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Isotropic Jacobi Fields, and Jacobi's Equations on Riemannian Homogeneous Spaces[1], [2]

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§ 1. Introduction and Preliminaries

Let M be a locally symmetric Riemannian manifold, i.e., assume the covariant derivative of the Riemannian curvature tensor vanishes identically on M. Then along any geodesic, Jacobi's equations of geodesic deviation assume an extremely simple form [10, p. 17], from which one derives a complete geometric picture of M. In particular one that knows if M is also complete and simply-connected then M can be represented as G/H, where G is the component of the identity of all isometries of M, and H is the compact isotropy group. Furthermore, if G/H is of strictly positive curvature, then all Jacobi fields along any geodesic emanating from $P_0=\pi(H)$, where $\pi: G \to G/H$ is the natural projection, are isotropic, i.e., they are generated by the 1-parameter subgroups of H. For the details, see [10].

In this paper we let M be a normal Riemannian homogeneous space G/H, (cf. definitions below) which is not necessarily symmetric, and devote our attention to the solving of Jacobi's equations on G/H. In particular we show that the solving of Jacobi's equations along any geodesic emanating from $P_0=\pi(H)$ is equivalent to solving two systems of ordinary homogeneous differential equations *with constant coefficients*. (cf. § 5 and equations (16)–(18)). A brief sketch of this approach has already appeared [10, p. 26–7] whereas here we give all the details. We then use these equations to find a point $Q \in Sp(2)/SU(2)$, which is conjugate to $P_0=\pi(SU(2))$, but not isotropically conjugate to P_0 (cf. Definition 2). In [2] we carry out a similar calculation for the space $SU(5)/(Sp(2)\times T)$ and show that there exists a point Q' in $SU(5)/(Sp(2)\times T)$ with the same property. Using Theorem 4 below, we are then able to state:

THEOREM 1: *Let G/H be a simply connected normal Riemannian homogeneous space of rank* 1 *such that every point Q conjugate to $P_0=\pi(H)$ is isotropically conjugate to P_0. Then G/H is homeomorphic to a Riemannian symmetric space of rank* 1.

We now turn to the basic definitions and notations used in the sequel:

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All manifolds will be of finite dimension $\geqslant 2$, and infinitely differentiable; and all parametrized curves will also be infinitely differentiable. If ϕ is a differentiable map of one manifold into another, then we write the induced linear map from the tangent space at x to the tangent space at $\phi(x)$ as $d\phi_x$.

If M is a manifold with affine connection Λ, we then denote covariant differentiation along a parametrized curve $x(t)$ in M by D/dt, and the torsion and curvature tensors of Λ by T and B, respectively. Γ will always denote an affine connection without torsion δ/dt covariant differentiation along a parametrized curve $x(t)$with respect to Γ, and R the curvature tensor of Γ. If M is a Riemannian manifold, we denote the induced inner product by $\langle\ ,\ \rangle$, the Levi-Civita connection by Γ, and R will then be the Riemannian curvature tensor.

G will always be a Lie group, H a closed subgroup, G/H the space of left cosets of H, $\pi: G \to G/H$ the natural projection given by $\pi(x) = xH$, $x \in G$, and τ the induced action of G on G/H given by $\tau(x)(yH) = xyH$, $x, y \in G$. The Lie algebras of G and H will be denoted by $\mathfrak{g}$ and $\mathfrak{h}$,respectively. An affine connection on G/H is said to be invariant, if it is invariant under $\tau(x)$ for all $x \in G$.

G/H is said to be a:

(a) *reductive homogeneous space* if the Lie algebra $\mathfrak{g}$ admits a vector space decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ such that $[\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{m}$. In this case $\mathfrak{m}$ is identified with the tangent space of $P_0 = \pi(H)$;

(b) *Riemannian homogeneous space* if G/H is a Riemannian manifold such that the metric is preserved by $\tau(x)$ for all $x \in G$;

(c) *normal Riemannian homogeneous space* if the metric on G/H is obtained as follows: Let there exist a positive definite inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ satisfying $\langle [x, y], z\rangle = \langle x, [y, z]\rangle$ for all $x, y, z \in \mathfrak{g}$, and let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{h}$. Then, the decomposition $(\mathfrak{g}, \mathfrak{h})$ is reductive, and the restriction of the inner product to $\mathfrak{m}$ (which is identified with the tangent space at $P_0 = \pi(H)$) induces a Riemannian metric on G/H (referred to as *normal*) by the action of G on G/H.

For any decomposition of $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, then for $Z \in \mathfrak{g}$, we let $Z_{\mathfrak{h}}$ and $Z_{\mathfrak{m}}$ denote the projections of Z onto $\mathfrak{h}$ and $\mathfrak{m}$, respectively.

The basic definitions and theorems concerning affine connections and homogeneous spaces can be found in [5; 7; 8]. Unless otherwise noted, we use the summation convention for repeated indices. Also the torsion and curvature tensors are minus those in [5; 7].

