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More Characteristic Classes for Spherical Fibre Spaces

by JAMES STASHEFF

The object of this paper is to continue Milnor's computation of $H^*(B_G; Z_p)$ [2], which is:

THEOREM A. *In dimensions less than* $2p(p-1)-1$ *the cohomology ring* $H^*(B_G; Z_p)$ *is isomorphic to a free commutative algebra generated by the Wu classes* q_i *and the Bockstein coboundaries* βq_i.

MILNOR computes using a Postnikov system for B_G. He denotes by $B^{[0,t-1]}$ the space obtained by killing all the homotopy groups $\pi_i(B_{SG})$ for $i \geqq t$.

THEOREM B (MILNOR). *If* $2 \leqq m < p$, *and* $r = 2(p-1)$, *the algebra* $H^*(B^{[0,mr-1]}; Z_p)$ *is isomorphic in dimensions less than* $pr = 2p(p-1)$ *to the free commutative algebra on generators*

$$\mathscr{P}^i \bar{q}, \beta \mathscr{P}^i \bar{q} \qquad i \geqq 0$$

and

$$\mathscr{P}^i k_m, \beta \mathscr{P}^i k_m \qquad i \geqq 0$$

where $\bar{q}$ *corresponds to the first Wu class and where* k_m *is the p primary component of the* $(mr+1)$*-dimensional k-invariant. If* $m < p-1$, $((m+1)\mathscr{P}^1 \beta - m \beta \mathscr{P}^1) k_m = 0$.

In the present paper we extend Milnor's computations, first by computing $H^*(B^{[0,mr]}; Z_p)$ for $m < p$ in dimensions $< 2pr$ and next in this range of dimensions for $m < 2p$. The change in dimensions must proceed the change in m in order to compute the k-invariants. The reason for stopping at level $2pr$ is partly expository; certain new ideas are fairly simple as needed for $p \leqq m < 2p$ but might be much more obscure if lost in the welter of bookkeeping required in higher dimensions.

THEOREM 1. *For* $n < 2pr$, *the algebra* $H^*(B^{[0,n]}; Z_p)$ *is isomorphic in dimensions* $< 2pr$ *to a free commutative algebra. The subalgebra which survives to* $H^*(B_G)$ *has generators which can be obtained by suitable elements of the Steenrod algebra A acting on generators*

q *of dimension* r	if	$n \geqq r$
y *of dimension* pr	if	$n \geqq 2r$
e_1 *of dimension* $pr-1$	if	$n \geqq pr-1$.

In proving the theorem, there will be given a specific set of generators modulo the image of later k-invariants.

THEOREM 2. *In dimensions* $< 2pr$, $H^*(B_G)$ *is isomorphic to a free commutative algebra on the Wu classes* q_i, *their Bocksteins* βq_i *and certain exotic classes* θe_1 *where* θ *ranges over an additive basis of* $A/A\mathscr{P}^1$.

1. Milnor's machine

Our method of attack is to use Milnor's approach as much as possible, feeding in additional information (such as secondary cohomology operations) when forced to. In the hopes that the present paper does not represent the limits of this approach, we identify components of the machine that seem conceptually significant.

In broadest outline, Milnor's approach is, inductively, to identify the k-invariant, compute the action of the Steenrod algebra on it and hence compute the transgression in the fibring and then compute H^* of the total space using the Serre spectral sequence. The latter step is, for Milnor, always of the following form.

PROPOSITION. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration. Assume F connected, B simply connected, both with homology of finite type. Let $H^*(F)$ and $H^*(B)$ be free commutative algebras on generators f_i, $i=1, \dots, m$ and $b_j, j=1, \dots, n$ such that $\tau(f_i)=b_i i=1, \dots, r$ and $\tau(f_i)=0$, $i=r+1, \dots, m$. Then $H^*(E)$ is free commutative on generators $p^*(b_i)$ $i=r+1, \dots, n$ and e_i, $i=r+1, \dots, m$ such that $i^*(e_i)=f_i$.* [TODA, 3, p. 105].

