{V^m}\cdots\int H_1 \eta^i n_i dA = -mH_1\int_{V^m}\cdots\int \eta^i n_i dA = 0.$$

Thus V^m is a pseudo-stationary hypersurface.

Conversely suppose $(\delta A/\partial\varepsilon)_{\varepsilon=0}=0$ for every variation with respect to a direction η^i such that $(\delta V/\partial\varepsilon)_{\varepsilon=0}=0$; we must prove that H_1 is constant. Let φ be an arbitrary function defined on V^m such that $\int\ldots\int_{V^m}\varphi\,dA=0$. We wish to show first that φ is in fact the normal component of a variation vector η^i such that $(\delta V/\partial\varepsilon)_{\varepsilon=0}=0$. Let us consider the family of hypersurfaces $\bar{x}^i(u^\alpha,\varepsilon)=x^i(u^\alpha)+\varphi\,n^i\,\varepsilon$, then from (2.4) we see

$$\left(\frac{\delta V}{\partial\varepsilon}\right)_{\varepsilon=0}=-\int\cdots\int_{V^m}\varphi\,n^i n_i\,dA=-\int\cdots\int_{V^m}\varphi\,dA=0\,.$$

Thus φ is the normal component of a variation vector such that $(\delta V/\partial\varepsilon)_{\varepsilon=0}=0$.

By hypothesis, V^m is pseudo-stationary, therefore it follows that

$$\left(\frac{\delta A}{\partial\varepsilon}\right)_{\varepsilon=0}=-m\int\cdots\int_{V^m}H_1\,\varphi\,dA=0\,.$$

Thus we have $\int\ldots\int_{V^m}H_1\,\varphi\,dA=0$. Also if h is an arbitrary constant, we have $\int\ldots\int_{V^m}h\,\varphi\,dA=0$, and hence for any function φ such that $\int\ldots\int_{V^m}\varphi\,dA=0$ and for any constant h, we obtain

$$\int\cdots\int_{V^m}(H_1-h)\,\varphi\,dA=0\,.$$

Now let h be the mean value of H_1:

$$h=\frac{1}{A}\int\cdots\int_{V^m}H_1\,dA\,,$$

then we have

$$\begin{aligned}\int\cdots\int_{V^m}(H_1-h)\,dA&=\int\cdots\int_{V^m}H_1\,dA-h\int\cdots\int_{V^m}dA\\&=\int\cdots\int_{V^m}H_1\,dA-h\cdot A=\int\cdots\int_{V^m}H_1\,dA-\int\cdots\int_{V^m}H_1\,dA=0\,.\end{aligned}$$

Consequently taking H_1-h for φ, we obtain

$$\int\cdots\int_{V^m}(H_1-h)^l\,dA=0\,.$$

Therefore $H_1\equiv h$, which concludes the proof.

This theorem is nothing but a generalization of the same theorem in an Euclidean space given already in [2], p. 19, and this proof follows the same argument as in [2].

A. D. Alexandrov has already proved the following result in his paper ([7], p. 304), where in the case of positive curvature, R^{m+1} shall be a sphere and V^m contained in a hemisphere of R^{m+1}:

THEOREM A. *If R^{m+1} has constant curvature and if V^m is a simple closed hypersurface with H_1 = constant, then V^m is a hypersphere.*

From this result, we have (under the same assumptions as above):

COROLLARY 3.1. *If V^m is a simple closed hypersurface in R^{m+1} with constant curvature, then V^m is a hypersphere if and only if V^m is a pseudo-stationary hypersurface.*

Now in R^{m+1}, let S be the collection of all closed orientable hypersurfaces V^m enclosing a fixed volume. Then the total area A of V^m is a function on S. Let V^m be a fixed hypersurface and consider a one parameter family of continuous and differentiable variations of V^m, indexed by a parameter ε. Let $V^m(\varepsilon)$ denote the varied hypersurface. Then we require that $V^m(0)=V^m$ and that for each ε, $V^m(\varepsilon)\in S$ (i.e. these variations are volume preserving).

The total area $A(\varepsilon)$ of $V^m(\varepsilon)$ is a differentiable function of ε. If $(\delta A/\partial\varepsilon)_{\varepsilon=0}=0$ for all volume preserving variations, then V^m is called a stationary hypersurface. Then we have

THEOREM 3.2. *If R^{m+1} admits a homothetic Killing vector field ξ^i ($\xi_{i;j}+\xi_{j;i}=2Cg_{ij}$, $C\neq 0$) and if V^m is a closed orientable hypersurface in R^{m+1}, then the first mean curvature H_1 of V^m is constant if and only if V^m is a stationary hypersurface.*