§ 2. Isotropic Jacobi Fields on Homogeneous Spaces

Let M be a manifold, Λ an affine connection on M, $\sigma: [0, \beta] \to M$ a geodesic in M, and $P_0 = \sigma(0)$. A geodesic variation of σ is a 1-parameter family of geodesics $x(s, \varepsilon)$ in M, where ε is the family parameter, and the geodesic σ corresponds to $\varepsilon = 0$. The vector field $\eta(s) \equiv \partial x/\partial\varepsilon(s, \varepsilon)|_{\varepsilon=0}$ will be called the *Jacobi field along σ induced by the* *variation* $x(s, \varepsilon)$ *of* σ. A point $Q=\sigma(s_0)$, $0<s_0<\beta$, on σ will be said to be *conjugate* to P_0 along σ if there exists a Jacobi field $\eta(s)\not\equiv 0$ along σ such that $\eta(0)=\eta(s_0)=0$.

Now we assume that M is a homogeneous space G/H, with an invariant affine connection Λ. Then $H\subset G$ is a group of affine transformations of G/H leaving $P_0=\pi(H)$ fixed. Hence for any $h\in H$, $\tau(h)$ maps geodesics emanating from P_0 into geodesics emanating from P_0. Let $h(\varepsilon)\subset H$ be a 1-parameter subgroup of H, $\sigma(s)$ a geodesic such that $\sigma(0)=P_0$. Then the action of $h(\varepsilon)$ on σ induces a geodesic variation of σ given by $x(x, \varepsilon)\equiv\tau(h(\varepsilon))(\sigma(s))$.

DEFINITION 1: The geodesic variation $x(s, \varepsilon)\equiv\tau(h(\varepsilon))(\sigma(s))$ will be called *isotropic*, and will be said to be induced by the 1-parameter subgroup $h(\varepsilon)$. Similarly, the Jacobi field η of an isotropic geodesic variation will also be called *isotropic.* (Clearly, $\eta(0)=0$.) When speaking of isotropic Jacobi fields, we shall always assume that they are not identically zero.

LEMMA 1: *If* $Q=\sigma(s_0)$ *is a zero of the isotropic Jacobi field induced by the* 1*-parameter subgroup* $h(\varepsilon)$ *of* H, *then all the geodesics* $\tau(h(\varepsilon))\,\sigma$ *meet at* Q; *and* Q *is conjugate to* P_0 *along each of the geodesics* $\tau(h(\varepsilon))\,\sigma$. *If* Q *is not a zero of a given isotropic Jacobi field,* η, *induced by the* 1*-parameter subgroup* $h(\varepsilon)\subset H$, *then for sufficiently small* ε, $Q\neq\tau(h(\varepsilon))(Q)$. [3, p. 326; 9, p. 116].

Proof: For all x, Lie's first theorem (applied to the transformation group $\tau(h(\varepsilon)):G/H\to G/H$) implies

$$\begin{aligned}\frac{\partial x}{\partial\varepsilon}(s, \varepsilon)|_{\varepsilon=\varepsilon_0} &= d\tau(h(\varepsilon_0))\left(\frac{\partial x}{\partial\varepsilon}(s, \varepsilon)|_{\varepsilon=0}\right)\\ &= d\tau(h(\varepsilon_0))(\eta(s)),\end{aligned}$$

for all ε_0, where $d\tau(h(\varepsilon_0))$ is the non-singular tangent map of $\tau(h(\varepsilon_0))$. Hence $\eta(s_0)=0$ implies $\partial x/\partial\varepsilon(s_0, \varepsilon)\equiv 0$, which implies the orbit of Q under $\tau(h(\varepsilon))$ is just Q. It is now also clear that Q is conjugate to P_0 along each geodesic of the variation. The second statement follows from the non-singularity of $d\tau(h(\varepsilon))$ for all ε.

DEFINITION 2: If $Q\in\sigma$ is the zero of an isotropic Jacobi field along, σ we say that Q is *isotropically conjugate to* P_0 *along* σ.

COROLLARY 1: *If* $Q\in G/H$ *is isotropically conjugate to* $P_0=\pi(H)$ *along* σ, *then* Q *is isotropically conjugate to* P_0 *along any geodesic (which is not left fixed by the* 1*-parameter subgroup in question) passing through* P_0 *and* Q.

LEMMA 2: *Let* G/H *be a reductive homogeneous space where* G *is a subgroup of* $GL(n)$, *the group of* $n\times n$ *nonsingular matrices, and let* Λ *be an invariant affine connection on* G/H, *whose geodesics emanating from* P_0, *are the images under the natural projection* $\pi:G\to G/H$ *of* 1*-parameter subgroups* $\exp(sX)$, *where* $X\in\mathfrak{m}$ *and* $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$,

$[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. *If the first zero of an isotropic Jacobi field along a geodesic σ emanating from $P_0 = \pi(H)$ has path parameter value α, then the set of numbers $k\alpha$, $k = \pm 1, \pm 2, \ldots$ constitute the complete set of zeros of this Jacobi field.*

