Operational Calculus for Two Commuting Closed Operators.

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1. Introduction

Functions of an endomorphism T of a complex Banach space $\mathfrak{X}$ can be defined through a proper homomorphic mapping Φ of an algebra H into the Banach algebra $B(\mathfrak{X})$ of all endomorphism of $\mathfrak{X}$. The elements of H are locally holomorphic complex functions $f(\lambda)$ defined on a neighborhood of the spectrum $\sigma(T)$ of T. If Φ is continuous and the image of λ under Φ is T, then Φ is unique and is represented by a Cauchy integral: $\Phi(f)=(2\pi i)^{-1}\int_\Gamma f(\lambda)\,(\lambda-T)^{-1}d\lambda$, where Γ is a suitably chosen curve in the complex plane. $\Phi(f)$ is a function of T, and the construction of functions of T by this procedure is known as operational calculus for bounded linear operators. The Cauchy formula has been used by DUNFORD [2, 3, 4] and by TAYLOR [10] to define functions of an endomorphism of $\mathfrak{X}$. Such a function could be defined also by a power series expansion in T, valid in a neighborhood of $\sigma(T)$, and the result is equivalent to the result obtainable with Cauchy's formula. However, if T is unbounded and closed, a generalization is possible only by use of the integral formula (TAYLOR [11], HILLE and PHILLIPS [7]). Another extension of the theory is the generalization to functions of more than one variable, which goes straightforward only in the case of commuting endomorphisms of $\mathfrak{X}$. Such functions have been introduced by SCHWARTZ [8] and WAELBROEK [13], where uniqueness is proven in the last cited work.

The theory presented in this note deals with functions of two (not necessarily bounded) commuting closed linear operators T_1 and T_2 on $\mathfrak{X}$ to itself. Again, the functions are defined through a proper homomorphism Φ of an algebra H into $B(\mathfrak{X})$. Here H consists of locally holomorphic complex functions $f(\lambda_1, \lambda_2)$ whose domain V is a neighborhood of $\sigma_e(T_1)\times\sigma_e(T_2)$, $\sigma_e(T_j)$ being the extended spectrum of T_j. This implies that V is unbounded, if T_1 or T_2 is unbounded. In principle, the main theorem describes the following facts: Provided that Φ transforms $(\lambda-\lambda_j)^{-1}$ into $(\lambda-T_j)^{-1}$ for $j=1, 2$ and for each λ in the resolvent set of T_j, Φ is unique and has the representation given by formula (1).

The operational calculus for closed linear operators plays an important role in the theory of Banach algebras [12, 14], in the spectral theory [4], the theory of semi-groups [7] and in quantum theory [1]. As an application we shall give a representation for polynomials in two endomorphisms of $\mathfrak{X}$. Furthermore, we derive formulas for the resolvents of the sum and the product of commuting closed linear transformations and show that the extended spectrum of the sum (product) is contained in the sum (product) of the extended spectra of the single transformations.

2. The operational calculus

Let V_1 and V_2 be Cauchy domains, open subsets of the extended complex plane $\mathbf{C}$ with a finite positive number of components whose boundaries are nonempty and are composed of a finite positive number of simple closed rectifiable curves. Let $V = V_1 \times V_2$ be the Cartesian product of V_1 and V_2 in $\mathbf{C}^2$. We denote by χ_X the characteristic function of a set X. Let $H(V)$ then be the complex algebra of all functions $f: \mathbf{C}^2 \to \mathbf{C}$ which are locally holomorphic in V and vanish in the complement of V with respect to $\mathbf{C}^2$. In $H(V)$ we use the ordinary definition of arithmetic operations. χ_V is the unit element of $H(V)$.

Now let U be a Cauchy domain and X a closed subset of U. Then there exists a Cauchy domain U', such that $X \subset U'$ and $\overline{U'} \subset U$ [11, Theorem 3.3]. The union Γ of all boundaries of U', with the usual positive orientation, we call an oriented envelope of X with respect to U.