Proof. Let V^m be given by (1.2) and suppose for simplicity that $V(0)=1$ and let $V^m(\varepsilon)$ be a variation of V^m; denote its total area and the total volume of the domain bounded by $V^m(\varepsilon)$ by $A(\varepsilon)$ and $V(\varepsilon)$ respectively. $V^m(\varepsilon)$ can be represented by

$$\bar{x}^i(u^\alpha,\varepsilon)=x^i(u^\alpha)+\eta^i(u^\alpha)\varepsilon+\cdots$$

for each value of ε near $\varepsilon=0$, where $\eta^i(u^\alpha)=(\partial\bar{x}^i/\partial\varepsilon)_{\varepsilon=0}$. Then from (2.7) and (2.8) we have

$$\left(\frac{\delta V}{\partial\varepsilon}\right)_{\varepsilon=0}=-\int\cdots\int_{V^m}\eta^i n_i\,dA\,,\qquad \left(\frac{\delta A}{\partial\varepsilon}\right)_{\varepsilon=0}=-m\int\cdots\int_{V^m}H_1\eta^i n_i\,dA\,.$$

Sufficiency in Theorem 3.2. is similar by proved as in Theorem 3.1; that is, suppose H_1 is constant and $\bar{x}^i(u^\alpha,\varepsilon)$ is a volume preserving variation of V^m then $(\delta V/\partial\varepsilon)_{\varepsilon=0}=-\int\ldots\int_{V^m}\eta^i n_i\,dA=0$ and hence

$$\left(\frac{\delta A}{\partial\varepsilon}\right)_{\varepsilon=0}=-m\int\cdots\int_{V^m}H_1\eta^i n_i\,dA=-mH_1\int\cdots\int_{V^m}\eta^i n_i\,dA=0\,.$$

Conversely, suppose $(\delta A/\partial\varepsilon)_{\varepsilon=0}=0$ for every volume preserving variation. Then we must show that H_1 is constant.

Let φ be an arbitrary function defined on V^m such that $\int\ldots\int_{V^m} \varphi\, dA = 0$; we wish to show first that φ is the normal component of a volume preserving variation. Consider the family of hypersurfaces

$$V^m(\varepsilon) : \bar{x}^i(u^\alpha, \varepsilon) = x^i(u^\alpha) + \varphi\, n^i \varepsilon, \tag{3.1}$$

then let $V(\varepsilon)$ denote the total volume of the domain bounded by the hypersurface $V^m(\varepsilon)$, then $V(0) = V = 1$; now the normal component of $(\partial \bar{x}^i/\partial \varepsilon)_{\varepsilon=0} = \varphi\, n^i$ is given by $(\partial \bar{x}^i/\partial \varepsilon)_{\varepsilon=0} n_i = \varphi\, n^i n_i = \varphi$. Hence, by virtue of (2.7) we have

$$\left(\frac{\delta V}{\partial \varepsilon}\right)_{\varepsilon=0} = -\int_{V^m}\!\!\cdots\!\int \left(\frac{\partial \bar{x}^i}{\partial \varepsilon}\right)_{\varepsilon=0} n_i\, dA = -\int_{V^m}\!\!\cdots\!\int \varphi\, dA = 0$$

by hypothesis. But the variation $\bar{x}^i(u^\alpha, \varepsilon)$ need not be volume preserving.

However by hypothesis, our space R^{m+1} admits an infinitesimal homothetic transformation given by (1.1) with the additional condition

$$\xi_{i;j} + \xi_{j;i} = 2C g_{ij} \qquad (C \neq 0,\ \text{constant}). \tag{3.2}$$

Let us choose a coordinate system such that the path of the infinitesimal transformation is the new x^1-coordinate curve, that is, a coordinate system in which the vector ξ^i has the components δ^i_1 (where δ^i_j denotes the Kronecker delta); then (1.1) becomes $x'^i = x^i + \delta^i_1\, \delta\tau$ and R^{m+1} admits a one-parameter continuous group G of transformations given by

$$x'^i = x^i + \delta^i_1 \tau. \tag{3.3}$$

Then in this new coordinate system, the condition (3.2) becomes as follows $\partial g_{ij}/\partial x^1 = 2\, C\, g_{ij}$. Therefore the metric tensor g_{ij} with respect to the new coordinate system has the form $g_{ij} = f_{ij}(x^2, \ldots, x^{m+1})\, e^{2Cx^1}$. Now we take the family of hypersurfaces

$$V^{*m}(\varepsilon) : x^{*i}(u^\alpha, \varepsilon) = \bar{x}^i(u^\alpha, \varepsilon) + \frac{1}{(m+1)C} \log \frac{1}{V(\varepsilon)}\, \delta^i_1\,; \tag{3.4}$$

we shall show that $V^{*m}(\varepsilon)$ is a volume preserving variation. Let $V^*(\varepsilon)$ be the total volume of the domain bounded by $V^{*m}(\varepsilon)$ and let n^{*i} and dA^* be a normal vector and an area element of the hypersurface $x^{*i}(u^\alpha, \varepsilon)$ respectively. Then from Corollary 2.1 and Corollary 2.2, we have

$$(m+1) C\, V^*(\varepsilon) = -\int_{V^{*m}(\varepsilon)}\!\!\cdots\!\int \delta^i_1\, n^*_i\, dA^* = -\int_{V^{*m}(\varepsilon)}\!\!\cdots\!\int n^*_1\, dA^*. \tag{3.5}$$