Proof: Since the geodesics emanating from $P_0 = \pi(H)$ are the projections of 1-parameter subgroups of G, the geodesic σ can be represented by the matrices $\sigma(s) = e^{sB}$, where $B \in \mathfrak{m}$. Let the 1-parameter subgroup $h(\varepsilon)$ be given by $h(\varepsilon) = e^{\varepsilon A}$, $A \in \mathfrak{h}$. Then (since $\tau(h)(xH) = hxh^{-1}H$ for all $h \in H$) the geodesic variation of σ induced by $h(\varepsilon)$ can be written as $x(s, \varepsilon) \equiv e^{\varepsilon A} e^{sB} e^{-\varepsilon A}$ which implies $\partial x / \partial \varepsilon (s, \varepsilon) = e^{\varepsilon A}(A e^{sB} - e^{sB} A) e^{-\varepsilon A}$, i.e., the zeros of $\eta(s) = \partial x / \partial \varepsilon (s, \varepsilon)|_{\varepsilon = 0}$ are given by the solutions of

$$A = e^{sB} A e^{-sB}. \tag{1}$$

One easily sees that if $s = \alpha$ solves (1), then $k\alpha$ solves (1) for all $k = \pm 1, 2, \ldots$. Also if α and α_0 solve (1) then so does $\alpha - \alpha_0$, which implies the lemma, cf. § 8.

We remark that the converse of Lemma 2 is false, i.e., there exist non-isotropic Jacobi fields whose zeros are the integral multiples of the first zero. cf. § 7.

Before turning to Jacobi's equations of geodesic deviation on G/H, we give a resume of results which we will need without providing any proofs. The reader is referred to [1, p. 182–6; 7; 8, p. 41–52].

§ 3. Invariant Affine Connections on Homogeneous Spaces

We first note that if Λ is an invariant affine connection on G/H, then its torsion and curvature tensors, T and B, respectively, are also invariant. We shall assume G/H to be reductive, with a fixed decomposition: $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, such that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. To any reductive homogeneous space there are associated two invariant affine connections which we now describe:

(A) Let $x(s)$ be any 1-parameter subgroup of G generated by an element $X \in \mathfrak{m}$ and let $x^*(s)$ be the image of $x(s)$ be the projection $\pi: G \to G/H$. Then there exists one and only one invariant affine connection without torsion, Γ – called the *connection of Cartan* – having the property that $x^*(s)$ described above is a geodesic.

(B) Let $x(s)$, $x^*(s)$ be as defined above and let $Y \in \mathfrak{m}$. Then there exists one and only one invariant affine connection, Λ – called the *canonical connection* – having the property that parallel displacement of Y at P_0 along the curve $x^*(s)$ is the same as the translation $d\tau(x(s))_{P_0}(Y)$ by the 1-parameter subgroup $x(s)$.

Note that by (A) and (B), the connections Λ and Γ have the same geodesics. From this one can easily show that

$$\Gamma = \Lambda - (\tfrac{1}{2})\, T, \tag{2}$$

whereby we mean that in any coordinate neighborhood (x^i), $i = 1, \ldots, n$ on G/H, $\Gamma^i_{jk} = \Lambda^i_{jk} - (\tfrac{1}{2})\, T^i_{jk}$, $i, j, k = 1, \ldots, n$.

LEMMA 3: *Let Λ and Γ be the canonical and Cartan connections, respectively. Then at $P_0=\pi(H)$, for $X, Y, Z\in\mathfrak{m}$ we have*

$$\begin{aligned} &T(X, Y) = [X, Y]_{\mathfrak{m}} \\ &B(X, Y)Z = [[X, Y]_{\mathfrak{h}}, Z] \\ &R(X, Y)Z = [[X, Y]_{\mathfrak{h}}, Z] + \tfrac{1}{2}[[X, Y]_{\mathfrak{m}}, Z]_{\mathfrak{m}} \\ &\qquad\qquad\qquad + \tfrac{1}{4}[[Z, X]_{\mathfrak{m}}, Y]_{\mathfrak{m}} + \tfrac{1}{4}[[Y, Z]_{\mathfrak{m}}, X]_{\mathfrak{m}}. \end{aligned} \tag{3}$$

Furthermore,

$$DT \equiv DB \equiv 0, \tag{4}$$

i.e., *the covariant derivatives of T and B vanish on all of G/H.*

LEMMA 4: *Let G/H be a compact normal Riemannian homogeneous space; then the Riemannian metric has the Cartan connection for its Levi-Civita connection. We shall henceforth refer to Γ as the metric-Cartan connection. Furthermore the Riemannian sectional curvature at $P_0=\pi(H)$ of the 2-section, μ, spanned by the orthonormal vectors X and $Y\in\mathfrak{m}$ is given by*

$$K(\mu) = \|[X, Y]_{\mathfrak{h}}\|^2 + \tfrac{1}{4}\|[X, Y]_{\mathfrak{m}}\|^2, \tag{5}$$

where $\| \ \|$ is the length of a vector in $\mathfrak{g}$ with respect to the inner product. In particular, G/H has strictly positive curvature if and only if $[X, Y]\neq 0$ for all linearly independent $X, Y\in\mathfrak{m}$.

Remark: Furthermore, it is known that G/H is a symmetric homogeneous space if and only if $[\mathfrak{m}, \mathfrak{m}]\subset\mathfrak{h}$, which implies G/H is a symmetric homogeneous space if and only if the Cartan and canonical connections coincide.