Since we are concerned with the fibrings $K(\pi_{mr}, mr) \to B^{[0,mr]} \to B^{[0,mr)}$, the generators of the fibre are easily labelled in terms of the Steenrod algebra acting on the fundamental class. The corresponding non-zero transgressions are helpfully labelled in terms of the Steenrod algebra acting on the k-invariant. In the range Milnor computes, the essence of his argument is that the generating classes which restrict non-trivially to the fibre at each stage are precisely the transgressions of generating classes in the fibre at the next stage. Certain exact sequences of Toda are relevant here. As we extend the range, we find this remains true *most* of the time; a major problem is that of keeping books on the few classes which are not disposed of so neatly. Let us look at $B^{[0,2r]}$ in detail.

2. $B^{[0,2r]}$

According to MILNOR, the first k-invariant h_2 is $(2\mathscr{P}^1\beta - \beta\mathscr{P}^1)\, q$ and the second, h_3, restricts to $(3\mathscr{P}^1\beta - 2\beta\mathscr{P}^1)\, u_2$ where q is the fundamental class in $K(Z_p, r)$ and u_2 is the fundamental class in $K(Z_p, 2r)$. According to TODA [4; I: Prop. 1.5], if A represents the Steenrod algebra the sequence $A \xrightarrow{3\mathscr{P}^1\beta - 2\beta\mathscr{P}^1} A \xrightarrow{2\mathscr{P}^1\beta - \beta\mathscr{P}^1} A$ is exact, where the maps indicate $\alpha \to \alpha(2\mathscr{P}^1\beta - \beta\mathscr{P}^1)$, etc. We compare this with our fibrings by

$$\begin{array}{ccccc} A & \xrightarrow{3\mathscr{P}^1\beta - 2\beta\mathscr{P}^1} & A & \xrightarrow{2\mathscr{P}^1\beta - \beta\mathscr{P}^1} & A \\ \downarrow & & \downarrow & & \downarrow \\ H^*(Z_p, 3r) & \underset{i^*\tau}{\rightarrow} & H^*(Z_p, 2r) & \underset{\tau}{\rightarrow} & H^*(Z_p, r) \end{array}$$

where the vertical arrows are $\alpha \to \alpha u_m$ $m=1, 2, 3$, $u_1 = q$. Thus generating classes in $B^{[0,2r]}$ which restrict non-trivially to the fibre $K(Z_p, 2r)$ are the transgressions of generating classes in $K(Z_p, 3r)$ except possibly when the class survives for unstable reasons, e.g.

$$\tau(\mathcal{P}^{p-2}u_2) = p\mathcal{P}^{p-1}\beta q - \beta\mathcal{P}^{p-1}q = -\beta(q^p) = 0.$$

Let y be a class which restricts to $\mathcal{P}^{p-2}u_2$. In dimensions $<2pr$, we find that $\mathcal{P}^1 y$ and $\beta\mathcal{P}^1 y$ restrict to $\mathcal{P}^{p-1}u_2$ and $\beta\mathcal{P}^{p-1}u_2$ and that these are the only classes in $B^{[0,2r]}$ which restrict non-trivially mod transgression from $K(Z_p, 3r)$.

On the other hand, there are classes in $K(Z_p, r)$ which survive to $B^{[0,2r]}$. Specifically in dimensions less than $p^2 r$ we have

$$\begin{array}{lll} \mathcal{P}^i\mathcal{P}^j q & j<p-1 & pj \leqq i < (p-1)(j+1) \\ \mathcal{P}^i\mathcal{P}^j \beta q & j<p-1 & pj \leqq i \leqq (p-1)(j+1) \\ \mathcal{P}^{(p-1)(j+1)}\beta\mathcal{P}^j q & j<p-1 & \\ \beta\mathcal{P}^{(p-1)(j+1)}\beta\mathcal{P}^j q & 0<j<p-1 & \end{array}$$

In dimensions $<2pr$ this reduces to

$$\begin{array}{llll} \mathcal{P}^i q & i<p & \mathcal{P}^i\mathcal{P}^1 q & p \leqq i < 2(p-1) \\ \mathcal{P}^i \beta q & i \leqq p-1 & \mathcal{P}^i\mathcal{P}^1\beta q & p \leqq i \leqq 2(p-1) \end{array}$$

We call these classes Wu generators; their relation to Wu classes is given by MILNOR in dimensions $<pr$ and in the remaining cases will be given in § 9. We have thus verified Theorem 1 for $B^{[0,2r]}$ and listed the generators which survive to $B^{[0,3r]}$.