On the other hand, from (3.4) we have the relations

$$\begin{aligned} g_{ij}(x^*) &= f_{ij}(x^{*2}, \ldots, x^{*m+1})\, e^{2Cx^{*1}} \\ &= f_{ij}(\bar{x}^2, \ldots, \bar{x}^{m+1})\, e^{2C\bar{x}^1} \cdot e^{(2/m+1)\log(1/V(\varepsilon))} = g_{ij}(\bar{x})\, e^{(2/m+1)\log(1/V(\varepsilon))}\,; \end{aligned}$$

thus we obtain

$$\sqrt{g(x^*)} = \sqrt{g(\bar{x})}\, e^{\log(1/V(\varepsilon))} = \frac{\sqrt{g(\bar{x})}}{V(\varepsilon)}. \tag{3.6}$$

Substituting (3.6) in (3.5) and making use of the relations

$$x^{*2}(u^\alpha, \varepsilon) = \bar{x}^2(u^\alpha, \varepsilon), \ldots, x^{*m+1}(u^\alpha, \varepsilon) = \bar{x}^{m+1}(u^\alpha, \varepsilon),$$

we see that

$$\int \cdots \int_{V^{*m}(\varepsilon)} n_1^* \, dA^* = \int \cdots \int_{V^m(\varepsilon)} \frac{n_1(\varepsilon)}{V(\varepsilon)} dA(\varepsilon)$$

and

$$(m+1)C\,V^*(\varepsilon) = -\frac{1}{V(\varepsilon)} \int \cdots \int_{V^m(\varepsilon)} \delta_1^i\, n_i(\varepsilon)\, dA(\varepsilon) = \frac{1}{V(\varepsilon)}(m+1)C\,V(\varepsilon) = (m+1)C.$$

Thus we have $V^*(\varepsilon) = 1$, therefore $V^{*m}(\varepsilon)$ is a volume preserving variation of V^m.

Now, since $(\delta V/\partial\varepsilon)_{\varepsilon=0} = 0$ it follows that

$$\left(\frac{\partial x^{*i}}{\partial \varepsilon}\right)_{\varepsilon=0} = \left(\frac{\partial \bar{x}^i}{\partial \varepsilon}\right)_{\varepsilon=0} = \varphi\, n^i,$$

and we have

$$\left(\frac{\partial x^{*i}}{\partial \varepsilon}\right)_{\varepsilon=0} n_i = \left(\frac{\partial \bar{x}^i}{\partial \varepsilon}\right)_{\varepsilon=0} n_i = \varphi.$$

Therefore φ is not only the normal component of $(\partial\bar{x}^i/\partial\varepsilon)_{\varepsilon=0}$ but is also the normal component of $(\partial x^{*i}/\partial\varepsilon)_{\varepsilon=0}$ and thus φ is the normal component of a volume preserving variation.

By hypothesis, since V^m is stationary, it follows that $(\delta A/\partial\varepsilon)_{\varepsilon=0} = -m \int \ldots \int_{V^m} H_1\, \varphi\, dA = 0$, thus $\int \ldots \int_{V^m} H_1\, \varphi\, dA = 0$. Also, if h is an arbitrary constant then $\int \ldots \int_{V^m} \varphi\, h\, dA = 0$ and hence for any function φ such that $\int \ldots \int_{V^m} \varphi\, dA = 0$ and for any constant h, $\int \ldots \int_{V^m} (H_1 - h)\, \varphi\, dA = 0$. Now let h be the mean value of H_1: $h = (1/A) \int \ldots \int_{V^m} H_1\, dA$ then we have $\int \ldots \int_{V^m} (H_1 - h)\, dA = 0$. Consequently we see $\int \ldots \int_{V^m} (H_1 - h)^2\, dA = 0$. Therefore $H_1 \equiv h$, which completes the proof.

From Theorem A and Theorem 3.2, we have the following corollary:

COROLLARY 3.2 *If R^{m+1} is an Euclidean space E^{m+1}, then a simple closed hypersurface with minimal hypersurface area enclosing a fixed volume is a hypersphere.*

This may be called a form of the isoperimetric theorem in E^{m+1}.

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