§ 4. The Jacobi Equations of an Affinely Connected Manifold

THEOREM 2: *Let M be an n-dimensional manifold with affine connection Λ, $\sigma: [0, \beta]\to M$ a geodesic; and let $x(s, \varepsilon)$ be a geodesic variation of σ, where s is the path parameter along each geodesic, ε is the family parameter, and $\sigma(s)=x(s, 0)$. Then the Jacobi field $\eta(s)=\partial x/\partial\varepsilon(s, \varepsilon)|_{\varepsilon=0}$ satisfies Jacobi's equation*

$$\frac{D^2}{ds^2}\eta + \frac{D}{ds}(T(\lambda, \eta)) + B(\lambda, \eta)\lambda = 0 \tag{6}$$

where $\lambda(s)$ is the velocity vector field of σ. [6, p. 33–4; 10, p. 126].

We note that (i) the vector field $s\lambda(s)$ is a solution of (6) satisfying $\eta(0)=0$; (ii) one can show that any vector field $\eta(s)$ solving (6) is the Jacobi field of a geodesic variation of σ; and (iii) if M is a Riemannian manifold and Γ the associated Levi-Civita connection, then $T\equiv 0$ and (6) reduces to the classical Jacobi equation

$$\frac{\delta^2}{\delta s^2}\eta + R(\lambda, \eta)\lambda = 0. \tag{7}$$

Henceforth we shall assume that $DT \equiv 0$, i.e., the covariant derivative of T vanishes identically on all of M. Then (6) reads as

$$\frac{D^2}{ds^2}\eta + T\left(\lambda, \frac{D}{ds}\eta\right) + B(\lambda, \eta)\,\lambda = 0. \tag{8}$$

Let $(e_\alpha(s))$, $\alpha = 1, \ldots, n$ be n linearly independent Λ-parallel vector fields along σ. Setting $\eta = \eta_\alpha e_\alpha$, $T(X, Y) = (T(X, Y))_\alpha e_\alpha$, $B(X, Y)\,Z = (B(X, Y)\,Z)_\alpha e_\alpha$, where X, Y, Z are any vector fields along σ, one then obtains

$$\eta''_\alpha + T_{\alpha\beta}\eta'_\beta + K_{\alpha\beta}\eta_\beta = 0, \tag{9}$$

where

$$\begin{aligned} T_{\alpha\beta} &= (T(\lambda, e_\beta))_\alpha \\ K_{\alpha\beta} &= (B(\lambda, e_\beta)\,\lambda)_\alpha, \end{aligned} \tag{10}$$

$\alpha, \beta = 1, \ldots, n$, and the prime denotes differentiation with respect to s. Note that $DT \equiv 0$ implies that $T_{\alpha\beta}$ is constant along any geodesic.

Suppose Γ is the connection on M defined by

$$\Gamma = \Lambda - (\tfrac{1}{2})\,T. \tag{11}$$

Then Γ has no torsion, and one readily shows that both Λ and Γ have the same geodesics.

LEMMA 5: *Let $(f_\alpha(s))$, $\alpha = 1, \ldots, n$ be a Γ-parallel n-frame along the geodesic σ. Then*

$$f_\alpha(s) = a_{\alpha\beta}(s)\, e_\beta(s) \tag{12}$$

implies

$$a'_{\alpha\beta}(s) = (-\tfrac{1}{2})\, a_{\alpha\gamma}(s)\, T_{\beta\gamma}(s). \tag{13}$$

Proof: Note that $\Gamma = \Lambda - (\frac{1}{2})T$ implies that $\delta\xi/ds = D\xi/ds - (\frac{1}{2})T(\xi, \lambda)$, where ξ is any vector field along σ. Then for each $\alpha = 1, \ldots, n$,

$$\begin{aligned} 0 &= \frac{\delta}{ds} f_\alpha \\ &= a'_{\alpha\beta}\, e_\beta + a_{\alpha\beta}\left(\frac{D}{ds} e_\beta - (\tfrac{1}{2})\, T(e_\beta, \lambda)\right) \\ &= \{a'_{\alpha\beta} + (\tfrac{1}{2})\, a_{\alpha\gamma}(T(\lambda, e_\gamma))_\beta\}\, e_\beta \\ &= \{a'_{\alpha\beta} + (\tfrac{1}{2})\, a_{\alpha\gamma} T_{\beta\gamma}\}\, e_\beta \end{aligned}$$

which implies the lemma by the linear independence of $(e_\beta(s))$, $\beta = 1, \ldots, n$ for all s.

LEMMA 6: *Let Γ be the Levi-Civita connection of a Riemannian metric on M, and let Γ be related to Λ by* (11), *where $DT \equiv 0$. Then it is possible to choose an orthonormal frame $(e_\alpha(s))$, $\alpha = 1, \ldots, n$, which is Λ-parallel along σ if and only if $(e_\alpha(0))$, $\alpha = 1, \ldots, n$, can* *be chosen orthonormal such that $T_{\alpha\beta}$ is skew-symmetric. Furthermore, when these conditions are satisfied $K_{\alpha\beta}$ can be chosen symmetric.*

Proof: If $(e_\alpha(0))$, $\alpha=1,\dots,n$, is an orthonormal frame such that $T_{\alpha\beta}$ is skew-symmetric, then setting $f_\alpha(0)=e_\alpha(0)$, $\alpha=1,\dots,n$, we see that $(f_\alpha(s))$, $\alpha=1,\dots,n$, is an orthonormal frame since parallel transport of vectors with respect to Γ preserves their inner product. By (13), since $T_{\alpha\beta}$ is a skew-symmetric matrix, $a_{\alpha\beta}(s)$ is an orthonormal matrix for all s. Therefore the frame $(e_\alpha(s))$ is orthonormal. The argument is reversible and the first part of the lemma is proven.