3. The inductive step

Consider the fibring $K(Z_p, mr) \to B^{[0,mr]} \to B^{[0,(m-1)r]}$ for $m<p$. According to MILNOR, the k-invariant $h_m \in H^{mr+1}(B^{[0,mr]}; Z_p)$ restricts to $(m\mathcal{P}^1\beta - (m-1)\beta\mathcal{P}^1)\, u_{m-1}$ in $K(Z_p, (m-1)r)$. According to Toda

$$A \xrightarrow{m\mathcal{P}^1\beta-(m-1)\beta\mathcal{P}^1} A \xrightarrow{(m-1)\mathcal{P}^1\beta-(m-2)\beta\mathcal{P}^1} A$$

is exact, so in the stable range the generating classes in $B^{[0,mr]}$ which restrict non-trivially to $K(Z_p, (m-1)r)$ are precisely those in the image of transgression from $K(Z_p, mr)$. By the stable range here we mean dimensions less than $p(m-1)r$. This is the standard "stable range mod p" in the sense of $\mathscr{C}$-theory for $K(Z_p, (m-1)r)$ since that space is $(m-1)r-1$-connected. Equivalently, this is the range in which $A \to H^*(Z_p, (m-1)r; Z_p)$ given by $\alpha \to \alpha u_{(m-1)r}$ is a monomorphism, and onto a generating set.

This observation together with the basic Proposition gives Theorem 1 for $B^{[0,n]}$, $n<(p-1)r$. The generators which survive are in fact precisely those already listed in $B^{[0,2r]}$.

4. The first exotic class

The next homotopy group after $\pi_{(p-1)r}$ is not π_{pr} but rather π_{pr-1}. GITLER and STASHEFF [1] have shown the k-invariant is zero, i.e. $B^{[0,pr-1]} \simeq B^{[0,(p-1)r]} \times K(Z_p, pr-1)$. A generator in cohomology corresponding to the fundamental class in $K(Z_p, pr-1)$ has been called e_1. Thus for $B^{[0,pr-1]}$, Theorem 1 is true if it is true for $B^{[0,(p-1)r]}$.

5. The k-invariant h_p

The homotopy group π_{pr} is Z_{p2}. We wish to show that $h_p \in H^{pr+1}(B^{[0,pr-1]}; Z_{p2})$ comes from a class which reduced mod p restricts to $\beta \mathscr{P}^1 u_{p-1}$ in $K(Z_p, (p-1)r)$. Let us go down to $B^{[0,(p-2)r]}$ and look at $\beta \mathscr{P}^1 h_{p-1}$. Since h_{p-1} restricts to $(-\mathscr{P}^1 \beta + 2\beta \mathscr{P}^1) u_{p-2}$, $\beta \mathscr{P}^1 h_{p-1}$ restricts to zero and hence comes from a primitive element in $H^*(B^{[0,(p-3)r]}; Z_p)$. The only primitive elements are generators and their p^i-powers; these never occur in dimensions congruent to 2 mod r. (This remark will continue to be true up to at least dimension $p^2 r$.) Thus $\beta \mathscr{P}^1 h_{p-1} = 0$. Let V_p be a class which restricts to $\beta \mathscr{P}_1 u_{p-1}$. Since $H^{pr+1}(B^{[0,pr-1]}; Z_p) \approx H^{pr+1}(B^{[0,(p-1)r]}; Z_p)$ via the projection map, Milnor's proof (3.10) goes through to show V_p can be chosen so that h_p reduced mod p is the image of V_p.

