On Finitely Generated Fuchsian Groups.

On Finitely Generated Fuchsian Groups[1])

by ALBERT MARDEN

We will prove the following two theorems.

THEOREM 1. *Let G be a finitely generated Fuchsian group in the unit disk Δ. Then* a) $\Omega = \Delta/G$ *is a Riemann surface of finite topological type, and* b) Δ *is ramified over at most a finite number of points* $\{p_i\}$ *of* Ω. *Conversely, if G satisfies* a) *and* b), *then G is finitely generated.*

THEOREM 2. *G is finitely generated if and only if every fundamental region P has a finite number of sides.*

DEFINITION. A fundamental region P for G is a connected open set in Δ which satisfies the following conditions.

1. Every point in Δ is equivalent under G to a point in $\bar{P}$ (= relative closure in Δ).
2. No two points in P are equivalent.
3. Each component of the relative boundary of P in Δ is an open Jordan arc or a Jordan curve and is the union of possibly infinitely many closed Jordan arcs, called sides (two sides can intersect only at a common end point).
4. The sides of P are arranged in pairs (s_i, s_i') where $(i) S_i(s_i') = s_i$ for some $S_i \in G$, (ii) $S_i \neq S_j^{\pm 1}$ for each $j \neq i$ with at most a finite number of exceptions, (iii) each side of P appears once and only once in the set $\{s_i, s_j'\}$.

Theorem 1 is well known and is fundamental in the theory of Fuchsian groups. By the use of variational methods it has been proven by AHLFORS [1] (in a more general form), BERS [2], and EARLE [3]. From a more general point of view it is a consequence of a theorem of SELBERG [8]. Theorem 2 is also known provided P satisfies the additional hypotheses that (a) *its sides are non-Euclidean line segments*, and (b) *only a finite number of images of P under G meet any given compact set in* Δ. In this form, a proof has recently been given by L. GREENBERG [5]. M. HEINS' proof [6] requires that P also be convex (in this paper HEINS also proves Theorem 1). The first proof of Theorem 1 and of Theorem 2 in the case of Poincaré normal polygons was given by FENCHEL and NIELSEN [4].

The purpose of this paper is to give direct, elementary proofs of Theorems 1 and 2 which are much shorter than those referred to above, and in the case of Theorem 2, more general as well. In fact our definition of fundamental region is, in a sense, the weakest one for which Theorem 2 is true: If P does not satisfy the principal condition 4

[1]) Supported in part by the National Science Foundation under grants GP 2280 and GP 3904.

(ii) then Theorem 2 is false in general. Our proofs are purely topological in character and make use of some elementary properties of surfaces of finite topological type. We will show that Theorem 1b is a simple consequence of the fact that a cycle in the exterior of a region which is homologous to a cycle in the interior is homologous to a cycle on the boundary. And Theorem 2 is a consequence of the fact that if there are infinitely many mutually disjoint simple closed curves not ~ 1 on a surface of finite type, then two of them bound an annular region.

Proof of a). Suppose $A_1, \ldots, A_m$ generate G and let π denote the projection $\Delta \to \Omega$. Assuming 0 is not a fixed point of G, denote by $[0, A_i(0)]$ the non-Euclidean line segment from 0 to $A_i(0)$ (actually any arc will do) and set $\alpha_i = \pi([0, A_i(0)])$, $1 \leq i \leq m$. We claim that the curves $\{\alpha_i\}$ generate the fundamental group of Ω with origin at $\pi(0)$ and consequently that Ω is of finite topological type.

If τ is a closed curve in Ω with initial point $\pi(0)$ there is a lift τ^* of τ in Δ with initial point 0 and end point $\tau^*(1)$. We may write $\tau^*(1) = B_m B_{m-1} \ldots B_1(0)$ where each B_j is some $A_i^{\pm 1}$. Consider the arc τ'^* in Δ from 0 to $\tau^*(1)$ obtained by joining the non-Euclidean line segments $[0, B_m(0)]$, $[B_m(0), B_m B_{m-1}(0)], \ldots, [B_m \ldots B_2(0), B_m \ldots B_1(0)]$; each of these segments projects onto some curve $\alpha_i^{\pm 1}$. Since τ^* is homotopic to τ'^*, τ is homotopic to $\pi(\tau'^*)$, that is τ is homotopic to a product of the α_i.

Proof of b). Let Ω_0 be a relatively compact subregion of Ω containing all the curves α_i such that each component of $\partial\Omega_0$ is a dividing cycle and no component of $\Omega - \Omega_0$ is compact. We claim that Δ is not ramified over $\Omega' = \Omega - \bar{\Omega}_0$.

Assume to the contrary that Ω is ramified of order $t \geq 2$ over $p \in \Omega'$. Let c_p be the oriented boundary of a disk about p in Ω', d_p a Jordan arc from $\pi(0)$ to c_p, and β the closed path from $\pi(0)$ along d_p to c_p, around c_p once, and back to $\pi(0)$ along d_p. We may assume that β does not pass through points over which Δ is ramified and intersects the curves α_i only a finite number of times.