Next, let $\mathfrak{X}$ be a complex Banach space, $\mathfrak{O}(\mathfrak{X})$ the class of all closed linear transformations with domain and range in $\mathfrak{X}$ and $B(\mathfrak{X})$ the complex Banach algebra of all endomorphisms of $\mathfrak{X}$ with identity I. Let T be an element of $\mathfrak{O}(\mathfrak{X})$ with domain $\mathfrak{D}(T)$, spectrum $\sigma(T)$ and resolvent set $\sigma(T)$. The extended spectrum $\sigma_e(T)$ is the set of all singular points of the resolvent $R(\lambda, T)$ of T in $\mathbf{C}$. By $\mathfrak{C}(\mathfrak{X})$ we denote the class of all pairs $\{T_1, T_2\}$ of operators of $\mathfrak{O}(\mathfrak{X})$ which satisfy $\mathfrak{D}(T_1) \subset \mathfrak{D}(T_2)$, $T_2[\mathfrak{D}(T_1)] \subset \mathfrak{D}(T_1)$ and commute on the set $\{x | x \in \mathfrak{D}(T_1), T_1 x \in \mathfrak{D}(T_2)\}$. If T_2 is bounded on $\mathfrak{X}$, this definition is equivalent to that of M. H. Stone [9, Definition 3]. Then we have

Lemma 1. *Let $\mathfrak{D}(T_1) \subset \mathfrak{D}(T_2)$ and $T_2[\mathfrak{D}(T_1)] \subset \mathfrak{D}(T_1)$. A necessary and sufficient condition that $R(\lambda_1, T_1)\, R(\lambda_2, T_2) = R(\lambda_2, T_2)\, R(\lambda_1, T_1)$ for each pair $\{\lambda_1, \lambda_2\} \in \varrho(T_1) \times \varrho(T_2)$ is that $\{T_1, T_2\} \in \mathfrak{C}(\mathfrak{X})$.*

Proof. If $\{T_1, T_2\} \in \mathfrak{C}(\mathfrak{X})$ we have $\{x | x \in \mathfrak{D}(T_1), T_1 x \in \mathfrak{D}(T_2)\} = \{x | x \in \mathfrak{D}(T_1), (\lambda_1 I - T_1) x \in \mathfrak{D}(T_2)\} = R(\lambda_1, T_1)[\mathfrak{D}(T_2)] = R(\lambda_1, T_1)\, R(\lambda_2, T_2)(\mathfrak{X})$, since the $R(\lambda_j, T_j)$ are one-to-one transformations of $\mathfrak{X}$ onto $\mathfrak{D}(T_j)$. For each $x \in \mathfrak{X}$

$$\begin{aligned} R(\lambda_2, T_2)\, R(\lambda_1, T_1)\, [(\lambda_1 I - T_1)(\lambda_2 I - T_2) - (\lambda_2 I - T_2)(\lambda_1 I - T_1)] \\ \times R(\lambda_1, T_1)\, R(\lambda_2, T_2)\, x \\ = [R(\lambda_1, T_1)\, R(\lambda_2, T_2) - R(\lambda_2, T_2)\, R(\lambda_1, T_1)]\, x\,. \end{aligned}$$

Hence a sufficient condition for the right-hand side to vanish is that T_1 and T_2 commute, i.e. $\{T_1, T_2\} \in \mathfrak{C}(\mathfrak{X})$. On the other hand, if the right-hand side is zero, again since the resolvents are one-to-one, we have $(T_1 T_2 - T_2 T_1)\, R(\lambda_1, T_1)\, R(\lambda_2, T_2) x = 0$. Thus $\{T_1, T_2\} \in \mathfrak{C}(\mathfrak{X})$ and this proves the necessary condition, q.e.d.

Finally, we denote by $\mathfrak{G}(\mathfrak{X}, V)$ the class of all elements of $\mathfrak{C}(\mathfrak{X})$ for which $\sigma_e(T_1) \times \sigma_e(T_2) \subset V$. Based on the preceding definitions we have

THEOREM 2. *For each pair* $\{T_1, T_2\} \in \mathfrak{G}(\mathfrak{X}, V)$ *there exists a proper homomorphism* Φ *of* $H(V)$ *into* $B(\mathfrak{X})$ *such that*

(i) $\Phi[(\lambda - \lambda_j)^{-1} \chi_V(\lambda_1, \lambda_2)] = R(\lambda, T_j), \quad \lambda \notin V_j, \quad j = 1, 2.$

(ii) *If a sequence* f_n *in* $H(V)$ *converges pointwise to* $f \in H(V)$, *the convergence being uniform on each compact subset of* V, *then this implies* $\lim_n \|\Phi(f_n) - \Phi(f)\| = 0$.