Because π_{pr} is Z_{p2}, we find it necessary to pay attention to higher order torsion. In particular we are interested in the second order Bockstein $\beta_2 h_p$. β_2 can be thought of as a secondary operation based on $\beta\beta = 0$. Alternatively we make use of the integral Bockstein $\bar{\beta}$ for the sequence $0 \to Z \to Z \to Z_p \to 0$ and the integral secondary Bockstein $\bar{\beta}_2$. That is $\bar{\beta}_2 = 1/p\bar{\beta}$ when such division by p is possible. Of course $\bar{\beta}_2$ is well defined only modulo $\bar{\beta}$. The mod p secondary Bockstein β_2 is just the mod p reduction of $\bar{\beta}_2$ and is well defined modulo β. Since the homotopy groups of B_G are all finite, $H^*(B^{[0,mr]}; Z)$ contains no elements of infinite order. Every element therefore has non-trivial β_i for some i or is in the image of β_i for some i.

The next section will be devoted to computing $\beta_2 h_p$. The reader who wishes to proceed to section 7 should take our word for it that:

THEOREM. In the fibring $K(Z_p, (p-1)r) \to B^{[0,(p-1)r]} \to B^{[0,(p-2)r]}$, we have $i^* \beta_2 V_p = \lambda \beta \mathscr{P}^1 \beta u_{p-1}$, $\lambda \neq 0 \in Z_p$.

6. Computation of $\beta_2 h_p$

The following result is of some use in computing secondary Bocksteins.

PROPOSITION. *Let $K(Z_p, q) \xrightarrow{i} Y \xrightarrow{p} X$ be a fibre space induced by $v \in H^{q+1}(X; Z_p)$. Let $u \in H^n(X; Z_p)$ be a class such that $p^* \beta u = 0$. Let θ be a stable primary cohomology operation such that $\theta v = \beta u \in H^{n+1}(X; Z_p)$. Then $i^* \beta_2 p^* u = \beta \theta \iota_q$ modulo $i^* \beta H^n(Y; Z_p)$.*

Proof. $K(Z_{p^2}, n)$ can be represented as a fibre space $K(Z_p, n) \to K(Z_{p^2}, n) \to K(Z_p, n)$ induced by $\beta\iota_n$. Since $p^*\beta u = 0$, a map representing u can be lifted to give a map of fibrings:

$$\begin{array}{ccccc}
K(Z_p, q) & \xrightarrow{w} & K(Z_p, n) & \searrow^{\beta} & \\
\downarrow & & \downarrow & & \searrow \\
Y & \longrightarrow & K(Z_{p^2}, n) & \xrightarrow{\beta_2} & K(Z_p, n+1) \\
\downarrow & & \downarrow & & \\
X & \xrightarrow{u} & K(Z_p, n) & &
\end{array}$$

The map β_2 is characterized by the commutativity of the triangle. Let w be the induced map of fibres. From the diagram we see $i^*\beta_2 p^* u = \beta w$. Transgression is natural with respect to maps of fibrings, so $\tau w = u^*\tau\iota_n = u^*\beta\iota n = \beta u = \theta v$. On the other hand $\tau\iota_q = v$ so $\tau(\theta\iota_q) = \theta v$. Thus $w - \theta\iota_q$ pulls back to $H^*(Y; Z_p)$. Now $\beta_2 p^* u$ is well defined modulo $i^*\beta H^*(Y; Z_p)$. On the other hand we have shown $\beta w = \beta\theta\iota_q$ modulo $\beta i^* H^*(Y; Z_p)$ so we are done.

This result may well be known to the practitioners of the art of secondary operations. Of course it is true of more general secondary operations. If β_2 is replaced by an operation ψ based on the relation $\alpha\gamma = 0$ in the Steenrod algebra, then we assume $p^*\gamma u = 0$ and $\theta v = \gamma u$ and conclude $i^*\psi p^* u = \alpha\theta\iota_q$ modulo $i^*\alpha H^*(Y; Z_p)$.