Let β^* denote the lift of β from 0; β^* is a Jordan arc but not a closed curve in Δ. We have seen above that β^* is homotopic to an arc β'^* such that $\beta' = \pi(\beta'^*)$ is a product of curves α_i. Consider the closed curve $\gamma^* = \beta'^* \beta^{*-1}$ and a relatively compact, simply connected region K containing γ^* with $\pi(\partial K) \cap c_p = \phi$ (the points $\pi^{-1}(p)$ are isolated in Δ). Remove from K the at most finite number of points which lie over p and denote the resulting region by K_1.

The collection of disjoint simple closed curves which comprise the components of $\pi^{-1}(c_p)$ in K_1 form a homology basis for K_1. Hence γ^*, viewed as a singular cycle, is homologous to a linear combination of these curves. In $\Omega - \{p\}$ this implies that $\gamma = \pi(\gamma^*) = \beta' \beta^{-1}$ is homologous to ntc_p for some integer n, possibly zero. In other words the cycle $(nt+1)\, c_p$ in $\Omega - \Omega_0$, with $nt+1 \neq 0$ since $t \geq 2$, is homologous in $\Omega - \{p\}$ to the cycle β' in Ω_0. Therefore $(nt+1)\, c_p$ must be homologous to a cycle in $\partial\Omega_0$ in contradiction to our choice of Ω_0.

The proof of the converse of Theorem 1 is standard and will be omitted.

Proof of Theorem 2. Let P be a fundamental region. Since G is generated by the transformations which pair the sides of P, the sufficiency is immediate.

Assume then that G is finitely generated but that P has an infinite number of sides. $\bar{P}$ is not compact in Δ for otherwise there would exist a sequence of points $\{z_j\}$ on distinct sides s_j of P which have a limit $p \in \Delta$ such that the sequence of conjugate points $\{z'_j\}$ on the conjugate sides s'_j also has a limit $p' \in \Delta$. If N is a neighborhood of p and N' of p' then for infinitely many distinct $S_j \in G$, $S_j(N') \cap N \neq \phi$, in violation of the discontinuity of G. In addition P is simply connected for otherwise there would be a relatively compact component of $\Delta - \bar{P}$ which would contain an image of P.

The hypotheses also imply that $S_i(P)$ is adjacent to P along s_i. Hence if $j \neq i$, $S_j(\bar{P})$ cannot intersect s_i at an interior point unless $S_j = S_i$.

In the remainder of the proof all arcs drawn in a region $\bar{R} \subset \bar{P}$ will be understood to be Jordan arcs which are contained in R except for their end points. In addition we can choose, and will only deal with, an infinite sequence of pairs $\{(s_i, s'_i)\}$ such that $S_i \neq S_j^{\pm 1}$ for all $j \neq i$.

Case 1. There exists an arc τ in $\bar{P}$ which divides P into two regions P_1, P_2 with the following property. There exists an infinite number of points $\{z_i\}$ in $\bar{P}_1$, each z_i an interior point of a side of P, but no two points on the same side, such that the conjugate points $\{z'_i\}$ are in $\bar{P}_2$.

Draw arcs γ_1 in $\bar{P}_1$ from z_1 to z_2 and γ'_1 in $\bar{P}_2$ from z'_2 to z'_1. γ_1 divides P_1 into two regions at least one of which, say P_{11}, contains infinitely many points z_i in its boundary, and γ'_1 divides P_2 into two regions one of which, say P_{21}, contains infinitely many points conjugate to those z_i in $\bar{P}_{11}$. Eliminate all z_i which are not in P_{11} and which do not have their conjugates in P_{21}. Draw the arcs γ_2 in $\bar{P}_{11}$ from z_3 to z_4 and γ'_2 in $\bar{P}_{21}$ from z'_4 to z'_3, etc. Thus we can find two infinite sequences of mutually disjoint arcs $\{\gamma_i\}$ in $\bar{P}_1$ and $\{\gamma'_i\}$ in $\bar{P}_2$ such that γ_i is an arc from z_{2i-1} to z_{2i} and γ'_i an arc from z'_{2i} to z'_{2i-1}.

In addition by a suitable choice of subsequence we may assume that either property A holds for all γ_i, γ_j or property A holds for no γ_i, γ_j in the infinite sequence $\{\gamma_i\}$:

Property A. γ_i separates γ_j from τ or γ_j separates γ_i from $\tau (i \neq j)$

To prove this apply the following procedure inductively. Assume that by relabeling, all the arcs γ_j, $0 < j \leq n$, have property A with respect to all the arcs in $\{\gamma_i\}$, $1 \leq i \leq \infty$, but none of the arcs δ_j, $0 < j \leq m$, has property A with respect to any of the arcs in $\{\gamma_i, \delta_k\}$, $1 \leq i < \infty$, $0 < k \leq m$ $(m, n \geq 0)$. Consider γ_{n+1}. If infinitely many arcs in $\{\gamma_i\}$ have property A with respect to γ_{n+1} (the arcs γ_i, $0 < i \leq n$, do), eliminate those arcs in $\{\gamma_i\}$ which do not, relabel, and move on to γ_{n+2}. If this is not the case, eliminate the finite number of arcs in $\{\gamma_i\}$ which do have property A with respect to γ_{n+1}, set $\delta_{m+1} = \gamma_{n+1}$, relabel, and move on to the new γ_1.