(iii) Φ *is unique and is defined by*

$$\begin{aligned}\Phi(f) = f(\infty, \infty)\, I &+ \frac{1}{2\pi i} \int_{\Gamma_1} f(\lambda_1, \infty)\, R(\lambda_1, T_1)\, d\lambda_1 \\ &+ \frac{1}{2\pi i} \int_{\Gamma_2} f(\infty, \lambda_2)\, R(\lambda_2, T_2)\, d\lambda_2 \\ &+ \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} f(\lambda_1, \lambda_2)\, R(\lambda_1, T_1)\, R(\lambda_2, T_2)\, d\lambda_1\, d\lambda_2,\end{aligned} \tag{1}$$

where Γ_j, $j = 1, 2$ *is an oriented envelope of* $\sigma_e(T_j)$ *with respect to* V_j, *containing* $\lambda = \infty$ *in its interior if* V_j *is unbounded.*

Proof. Since the integrands of (1) are locally holomorphic in $V \cap [\varrho(T_1) \times \varrho(T_2)]$ the integrals exist and are independent of the choice of Γ_1 and Γ_2. Clearly Φ, defined by (1), is linear. In order to demonstrate that it is a homomorphism we take $f, g \in H(V)$ and for $j = 1,2$ two oriented envelopes Γ_j and Γ_j' of $\sigma_e(T_j)$ with respect to V_j such that the open set bounded by Γ_j contains Γ_j'. Then by (1) with $c = (2\pi i)^{-1}$,

$$\Phi(f)\, \Phi(g) - \Phi(f\, g) = \sum_{k=1}^{8} A_k, \tag{2}$$

where

$$A_1 = c \int_{\Gamma_1} [f(\lambda_1, \infty)\, g(\infty, \infty) + f(\infty, \infty)\, g(\lambda_1, \infty) - (f\, g)(\lambda_1, \infty)]\, R(\lambda_1, T_1)\, d\lambda_1$$

$$A_2 = c \int_{\Gamma_2} [f(\infty, \lambda_2)\, g(\infty, \infty) + f(\infty, \infty)\, g(\infty, \lambda_2) - (f\, g)(\infty, \lambda_2)]\, R(\lambda_2, T_2)\, d\lambda_2$$

$$A_3 = c^2 \int_{\Gamma_1} \int_{\Gamma_1'} f(\lambda_1, \infty)\, g(\mu_1, \infty)\, R(\lambda_1, T_1)\, R(\mu_1, T_1)\, d\lambda_1\, d\mu_1$$

$$A_4 = c^2 \int_{\Gamma_2} \int_{\Gamma_2'} f(\infty, \lambda_2)\, g(\infty, \mu_2)\, R(\lambda_2, T_2)\, R(\mu_2, T_2)\, d\lambda_2\, d\mu_2$$

$$\begin{aligned}A_5 = c^2 \int_{\Gamma_1} \int_{\Gamma_2} [&f(\infty, \lambda_2)\, g(\lambda_1, \infty) + f(\lambda_1, \infty)\, g(\infty, \lambda_2) + f(\infty, \infty)\, g(\lambda_1, \lambda_2) \\ &+ f(\lambda_1, \lambda_2)\, g(\infty, \infty) - (f\, g)(\lambda_1, \lambda_2)]\, R(\lambda_1, T_1)\, R(\lambda_2, T_2)\, d\lambda_1\, d\lambda_2\end{aligned}$$

$$A_6 = c^3 \int_{\Gamma_1} \int_{\Gamma_1'} \int_{\Gamma_2} [f(\mu_1, \infty) g(\lambda_1, \lambda_2) + f(\lambda_1, \lambda_2) g(\mu_1, \infty)]$$
$$\times R(\lambda_1, T_1) R(\mu_1, T_1) R(\lambda_2, T_2) \, d\lambda_1 \, d\mu_1 \, d\lambda_2$$

$$A_7 = c^3 \int_{\Gamma_1} \int_{\Gamma_2} \int_{\Gamma_2'} [f(\infty, \mu_2) g(\lambda_1, \lambda_2) + f(\lambda_1, \lambda_2) g(\infty, \mu_2)]$$
$$\times R(\lambda_1, T_1) R(\lambda_2, T_2) R(\mu_2, T_2) \, d\lambda_1 \, d\lambda_2 \, d\mu_2$$

$$A_8 = c^4 \int_{\Gamma_1} \int_{\Gamma_2} \int_{\Gamma_1'} \int_{\Gamma_2'} f(\lambda_1, \lambda_2) g(\mu_1, \mu_2) R(\lambda_1, T_1) R(\lambda_2, T_2) R(\mu_1, T_1)$$
$$\times R(\mu_2, T_2) \, d\lambda_1 \, d\lambda_2 \, d\mu_1 \, d\mu_2 .$$

In some terms we have changed the order of integration and used the commutativity of the resolvents (Lemma 1). By [11, p. 196] we have for $j = 1, 2$ and any $h \in H(V)$ as a result of the method of residues

$$c \int_{\Gamma_j} \frac{h(\mu_j, \cdot)}{\mu_j - \lambda_j} \, d\mu_j = h(\lambda_j, \cdot) - h(\infty, \cdot), \qquad \lambda_j \in \Gamma_j'$$

and

$$c \int_{\Gamma_j'} \frac{h(\mu_j, \cdot)}{\mu_j - \lambda_j} \, d\mu_j = - h(\infty, \cdot), \qquad \lambda_j \in \Gamma_j .$$