To prove the second part of the lemma, one notes that by direct calculation, using $DT\equiv 0$, that

$$R(X,Y)Z = B(X,Y)Z + \tfrac{1}{2}T(T(X,Y),Z) + \tfrac{1}{4}T(T(Z,X),Y) + \tfrac{1}{4}T(T(Y,Z),X). \tag{14}$$

We choose $(e_\alpha(s))$, $\alpha=1,\dots,n$, to be an orthonormal frame parallel with respect to Λ. The orthonormality will then imply that

$$\left.\begin{aligned} T_{\alpha\beta} &= \langle e_\alpha, T(\lambda, e_\beta)\rangle \\ K_{\alpha\beta} &= \langle e_\alpha, B(\lambda, e_\beta)\,\lambda\rangle. \end{aligned}\right\} \tag{15}$$

Therefore by the skew-symmetry and symmetry properties of T, B, and R we have

$$\begin{aligned} K_{\alpha\beta} &= \langle e_\alpha, B(\lambda, e_\beta)\,\lambda\rangle \\ &= \langle e_\alpha, R(\lambda, e_\beta)\,\lambda\rangle + \tfrac{1}{4}\langle e_\alpha, T(\lambda, T(\lambda, e_\beta))\rangle \\ &= \langle e_\alpha, R(\lambda, e_\beta)\,\lambda\rangle + \tfrac{1}{4}\langle e_\alpha, T(\lambda, e_\gamma)\rangle\, T_{\gamma\beta} \\ &= \langle e_\alpha, R(\lambda, e_\beta)\,\lambda\rangle + \tfrac{1}{4}T_{\alpha\gamma}T_{\gamma\beta}. \end{aligned}$$

which implies the lemma by the symmetry of $\langle e_\alpha, R(\gamma, e_\beta)\gamma\rangle$ and skew-symmetry of $T_{\alpha\beta}$.

DEFINITION 3: A connection Λ on M is said to be *locally reductive* if $DT\equiv DB\equiv 0$ on all of M.

THEOREM 3: *Let M be a compact orientable Riemannian manifold with Levi-Civita connection Γ and a locally reductive connection Λ related to Γ by* (11). *Then $T_{\alpha\beta}$ and $K_{\alpha\beta}$ of* (9) *(with respect to Λ) can be chosen skew-symmetric and symmetric, respectively.*

Proof: Since Λ is naturally reductive, by a theorem of Nomizu [8, p. 60], M can be locally represented about any $P_0\in M$ as a reductive homogeneous space G/H with Λ for its canonical connection and $P_0=\pi(H)$. Let $x(s)$ be the 1-parameter subgroup of G projecting onto σ, where $\sigma(0)=P_0$. The elements of $x(s)$ leave Λ, and therefore T, and therefore Γ, invariant which implies by a theorem of Yano [11] that $x(s)$ is a 1-parameter group of isometries. Therefore if (e_α) is an orthonormal frame at $P_0=\pi(H)$, then the subgroup $x(s)$ moves (e_α), preserving orthonormality ,in parallel manner with respect to Λ, the canonical connection – and the theorem is proven by Lemma 5.

§ 5. Jacobi's Equations in a Normal Riemannian Homogeneous Space

If G/H is a normal Riemannian homogeneous space which is not symmetric, then contrary to the symmetric case, Jacobi's equations (7) when written with respect to the Levi-Civita connection does not necessarily have constant coefficients $R_{\alpha\beta}$ nor is there necessarily separation of variables as in equations (6), (7) of [10]. In order to solve Jacobi's equations we therefore turn to the canonical connection.

The metric-Cartan and the canonical connection have the same geodesics, and therefore equations (7) and (8) written in the metric-Cartan and canonical connections, respectively, are identical – as one can verify directly by substituting one into the other using (11) and (14). Since, as noted in Theorem 3, the frame $(e_\alpha(s))$ can be chosen orthonormal and transported in parallel manner with respect to the canonical connection, the solutions of (9) when written with respect to the canonical connection can be made to differ from the solutions of (7) written with respect to the metric-Cartan connection by multiplication by the orthogonal matrix obtained by solving (13) (at worst they differ by a non-singular matrix.) Hence one obtains the conjugate points of the metric-Cartan connection by solving the Jacobi equations in the canonical connection which (since $DT \equiv DB \equiv 0$) has only constant coefficients.

