Some Remarks concerning Surfaces in Three-space.

Autor(en): **Hawley, Newton S.**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **42 (1967)**

PDF erstellt am: **17.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-32144

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.
Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.
Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Ein Dienst der *ETH-Bibliothek*
ETH Zürich, Rämistrasse 101, 8092 Zürich, Schweiz, www.library.ethz.ch

http://www.e-periodica.ch

Some Remarks Concerning Surfaces in Three-space

In Memoriam For Anja Hopf

by Newton S. Hawley[1])
(Stanford University and Università di Pisa)

1. Introduction

In this paper a class of complete surfaces will be defined by means of certain quantities related to the first and second fundamental forms of the surfaces in question. It will then be shown that this class is, in fact, just the class of oval cylinders in three space.

It is advisable, however, to develop first a terminology which is sufficiently precise for this purpose.

The term *surface* will always denote, in this paper, a non-singular complete surface of class C^3. The surface in question will be designated by S, and it will be assumed that S is a C^3 surface in the ordinary euclidean three-space R^3. The Gaussian curvature of S will be designated by the symbol K, the mean curvature by H, etc., in the usual classical notation. The universal covering surface of S will be denoted by $\tilde{S}$.

Although $\tilde{S}$ is an abstract surface, the projection mapping

$$p:\tilde{S} \to S$$

can be used to carry the first and second fundamental forms of S, as well as H, K, etc., from S back onto $\tilde{S}$. Thus $\tilde{S}$ can be dealt with almost as if it were a surface in R^3.

Extensive use will be made of *isothermal parameters* introduced on S and on $\tilde{S}$, and in particular the fundamental forms will be expressed in terms of such parameters. For example, if $z=x+iy$ is an isothermal parameter then the first fundamental form I is expressed by

$$\mathrm{I} = E\,|dz|^2 \quad (\text{since } E = G \quad \text{and} \quad F = 0).$$

(For all classical notions and notations see [2].)

For later convenience, in stating some of the results, some non-standard definitions will be made.

Definition 1: A surface S will be called *anticonvex* if $K \leqslant 0$ everywhere on S.

Definition 2: A surface S will be called *antiminimal* if there is a constant $a \neq 0$ such that $H \geqslant 1/a^2 > 0$ everywhere on S.

[1]) This work was supported in part by National Science Foundation grant GP 4069 at Stanford University.

DEFINITION 3: A plane curve will be called an *oval* if it is of class C^3, complete, simple, and if its curvature $k \geqslant k_0 > 0$, i.e. is bounded away from zero.

(See the remarks in the last section of the paper concerning this rather strange definition of an oval.)

Important use will be made also of the following statement, which is set forth as a lemma.

LEMMA 1. *If a function is defined and subharmonic in the entire plane, and if it is bounded above then it is a constant.*

(The proof of this lemma is an elementary exercise which has no place here.)

Even though the following statement is simple, and its proof almost immediate, it is presented as a lemma because it embodies the central idea of this paper.

LEMMA 2. *Let S be an anticonvex surface in R^3 such that $\tilde{S}$ is of parabolic type and admits a global isothermal parameter z in terms of which E is bounded; then S is developable*, i.e. $K \equiv 0$.

Proof. Since $K \leqslant 0$ everywhere on S, the Gauss equation, viz.

$$\Delta \log E = -2EK$$

implies that $\Delta \log E \geqslant 0$ everywhere on S, i.e. $\log E$ is subharmonic on S, and therefore on $\tilde{S}$ as well. Since $\tilde{S}$ is conformally equivalent to the entire plane, $\log E$ can be considered to be subharmonic in the entire plane. But E is bounded, by hypothesis, hence $\log E$ is bounded above. Lemma 1 thus implies that $\log E$ is constant, and another application of the Gauss equation shows that $K \equiv 0$ on S, which proves the lemma.

It should be remarked also that since S is complete and non singular it can be neither a tangent developable nor a cone, therefore it must be a cylinder over a plane curve.

2. The quadratic differential Q

A quantity will now be introduced which was first utilized by H. HOPF in order to show that a compact surface of genus zero and constant mean curvature must be a sphere [3].

Let z be an isothermal parameter on S (or on $\tilde{S}$), then the second fundamental form II can be written in terms of z and $\bar{z}$. Since II is real the expression will be of the form

$$\mathrm{II} = Q\,dz^2 + 2P\,dz\,d\bar{z} + \bar{Q}\,d\bar{z}^2,$$

where P is real. The coefficient Q is the quantity in which the interest here is centered.