We apply the proposition to $K(Z_p, 2r) \to B^{[0,2r]} \to B^{[0,r]}$ with $u = \mathscr{P}^{p-1}\beta q$. We conclude $i^*\beta_2 p^* \mathscr{P}^{p-1}\beta q = \beta\mathscr{P}^{p-2}\beta u_2$.

In lower dimensions, the torsion is easy to handle. MILNOR lists explicit generators for $H^*(B^{[0,mr]})$ and we can see they are paired by β, i.e. as far as generators are concerned: $\ker\beta = \text{image } \beta$. Thus the first higher order torsion is $\beta_2(q^p) \neq 0$. In $B^{[0,2r]}$ we have y which restricts to $\mathscr{P}^{p-2}u_2$ and βy which restricts to $\beta\mathscr{P}^{p-2}u_2$. However in $B^{[0,2r]}$, the restriction of the transgression of $\mathscr{P}^{p-3}u_3$ is $-2\beta\mathscr{P}^{p-2}u_2$ so that in $B^{[0,3r]}$, βy must come from $B^{[0,r]}$.

LEMMA. y can be chosen so that in $B^{[0,3r]}$ βy can be used in place of $\mathscr{P}^{p-1}\beta q$ as a generator.

Proof. In B_G we have the *Wu* formula $\beta q_p = \mathscr{P}^{p-1}\beta q_1 + \text{poly}(q_i, \beta q_i, i<p)$. We know that q_p must be of the form $\lambda y + \mu\beta e_1 +$ poly $(q_i, \beta q_i, i < \text{p})$. By changing our choice of y we can assume $q_p = \lambda y + \mu\beta e_1$. Since $\beta q_p \neq 0$ or since $q_p \neq 0$ in B_U, we know λ can be assumed to be 1. Thus we have

$$\beta y = \mathscr{P}^{p-1}\beta q_1 + \text{poly}(q_i, \beta q_i, i < p). (*)$$

This establishes the lemma.

We study this relation further. Since in $B^{[0,2r]}$, βy restricts to $\beta\mathscr{P}^{p-2}u_2$, this relation (*) does not hold in $B^{[0,2r]}$ but must hold in $B^{[0,3r]}$. Therefore in $B^{[0,3r]}$ we have $\bar{\beta}\mathscr{P}^{p-1}\beta q_1 = -\bar{\beta}$ poly$(q_i, \beta q_i, i<p)$. Since the left hand side is primitive, the right hand side must be also and therefore is zero; so $\bar{\beta}\mathscr{P}^{p-1}\beta q_1$ lifts to zero in $B^{[0,3r]}$. Now

in $B^{[0,2r]}$ we have seen $\beta_2 \mathscr{P}^{p-1} \beta q$ restricts to $\beta \mathscr{P}^{p-2} \beta u_2$. In other words, $\bar{\beta} \mathscr{P}^{p-1} \beta q$ is p times something, call it z, which restricts to $\lambda_2 \bar{\beta} \mathscr{P}^{p-2} \beta u_2$ where λ_2 is not divisible by p. For $\bar{\beta} \mathscr{P}^{p-1} \beta q$ to lift to zero in $B^{[0,3r]}$, it must be killed by the transgression of some multiple of $\bar{\beta} \mathscr{P}^{p-3} u_3$, since there is no other potential assassin. Now $\beta \mathscr{P}^{p-2} \beta u_2$ is killed by $\mathscr{P}^{p-3} \beta u_3$ but there is no integral class in $K(Z_p, 3r)$ which can kill $\bar{\beta} \mathscr{P}^{p-2} \beta u_2$ so z survives to $B^{[0,3r]}$ but its mod p reduction lifts to zero, i.e. z lifts to p times something which must restrict non-trivially to the fibre, which is to say to $\lambda_3 \bar{\beta} \mathscr{P}^{-3} \beta u_3$ for some λ_3 not divisible by p.

LEMMA. $\beta_2 \mathscr{P}^{p-m} h_m$ restricts to $\lambda_{m-1} \beta \mathscr{P}^{p-m+1} \beta u_{m-1}$ for $\lambda \neq 0 \in Z_p$.