To be more explicit, we shall let $e_\alpha(0) = Q_\alpha$, $\alpha = 1, \dots, n$, where (Q_α) is the natural basis of $\mathfrak{m}$, and where the coordinates about the identity e of G are chosen such that (Q_α) is an orthonormal basis of $\mathfrak{m}$. Let $x^*(s)$ be a 1-parameter subgroup of G such that $\pi(x(s)) = \sigma(s)$ is the geodesic σ. For each $\alpha = 1, \dots, n$, let $e_\alpha(s) = dx(s)_{P_0}(Q_\alpha)$, $P_0 = \pi(H)$; then by the definition of the canonical connection, $e_\alpha(s)$ is a Λ-parallel vector field along σ. Since G is a group of isometries, $e_\alpha(s))$ form an orthonormal frame for each s. By Lemma 3, (15), and the natural reductivity of Λ, we have

$$\eta_\alpha'' + \langle Q_\alpha, [\lambda, Q_\beta]_{\mathfrak{m}} \rangle \eta_\beta' + \langle Q_\alpha, [[\lambda, Q_\beta]_{\mathfrak{h}}, \lambda] \rangle \eta_\beta = 0, \tag{16}$$

$\alpha, \beta = 1, \dots, n$, where λ is the unit velocity vectory of σ at P_0. To solve Jacobi equations relative to the Riemannian parallel frame $f_\alpha(s)$, $\alpha = 1, \dots, n$, satisfying $f_\alpha(0) = Q_\alpha$, one lets $A = (a_{\alpha\beta})$ be the matrix defined by

$$A = \exp\left(\frac{s}{2}\mathcal{T}\right) \tag{17}$$

where $\mathcal{T} = (T_{\alpha\beta}) = (\langle Q_\alpha, [\lambda, Q_\beta]_{\mathfrak{m}} \rangle)$. Then setting $\eta = \hat{\eta}_\alpha f_\alpha$, one has by Lemma 5,

$$\hat{\eta}_\alpha(s) = a_{\beta\alpha}(s)\, \eta_\beta(s). \tag{18}$$

§ 6. The Homogeneous Space $Sp(2)/SU(2)$

THEOREM 4: *If G/H is a simply connected normal Riemannian homogeneous space of dimension $\geqslant 2$ having strictly positive sectional curvature, then with two exceptions*

$Sp(2)/SU(2)$, $SU(5)/(Sp(2)\times T)$, G/H is *homeomorphic to a Riemannian symmetric space of rank* 1 [1, p. 226] (*"Rank* 1*" is equivalent to saying that* $X, Y\in\mathfrak{m}$, $[X, Y]=0$ *implies* X *and* Y *are linearly dependent.*)

We shall consider the example $M=Sp(2)/SU(2)$, where $Sp(2)$ is the symplectic 2-group and $SU(2)$ is the special unitary 2-group. Now an element of the Lie algebra $\mathfrak{Sp}(2)$ is skew-Hermitian of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ -\bar{a}_{12} & -a_{11} & \bar{a}_{14} & -\bar{a}_{13} \\ -\bar{a}_{13} & -a_{14} & a_{33} & a_{34} \\ -\bar{a}_{14} & a_{13} & -\bar{a}_{34} & -\bar{a}_{33} \end{pmatrix}$$

where a_{11}, a_{33} are pure imaginary, and the rest are arbitrary complex numbers. Let S_i, $i=1,\dots,10$ be matrices in $\mathfrak{Sp}(2)$ such that

$$\begin{array}{llll}
S_1: a_{11}=-a_{22}=i, & & & \text{otherwise} \quad a_{ij}=0 \\
S_2: a_{33}=-a_{44}=i, & & & \text{———} \\
S_3: a_{12}=-a_{21}=1, & & & \text{———} \\
S_4: a_{12}=a_{21}=i, & & & \text{———} \\
S_5: a_{34}=-a_{43}=1, & & & \text{———} \\
S_6: a_{34}=a_{43}=i, & & & \text{———} \\
S_7: a_{13}=-a_{31}=-a_{24}=a_{42}=1, & & & \text{———} \\
S_8: a_{13}=a_{31}=a_{24}=a_{42}=i, & & & \text{———} \\
S_9: a_{14}=-a_{41}=a_{23}=-a_{32}=1, & & & \text{———} \\
S_{10}: a_{14}=a_{41}=-a_{23}=-a_{32}=i, & & & \text{———}
\end{array}$$

Setting

$$\begin{array}{ll}
Q_1=\tfrac{1}{2}(S_1-3S_2) & Q_2=\sqrt{\tfrac{5}{2}}\,S_3 \\
Q_3=\sqrt{\tfrac{5}{2}}\,S_4 & Q_4=1/\sqrt{2}(\sqrt{3}\,S_5-S_7) \\
Q_5=1/\sqrt{2}(\sqrt{3}\,S_6-S_8) & Q_6=\sqrt{\tfrac{5}{2}}\,S_9 \\
Q_7=\sqrt{\tfrac{5}{2}}\,S_{10} & Q_8=\tfrac{1}{2}(3S_1+S_2) \\
Q_9=S_5+\sqrt{\tfrac{3}{2}}\,S_7 & Q_{10}=S_6+\sqrt{\tfrac{3}{2}}\,S_8
\end{array}$$

we have (i) $\{Q_1,\dots,Q_{10}\}$ are linearly independent and therefore form a basis of $\mathfrak{Sp}(2)$. (ii) Furthermore, if for an inner product on $\mathfrak{Sp}(2)$ we take $\langle A, B\rangle=-\tfrac{1}{5}\operatorname{trace}(AB)$, then $\{Q_1,\dots,Q_{10}\}$ is an orthonormal basis of $\mathfrak{Sp}(2)$; also (iii) the inner product is invariant under $Ad(Sp(2))$. (iv) Finally, one can show that $\mathfrak{h}=$ linear span $\{Q_8, Q_9, Q_{10}\}$ is Lie isomorphic to $\mathfrak{SU}(2)$ and therefore the group H generated by H is analytically isomorphic to $SU(2)$.