The transformation properties of Q are immediately evident; for if w is another isothermal parameter then, in terms of w and $\bar{w}$, the expression of II is,

$$\mathrm{II} = Q^* \, dw^2 + 2P^* \, dw \, d\bar{w} + \bar{Q}^* \, d\bar{w}^2,$$

which shows that

$$Q \, dz^2 = Q^* \, dw^2,$$

i.e. Q transforms like a quadratic differential. It may be useful to point out that, in the classical notation, Q is given by,

$$Q = \tfrac{1}{4}(L - N - 2iM), \tag{1}$$

and P is given by,

$$P = \tfrac{1}{2} E H.$$

Thus the first and second fundamental forms are related as follows:

$$\mathrm{II} - H \, \mathrm{I} = 2 \, \mathrm{Re}\{Q \, dz^2\}.$$

Use will be made also of the operators

$$\partial = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right) \quad \text{and} \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)$$

expressed in terms of the parameter $z = x + iy$.

The quantity Γ, defined by

$$\Gamma = -\,\partial(\log E) = -\,(\log E)_z,$$

clearly transforms like a connection (see [5]), and in terms of this Γ a covariant differentiation operator D can be defined by

$$D = \partial + \Gamma,$$

i.e. if V is a (covariant) vector field on S, its covariant derivative is given by

$$DV = \partial V + \Gamma V = V_z + \Gamma V.$$

Let the surface S, parameterized by z, be given in R^3 by the vector equation

$$X = X(z, \bar{z});$$

then $X_z = \partial X$ and $X_{\bar{z}} = \bar{\partial} X$ are tangent vectors.

The coefficient E, of the first fundamental form, and the unit normal vector $\mathfrak{N}$ are given by

$$E = 2 X_z \cdot X_{\bar{z}}, \quad \mathfrak{N} = \frac{X_z \times X_{\bar{z}}}{i \, |X_z \times X_{\bar{z}}|}.$$

It is easily deduced that

$$D X_z = Q \, \mathfrak{N}, \tag{2}$$

and

$$\bar{\partial} X_z = P \mathfrak{N}.$$

The classical equations of Codazzi and Gauss can be written as,

$$DP = \bar{\partial} Q \qquad \text{(Codazzi)},$$
$$H \bar{\partial} \Gamma = K P \quad \text{(Gauss)},$$

but it is perhaps more useful here to write them in the form,

$$Q_{\bar{z}} = \tfrac{1}{2} E H_z \quad \text{(Codazzi)}, \tag{3}$$

and

$$\Gamma_{\bar{z}} = \tfrac{1}{2} E K \quad \text{(Gauss)}.$$

With this notation established a necessary definition can now be made.

DEFINITION 4. A surface S will be called "Q-bounded" if its universal covering surface $\tilde{S}$ admits a global isothermal parameter z such that the Q, expressed in terms of this z, is bounded on $\tilde{S}$.

3. The theorem

The necessary definitions have been made, the necessary terminology has been established and now the principal result can be stated.

THEOREM. *If the surface S is anticonvex, antiminimal, and Q-bounded, it is an oval cylinder.*

(An oval cylinder is, of course, a cylinder over an oval, as defined in Definition 3.)

Proof of theorem. Since S is Q-bounded there is a number b such that $|2Q| \leqslant b^2$. From equation (1) it follows that

$$|2Q|^2 = E^2 (H^2 - K).$$

Therefore the inequality $K \leqslant 0$ implies

$$E \leqslant \frac{b^2}{H} \leqslant a^2 b^2.$$

It must now be shown that $\tilde{S}$ is parabolic.

If $\tilde{S}$ were not parabolic it would be conformally equivalent to the unit disk $\mathscr{D}^*$ in the w-plane, i.e. the set of points w such that $|w| < 1$. In this case w also could be taken to be a global isothermal parameter on $\tilde{S}$. Let the first fundamental form I, expressed in terms of w, be denoted by

$$\mathrm{I} = ds^2 = E^* |dw|^2;$$

then it follows that

$$E^* = E\left|\frac{dz}{dw}\right|^2.$$

Since z is a global uniformizing parameter on $\tilde{S}$ it defines a conformal mapping of $\mathscr{D}^*$ in the w-plane onto a simply connected domain $\mathscr{D}$ in the z-plane. (And $\mathscr{D}$ must have at least two boundary points if it is assumed that $\tilde{S}$ is not parabolic.)

Therefore there exists a point w_0, with $|w_0|=1$, such that the quantity $|dz/dw|$ remains bounded near w_0. To be more precise, let $\mathscr{D}^*(w_0, \varepsilon)$ denote the set of points w such that

$$|w| < 1, \quad \text{and} \quad |w - w_0| < \varepsilon.$$

Then there exists a w_0 and an $\varepsilon > 0$ such that $|dz/dw|$ remains bounded in $\mathscr{D}^*(w_0, \varepsilon)$. (For if not the circle $|w|=1$ would be a natural boundary for the holomorphic function z; therefore z could not give a conformal mapping of $\mathscr{D}^*$ onto $\mathscr{D}$.) Letting c be a number such that $|dz/dw| \leqslant c$ in $\mathscr{D}^*(w_0, \varepsilon)$, it follows that

$$E^* = E\left|\frac{dz}{dw}\right|^2 \leqslant a^2 b^2 c^2$$

in $\mathscr{D}^*(w_0, \varepsilon)$.