Proof. $\mathscr{P}^{p-m} h_m$ restricts to $\beta \mathscr{P}^{p-m+1} u_{m-1}$, but $\bar{\beta} \mathscr{P}^{p-m+1} u_{m-1}$ transgresses to $\bar{\beta} \mathscr{P}^{p-m+1} h_{m-1}$ which by induction is non-trivial. Therefore $\mathscr{P}^{p-m} h_m$ is not the reduction of an integral class and so $\bar{\beta} \mathscr{P}^{p-m} h_m$ must be non-trivial. Since $\beta \mathscr{P}^{p-m} h_m$ is zero, $\bar{\beta} \mathscr{P}^{p-m} h_m$ must be divisible by p. In this dimension, the image of $B^{[0,(m-2)r]}$ is all of order p (p times $\bar{\beta}_2 \mathscr{P}^{p-m+1} h_{m-1}$ has been killed by $\bar{\beta} \mathscr{P}^{p-m+1} u_{m-1}$) so the only classes which can possibly have order greater than p are those which restrict non-trivially to the fibre, i.e. to $\lambda_{m-1} \bar{\beta} \mathscr{P}^{p-m+1} \beta u_{m-1}$. Since the fibre has no more than p-torsion in this dimension, the class in $B^{[0,(m-1)r]}$ has order at most p^2. Therefore $\beta_2 \mathscr{P}^{p-m} h_m$ is non-zero and restricts to $\lambda_{m-1} \beta \mathscr{P}^{p-m+1} \beta u_{m-1}$.

For $m=p$, we have a slight modification due to the fact that h_p is a class of $B^{[0,pr-1]}$, not $B^{[0,(p-1)r]}$. We can still show $\beta_2 V_p$ restricts to $\lambda_{p-1} \beta \mathscr{P}^1 u_{p-1}$. Since some choice of V_p lifts to h_p, $\beta_2 h_p$ is the image of $\beta_2 V_p$.

7. $H^*(B^{[0,n]})$ for $n=(p-1)r$ and $pr-1$

So far we have used Toda's exact sequence for the Steenrod algebra and the fact that the Steenrod algebra approximates $H^*(Z_p, n; Z_p)$. What of $H^*(Z_{p2}, n; Z_p)$? The description is formally the same as for (Z_p, n) except that for every element θ of the Steenrod algebra such that $\theta \beta \iota_n$ is a generator of $H^*(Z_p, n; Z_p)$, the class $\theta \beta_2 \iota_n$ appears as a generator of $H^*(Z_{p2}, n; Z_p)$, which thus stably is additively isomorphic to $A/_{A\beta} + (A/_{A\beta}) \beta_2$. Consider

$$\begin{array}{ccccc} A/A\beta + A/A\beta & \xrightarrow{\beta \mathscr{P}^1 - \beta \mathscr{P}^1 \beta} & A & \xrightarrow{-\mathscr{P}^1 \beta + 2\beta \mathscr{P}^1} & A \\ \downarrow & & \downarrow & & \downarrow \\ H^*(Z_p 2, pr) & \underset{i^* \tau}{\rightarrow} & H^*(Z_p, (p-1) r) & \underset{\tau}{\rightarrow} & H^*(Z_p, (p-2) r) \end{array}$$

where the vertical maps are as before except for the first which is $(\alpha_1, \alpha_2) \rightarrow \alpha_1 \iota_{pr} - 1/\lambda_{p-1}(\alpha_2 \beta_2 \iota_{pr})$. Our computations of h_{p-1}, h_p and $\beta_2 h_p$ give commutativity of the diagram. The top line is exact by TODA. [There is a crucial misprint in Toda but the dual sequence (Proposition 1.1) is correct and gives the above.] We conclude that in the stable range (i.e. dimension less than $p(p-1) r$) the generators in $B^{[0,(p-1)r]}$ which restrict non-trivially to the fibre are the transgressions of generators in $K(Z_{p2}, pr)$. Thus we have verified Theorem 1 for $n=(p-1)\,r$, and the surviving generators remain the same. Since the k-invariant for π_{pr-1} is trivial, we easily check Theorem 1 for $n=pr-1$.