Consider the sequence of mutually disjoint simple closed curves $\{\alpha_i = \pi(\gamma_i \cup \gamma'_i)\}$ in Ω. No α_i can be homotopic to 1 in $\Omega - \{p_i\}$. For otherwise $\gamma_i \cup S_{2i}(\gamma'_i)$ is a simple closed curve through z_{2i-1} and z_{2i} and hence $S_{2i-1} = S_{2i}$, which is impossible. Therefore since $\Omega - \{p_i\}$ has finite topological type, there exist three of the curves α_i, say $\alpha_1, \alpha_2, \alpha_3$, such that α_1 and α_2 bound an annular region K in $\Omega - \{p_i\}$ and α_3 separates the contours of K. We can also choose K so that $\pi(\tau) \cap K = \phi$.

The arcs γ_1, γ_2 divide P_1 into three regions. If two of these regions lie over K then γ_3 is contained in one of them, but not in the other, so that γ_3 satisfies property A with only one of the arcs γ_1, γ_2 in contradiction to our selection of $\{\gamma_i\}$. We conclude that γ_1 and γ_2 bound a region R in P_1 which lies over K and contains γ_3.

In P_2, γ_1' and γ_2' bound one or two regions which lie over K. One of these regions contains γ_3' and therefore γ_3' bounds a region R_1' with, say, γ_1' which lies over $K(\pi(R_1') \subset K)$. We can draw an arc δ' in R_1' from z_3' to one of z_1', z_2', say to z_1', such that δ' does not separate γ_1' from γ_3' in R_1'. In R draw an arc δ from z_1 to z_3 which does not otherwise intersect γ_3.

In the annular region $K_1 \subset K$ which is bounded by α_1 and α_3, the simple closed curve $\beta = \pi(\delta \cup \delta')$ does not separate α_1 and α_3. Hence β is homotopic to 1 in K_1 which we have seen above, is impossible.

CASE 2. No such τ exists. In this case draw an arc γ_1 from z_1 to z_1'. There exist infinitely many points $\{z_i\}$ such that γ_1 does not separate z_i from z_i' in P. Draw γ_2 from z_2 to z_2' which is disjoint from γ_1. Again there are infinitely many points $\{z_i\}$ such that neither γ_1 nor γ_2 separates z_i from z_i'. Thus we can find an infinite sequence of disjoint arcs $\{\gamma_i\}$ such that γ_i runs from z_i to z_i'.

The simple closed curves $\alpha_i = \pi(\gamma_i)$ are mutually disjoint and hence, as above, there are three of them, say $\alpha_1, \alpha_2, \alpha_3$, such that α_1 and α_2 bound an annular region K in $\Omega - \{p_i\}$ and α_3 separates the boundary components of K. γ_1 and γ_2 divide P into three regions, one or two of which lie over K and of these, one contains γ_3. Hence one of the pairs (γ_1, γ_2), (γ_1, γ_3), (γ_2, γ_3), say (γ_1, γ_2), bounds a subregion R of P which lies over K. Interchange z_2 and z_2' if necessary so that α_1 is homologous to α_2. Draw an arc δ in $\bar{R}$ from z_1' to z_2'; then α_1 is homotopic to $\pi(\delta)\,\alpha_2\,\pi(\delta)^{-1}$. But then the arc $S_2(\delta^{-1})$ in $\Delta - P$ from z_2 must terminate at z_1 which implies that $S_2 = S_1$, a contradiction.

Remark. If P also satisfies hypothesis (b) a much simpler proof can be given. The sequence $\{z_n\}$ can be chosen so that $\{\pi(z_n)\}$ approaches an ideal boundary component I of Ω. By using the fact that if α is a simple closed curve surrounding I then $\pi^{-1}(\alpha)$ divides P into a finite number of components, the arcs γ_i, γ_i' of Cases 1 and 2 can be chosen directly so that the curves α_i approach I in such a way that α_i bounds an annular region A_i in $\Omega - \{p_i\}$ with α_{i+1} that doesn't contain α_j for $j > i+1$. It follows that the pairs (γ_i, γ_{i+1}), $(\gamma_i', \gamma_{i+1}')$ (just the former in Case 2) each bound a region in P which lies over A_i, since α_{i+2} cannot be connected to α_i without crossing α_{i+1}. The proof is now completed as above.

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University of Minnesota

Received July 27, 1966