The decomposition we have chosen follows Berger [1, p. 234]. Note that the representation of $\mathfrak{SU}(2)$ in $\mathfrak{Sp}(2)$ is not the canonical one, but rather an irreducible one. Berger also shows that if $X, Y\in\mathfrak{m}\subset\mathfrak{Sp}(2)$, $\mathfrak{m}=\operatorname{span}\{Q_1,\dots,Q_7\}$, and X, Y are linearly independent then $[X, Y]\neq 0$; therefore by Lemma 4, $M=Sp(2)/SU(2)$ has strictly positive curvature. Note that Berger's theorem says that M cannot be given *any* metric so that with respect to the metric it becomes a locally symmetric space. (To see that M is not symmetric with respect to the given metric, one can easily show that $[\mathfrak{m}, \mathfrak{m}]\not\subset\mathfrak{h}$.)

We note that the pinching of $Sp(2)/SU(2)$ has been calculated by ELIASSON, and is $\frac{1}{37}$. [4]

§ 7. Two Geodesics in $Sp(2)/SU(2)$

We shall now show that the geodesic σ generated by $Q_2\in\mathfrak{Sp}(2)$ has a Jacobi field which vanishes at $s=\alpha$ along the geodesic and does not vanish at $s=2\alpha$; also no other Jacobi field along the geodesic vanishes at $s=\alpha$. Hence, by Lemma 2, there exists a non-isotropic Jacobi field vanishing at a point where no isotropic Jacobi field vanishes.

Since $Sp(2)/SU(2)$ is not Riemannian symmetric, we shall solve Jacobi's equations written with respect to the canonical connection. Then for the values of $T_{\alpha\beta}$, $K_{\alpha\beta}$ along σ, we have $T_{13}=-T_{31}=-T_{46}=T_{64}=T_{57}=-T_{75}=1$ and the rest of the T's are zero; an d$K_{33}=9, K_{66}=K_{77}=\frac{5}{2}$ otherwise $K_{\alpha\beta}=0$. To solve (9), we solve the associated eigenvalue problem:

$$
\begin{aligned}
0 &= \det(\lambda^2 I + \lambda T + K) \\
&= \begin{vmatrix}
\lambda^2 & 0 & \lambda & 0 & 0 & 0 & \lambda \\
0 & \lambda^2 & 0 & 0 & 0 & 0 & 0 \\
-\lambda & 0 & \lambda^2+9 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda^2 & 0 & -\lambda & 0 \\
0 & 0 & 0 & 0 & \lambda^2 & 0 & \lambda \\
0 & 0 & 0 & \lambda & 0 & \lambda^2+\frac{5}{2} & 0 \\
0 & 0 & 0 & 0 & -\lambda & 0 & \lambda^2+\frac{5}{2}
\end{vmatrix} \\
&= \lambda^8(\lambda^2+10)(\lambda^2+\tfrac{5}{2})^2 .
\end{aligned}
$$

Using the method of undertermined coefficients one shows that the general solutions of (9) along σ, such that $\eta(0)=0$ can be written as:

$$
\begin{aligned}
\eta_1(s) &= \alpha_1(1-\cos\sqrt{10}\,s) + \alpha_2(9\sqrt{10}\,s + \sin\sqrt{10}\,s) \\
\eta_2(s) &= \alpha_3\, s \\
\eta_3(s) &= \alpha_1(-\sqrt{10}\sin\sqrt{10}\,s) + \alpha_2\{\sqrt{10}(1-\cos\sqrt{10}\,s)\} \\
\eta_4(s) &= \alpha_4(1-\cos\sqrt{\tfrac{5}{2}}\,s) + \alpha_5\left(\frac{3\sqrt{5}}{2\sqrt{2}}\,s + \sin\sqrt{\tfrac{5}{2}}\,s\right) \\
\eta_5(s) &= \alpha_6(1-\cos\sqrt{\tfrac{5}{2}}\,s) + \alpha_7\left(\frac{3\sqrt{5}}{2\sqrt{2}}\,s + \sin\sqrt{\tfrac{5}{2}}\,s\right) \\
\eta_6(s) &= \alpha_4(\sqrt{\tfrac{5}{2}}\sin\sqrt{\tfrac{5}{2}}\,s) + \alpha_5\{-\sqrt{\tfrac{5}{2}}(1-\cos\sqrt{\tfrac{5}{2}}\,s)\} \\
\eta_7(s) &= \alpha_6(\sqrt{\tfrac{5}{2}}\sin\sqrt{\tfrac{5}{2}}\,s) + \alpha_7\{-\sqrt{\tfrac{5}{2}}(1-\cos\sqrt{\tfrac{5}{2}}\,s)\}
\end{aligned}
\tag{19}
$$