Since w is a mapping of $\tilde{S}$ onto $\mathscr{D}^*$, S can be considered as arising from $\mathscr{D}^*$ by the identification of points under the action of some Fuchsian group (see [1] or [4]). Let S_0 denote a fundamental domain for S under the action of this group, and let S_0 be chosen so that w_0 is on its boundary. (Where w_0 is the point whose existence was established above.) And let λ_0 be a half-open circular arc which has w_0 as its missing endpoint and which lies entirely within $\mathscr{D}^*(w_0, \varepsilon) \cap S_0$. (Such a λ_0 exists since S_0 is bounded by circular arcs.)

If $\lambda = p\lambda_0$ is the projection of λ_0 into S then

$$\int_\lambda ds = \int_{\lambda_0} \sqrt{E^*}\,|dw| \leqslant a\,b\,c \int_{\lambda_0} |dw| < 2\pi\,a\,b\,c.$$

On the other hand $\int_\gamma ds$ diverges since S is complete and λ "goes to infinity". This give a contradiction, therefore S must be parabolic.

Lemma 2 can now be applied to conclude that S is a cylinder over a plane curve. That this plane curve is an oval follows from the fact that its curvature $k \geqslant 2/a^2 > 0$ at each point, which in turn follows from the assumption on H. Thus the theorem is proved.

It should be remarked that the theorem is sharp, in the sense that every oval cylinder satisfies the hypotheses of the theorem. Thus we have established the identity of the class of oval cylinders with the class of anticonvex, antiminimal, Q-bounded surfaces.

4. A corollary and some concluding remarks

The following questions will naturally arise.

"Why formulate a theorem for Q-bounded surfaces?"

"What does this condition mean?" Unfortunately no fully satisfactory answer will be found here. Equation (2) – which was included as a partial answer to these questions – cannot be considered as a sufficient reply. For equation (2), even though it gives a geometric interpretation of Q, fails to enable one to determine effectively which surfaces are Q-bounded.

But the following corollary can be put forward also as partial justification for the notion of Q-boundedness.

COROLLARY: *If S is anticonvex and $H \neq 0$ is constant on S, then S is a circular cylinder.*

Proof. Since H is a constant, different from zero, and $K \leqslant 0$, there are no umbilical points on S or on $\tilde{S}$. Therefore curvature coordinates define a global isothermal parameter on $\tilde{S}$. And Q, which is analytic on $\tilde{S}$ by equation (3), is constant in these coordinates since $M = -2\,\mathrm{Im}\{Q\}$ is identically zero in curvature coordinates. Therefore Q is bounded and the theorem applies. The oval is a circle since H is constant. This establishes the corollary.

Some remarks should be added here concerning the peculiar definition of an oval given in the introduction. It is intuitively clear that Definition 3 is equivalent to the usual definition of an oval as a closed convex, plane curve of class C^3 whose curvature does not vanish; and in fact the two definitions are rigorously equivalent.

However the proof is both long (elementary, but long) and uninteresting, and since no use is made here of the usual definition, it was deemed advisable to proceed as has been done. The formulation of the definition of an oval was made to tie in as easily as possible with the method of proof of the theorem.

Finally it should be noted that the condition of Q-boundedness is stronger than necessary to prove a theorem like the one above; and the condition that H be bounded away from zero could be relaxed somewhat also. These conditions were used to establish the boundedness of E which, in turn, is more than is necessary, since lemma 2 can easily be strengthened. One only needs to know that E grows sufficiently slowly in order to conclude from its subharmonisity that it is constant.

(Added May 22, 1967)

It has been called to the attention of the author, that a paper: "Complete Surfaces in E^3 with Constant Mean Curvature", by T. KLOTZ and R. OSSERMAN, Comment.

Math. Helv. *41* (1966–67), 313–318, has appeared subsequent to the submission of the present paper. A theorem in the Klotz–Osserman paper contains the corollary given above. Not only was their result submitted earlier, but the method of proof is quite different and therefore should be of interest to any reader of this present paper.

REFERENCES

[1] Appel, P. and Goursat, E., *Theorie des fonctiones algébriques*, t. *II.* Paris 1930.

[2] Bianchi, L., *Lezioni di Geometria Differenziale*, vol. *I.* Pisa 1922.

[3] Hopf, H., *Über Flächen mit einer Relation zwischen den Hauptkrümmungen*, Math. Nachr. *4* (1951), 232–249.

[4] Lehner, J., *Discontinuous Groups and Automorphic Functions*, Amer. Math. Soc. Survey *VIII*, Providence 1964.

[5] Schiffer, M. and Hawley, N. S., *Connections and conformal mapping*, Acta Math., *107* (1962), 175–274.

Received December 2, 1966