8. $n \geqq pr$

In $B^{[0,(p-1)r]}$ we can compute directly that the restrictions of $\mathscr{P}^1\beta_2 h_p$ and $\lambda_{p-1}\beta\mathscr{P}^1 h_p$ agree. Milnor's remarks again apply to show $(\mathscr{P}^1\beta_2-\lambda_{p-1}\beta\mathscr{P}^1)\,h_p=0$. We set $V_{p+1}=\mathscr{P}^1\beta_2 u_p-\lambda_{p-1}\beta\mathscr{P}^1 u_p$ and would like to show that h_{p+1} restricts to a non-zero multiple of V_{p+1}.

Unfortunately the next homotopy group is not $\pi_{(p+1)r}$ but rather $\pi_{(p+1)r-2}$. If β is a generator of π_{pr-1} and α_1 is a generator of the stable group $\pi_{n+r-1}(S^n)$, then $\pi_{(p+1)r-2}(B_G)$ is generated by $\beta\alpha_1$ [4, II: Theorem 4.15]. Since α_1 is detected by $\mathscr{P}^1$, the k-invariant for $\pi_{(p+1)r-2}$ must restrict to $\mathscr{P}^1 e_1$ in $K(Z_p, pr-1)$. [A direct cohomological argument can be derived from Toda.] Since $H^*(B^{[0,pr]})$ has no cohomology in dimensions congruent to -1 mod r except for classes involving e_1, $\mathscr{P}^1 e_1$ and qe_1 are the only classes in dimension $(p+1)\,r-1$. Since the k-invariant is primitive and e_1 is, the k-invariant must be $\mathscr{P}^1 e_1$.

Modulo classes which restrict non-trivially to $K(Z_{p^2}, pr)$ or $K(Z_p, (p+1)\,r-2)$, $H^*(B^{[0,(p+1)r-2})$ is freely generated by the images of generators of $H^*(B^{[0,(p-1)r]})$ and classes corresponding to an additive basis of $(A/A\mathscr{P}^1)\,e_1$.

Now let us look at h_{p+1}, the k-invariant for $\pi_{(p+1)r}$.

LEMMA. h_{p+1} can be used as a generator of $H^*(B^{[0,(p+1)r-2]})$ in place of a class which restricts to V_{p+1}.

Proof. We apply Milnor's argument (3.10) after first noting that $H^{(p+1)r+1}(K(Z_p,(p+1)\,r-2)=0$ so that h_{p+1} is in the image of $B^{[0,pr]}$. If h_{p+1} were not the image of $\bar{V}_{p+1}$ for some $\bar{V}_{p+1}$ which restricts to $(\mathscr{P}^1\beta_2-\lambda_{p-1}\beta\mathscr{P}^1)\,u_p$, it would be the image of a primitive class in $B^{[0,pr-1]}$. The only such classes are $\beta\mathscr{P}^1\beta e_1$ and possibly polynomials in $\beta\mathscr{P}^1 y$ and the other Wu generators. Since h_{p+1} is in dimension $(p+1)\,r+1$, every term of the polynomial must contain a bockstein. In $B_U^{(0,(p+1)r-1)}$, Bocksteins are zero in dimensions $<(p+1)\,r$ since they are zero in B_U and in dimension $(p+1)\,r$ as observed by MILNOR. Milnor's argument (Lemma 3) again shows h_{p+1} does not go to zero in $B_U^{[0,(p+1)r-1]}$ so the lemma follows.