where $\alpha_1, \dots, \alpha_7$ are arbitrary constants. We consider the set of points of σ conjugate to $P_0 = \pi(H)$ coming from the Jacobi fields spanned by the two solutions corresponding to $\alpha_\kappa = \delta_{1\kappa}$ and $\alpha_\kappa = \delta_{2\kappa}$, $\kappa = 1, \dots, 7$. Then the arc length at the conjugate points are the zeros of the determinant

$$\begin{vmatrix} 1 - \cos\sqrt{10}\,s & 9\sqrt{10}\,s + \sin\sqrt{10}\,s \\ -\sin\sqrt{10}\,s & 1 - \cos\sqrt{10}\,s \end{vmatrix}.$$

Setting $u = \sqrt{10}\,s$, the problem becomes that of finding the zeros of $f(u) =$ 1-$\cos u + \frac{9}{2} u \sin u$. Clearly $f(2\pi k) = 0$ for all $k = 1, 2, \dots$. Now $f'(u) = \frac{1}{2}\{11 \sin u + 9u \cos u\}$, which implies $f'(2\pi k) = 9\pi k > 0$ for all $k = 1, 2, \dots$. Also there exists an $\varepsilon > 0$ such that: $0 < u < \varepsilon$ implies $f(u) > 0$, and therefore there exists α, $0 < \alpha < 2\pi$ such that $f(\alpha) = 0$. Of course, $\alpha \neq \pi/2$, π, $3\pi/2$. We now show that $f(2\alpha) \neq 0$:

$$f(2\alpha) = 2 \sin\alpha \{\sin\alpha + 9\alpha \cos\alpha\}, \text{ and therefore } f(2\alpha) = 0 \text{ implies } \alpha = -\frac{\tan\alpha}{9},$$

which implies $f(\alpha) = -(\cos\alpha - 1)^2/(2\cos\alpha) \neq 0$, which implies a contradiction. Therefore, any Jacobi field η with $(\alpha_1)^2 + (\alpha_2)^2 > 0$ such that $\eta(\alpha) = 0$ is not a isotropic. By looking at (19), one can show that the number of linearly independent Jacobi fields vanishing at α is exactly one. Since this Jacobi field is non-isotropic, the point $\sigma(\alpha)$ is also non-isotropic.

Now we show that the converse of Lemma 2 is false (§2). Indeed, for the geodesic σ generated by $Q_1 \in \mathfrak{S}_{\mathfrak{q}}(2)$ the constants $T_{\alpha\beta}$ and $K_{\alpha\beta}$ are: $-T_{23} = T_{32} = T_{45} = -T_{54} = T_{67} = -T_{76} = 1$; $K_{44} = K_{55} = 6$; and the rest are zero. The associated characteristic polynomial is then:
$\lambda^6 (\lambda^2 + 1)^2 (\lambda^2 + 4)(\lambda^2 + 9)$; and a basis of solutions of (9) such that $\eta(0) = 0$ is given by

$$\begin{aligned}
\eta_1(s) &= \alpha_1 s \\
\eta_2(s) &= \alpha_2(1 - \cos s) - \alpha_3 \sin s \\
\eta_3(s) &= \alpha_2 \sin s + \alpha_3(1 - \cos s) \\
\eta_4(s) &= \alpha_4(\cos 2s - \cos 3s) + \alpha_5(\sin 2s - \sin 3s) \\
\eta_5(s) &= -\alpha_4(\sin 2s - \sin 3s) + \alpha_5(\cos 2s - \cos 3s) \\
\eta_6(s) &= \alpha_6(1 - \cos s) + \alpha_7 \sin s \\
\eta_7(s) &= -\alpha_6 \sin s + \alpha_7(1 - \cos s).
\end{aligned} \tag{20}$$

One sees that there are more than 3 linearly independent solutions whose zeros are integral multiples of a fixed real number; since $\dim SU(2) = 3$, there exists at least one such non-isotropic Jacobi field.

§ 8. Added in proof

The proof of Lemma 2 is incorrect, viz, the proof only concerns itself with what happens in G, *not* in G/H. The Jacobi field $\eta(s)$ of the variation is not given by

$\partial x/\partial\varepsilon(s, \varepsilon)|_{\varepsilon=0}$, as stated in the proof; it is given by $\eta(s)=d\pi(\partial x/\partial\varepsilon(s, \varepsilon)|_{\varepsilon=0})$. Hence there may be more zeros than we have accounted for.

Theorem 1 remains true nevertheless, since by the Remark at the end of [2], one can show that the isotropic Jacobi fields along $\pi(e^{sQ_2})$, $Q_2 \in \mathfrak{S}\mathfrak{p}\,(2)$ are spanned by the three Jacobi fields having constants of integration, cf. (19), $\alpha_k = c_j\delta_{jk}$, $j=1, 4, 7$, respectively, where c_j are constants. The rest of the discussion of $\pi(e^{sQ_2})$ follows as stated.

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