To compute $H^*(B^{[0,pr]})$, we turn to Toda's exact sequence

$$A\xrightarrow{\beta\mathscr{P}^1,\,\mathscr{P}^1}A/A\beta+A/A\beta\xrightarrow{\beta\mathscr{P}^1-\beta\mathscr{P}^1\beta}A$$

and conclude in the stable range that the generators in $B^{[0,pr]}$ which restrict non-trivially to $K(Z_p\,,pr)$ are the transgressions of generators of $K(Z_p, (p+1)\,r)$. From here on the argument is very similar to Milnor's for $B^{[0,mr]}$, $2\leqq m<p$, simplified by

being in the stable range and by using Toda's sequence. In general for $2 \leqq m < p$, h_{p+m} restricts to some non-zero multiple of

$$(m\mathscr{P}^1\beta - (m-1)\beta\mathscr{P}^1)\, u_{p+m-1} \quad \text{in} \quad K(Z_p, (p+m-1)\, r).$$

Finally we consider the exotic group π_{2pr-3} generated by $\beta\beta_1$ where $\beta_1 \in \pi_{pr-2+n}(S^n)$ corresponds to β. β_1 is detected by a secondary operation φ based on the relation $\mathscr{P}^{p-1}\mathscr{P}^1 = 0$. Since $\mathscr{P}^1 e_1$ has been killed by $\pi_{(p+1)r-2}$, φe_1 is defined and is the only primitive class in this dimension (congruent to $-1 \bmod r$). It is therefore the k-invariant.

From Toda's exact sequence [4; I: Prop. 1.6]

$$A \xrightarrow{\mathscr{P}^1} A \xrightarrow{\mathscr{P}^{p-1}} A \xrightarrow{\mathscr{P}^1} A$$

we can conclude that in dimensions less than $2pr$, $H^*(B^{[0,\, 2pr-3]})$ is freely generated by

$$\begin{array}{llllll} \mathscr{P}^i q & i < p-1 & \mathscr{P}^i\mathscr{P}^1 q & p \leqq i < 2(p-1) & y, \mathscr{P}^1 y, \beta\mathscr{P}^1 y \\ \mathscr{P}^i \beta q & i \leqq p-1 & \mathscr{P}^i\mathscr{P}^1\beta q & p \leqq i \leqq 2(p-1) & \end{array}$$

and $\{\theta e_1\}$ where θ runs over an additive basis of $A/A\mathscr{P}^1$. Since the next homotopy group is π_{2pr}, the same statement holds for $H^*(B_G)$ in dimensions $< 2pr$.

9. Wu classes

THEOREM 2. In dimensions $< 2pr$, $H^*(B_G)$ is freely generated by the Wu classes q_i, their Bocksteins βq_i and $\{\theta e_1\}$ where θ runs over an additive basis of $A/A\mathscr{P}^1$.

Proof. As MILNOR has observed, q_{j+1} can replace $\mathscr{P}^j q$ as a generator for $j < p-1$. By the same reasoning, q_{i+j+1} can replace $\mathscr{P}^i\mathscr{P}^j q$ for $j < p-1$ since $\mathscr{P}^i q_{j+1} = (-1)^i \binom{(j+1)(p-1)-1}{i} q_{i+j+1}$ + polynomial in lower q_k and the binomial coefficient is non-zero for $pj \leqq i < (p-1)(j+1)$, $j < p-1$. This gives us q_k for $1 \leqq k \leqq p-1$ and $p+2 \leqq k \leqq 2p-1$. The Wu class q_p is independent of q_i for $i < p$ in B_U and hence must be of the form $\lambda y + \text{poly}(q_i, \beta q_i, \beta e_1)$ with $\lambda \neq 0$, so q_p can replace y. Again $\mathscr{P}^1 q_p = q_{p+1}$ modulo lower q_k so q_{p+1} can be used in place of $\mathscr{P}^1 y$. Similar computations show that $\mathscr{P}^i\mathscr{P}^j\beta q_1$ can be replaced by βq_{i+j+1} so as to give us βq_k for $1 \leqq k \leqq p$ and $p+2 \leqq k \leqq 2p$. The remaining βq_{p+1} appears to replace $\beta\mathscr{P}^1 y$.

10. The case $p=3$

Certain modifications are necessary when $p=3$. The computation of $H^*(B^{[0,\, 2r]}; Z_p)$ is altered because $\beta V_3 = 0$ and $\beta_2 V_3$ must be used instead, but the general remarks about h_p still apply and the classes y, $\mathscr{P}^1 y$, $\beta\mathscr{P}^1 y$ appear just as for $p > 3$. The results are isomorphic.

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