Using this and the resolvent equation $(\mu_1 - \lambda_1) R(\lambda_1, T_1) R(\mu_1, T_1) = R(\lambda_1, T_1) - R(\mu_1, T_1)$ we get

$$A_3 = c^2 \int_{\Gamma_1} R(\lambda_1, T_1) f(\lambda_1, \infty) \, d\lambda_1 \int_{\Gamma_1'} \frac{g(\mu_1, \infty)}{\mu_1 - \lambda_1} \, d\mu_1$$
$$+ c^2 \int_{\Gamma_1'} R(\mu_1, T_1) g(\mu_1, \infty) \, d\mu_1 \int_{\Gamma_1} \frac{f(\lambda_1, \infty)}{\lambda_1 - \mu_1} \, d\lambda_1$$
$$= - A_1$$

and similarly $A_4 = -A_2$. In the same manner we obtain

$$A_8 = c^3 \int_{\Gamma_1} \int_{\Gamma_1'} \int_{\Gamma_2} [f(\lambda_1, \lambda_2) g(\mu_1, \lambda_2) - f(\lambda_1, \infty) g(\mu_1, \lambda_2) - f(\lambda_1, \lambda_2) g(\mu_1, \infty)]$$
$$\times R(\lambda_1, T_1) R(\mu_1, T_1) R(\lambda_2, T_2) \, d\lambda_1 \, d\mu_1 \, d\lambda_2 ,$$

hence

$$A_6 + A_8 = c^3 \int_{\Gamma_1} \int_{\Gamma_1'} \int_{\Gamma_2} f(\lambda_1, \lambda_2)\, g(\mu_1, \lambda_2)\, R(\lambda_1, T_1)\, R(\mu_1, T_1)\, R(\lambda_2, T_2)\, d\lambda_1\, d\mu_1\, d\lambda_2$$

$$= c^2 \int_{\Gamma_1} \int_{\Gamma_2} [(f\,g)(\lambda_1, \lambda_2) - f(\infty, \lambda_2)\, g(\lambda_1, \lambda_2) - f(\lambda_1, \lambda_2)\, g(\infty, \lambda_2)]$$

$$\times R(\lambda_1, T_1)\, R(\lambda_2, T_2)\, d\lambda_1\, d\lambda_2 .$$

Since analogously

$$A_7 = c^2 \int_{\Gamma_1} \int_{\Gamma_2} [-f(\infty, \lambda_2)\, g(\lambda_1, \infty) - f(\lambda_1, \infty)\, g(\infty, \lambda_2) - f(\infty, \infty)$$

$$\times g(\lambda_1, \lambda_2) - f(\lambda_1, \lambda_2)\, g(\infty, \infty)$$

$$+ f(\infty, \lambda_2)\, g(\lambda_1, \lambda_2) + f(\lambda_1, \lambda_2)\, g(\infty, \lambda_2)]\, R(\lambda_1, T_1)\, R(\lambda_2, T_2)\, d\lambda_1\, d\lambda_2 ,$$

we have $A_7 + A_5 = -A_6 - A_8$, so that the right-hand side of (2) vanishes and $\Phi(fg) = \Phi(f)\, \Phi(g)$.

Next, to prove (i) we write down two well-known relations from the operational calculus in one variable [7, Theorem 5.11.2]

$$c \int_{\Gamma_j} R(\lambda_j, T_j)\, d\lambda_j = [1 - \chi_{V_j}(\infty)]\, I \tag{3}$$

and [11, Theorem 7.4] (after deforming the path of integration)

$$c \int_{\Gamma_j} (\lambda - \lambda_j)^{-1}\, R(\lambda_j, T_j)\, d\lambda_j = R(\lambda, T_j), \qquad \lambda \notin V_j . \tag{4}$$

Using equations (1), (3) and (4) we get

$$\Phi[(\lambda - \lambda_1)^{-1}\, \chi_V(\lambda_1, \lambda_2)] = \chi_{V_2(\infty)}\, R(\lambda, T_1) + [1 - \chi_{V_2(\infty)}]\, R(\lambda, T_1) = R(\lambda, T_1)$$

and

$$\Phi[(\lambda - \lambda_2)^{-1}\, \chi_V(\lambda_1, \lambda_2)] = \chi_{V_1(\infty)}\, R(\lambda, T_2) + [1 - \chi_{V_1(\infty)}]\, R(\lambda, T_2) = R(\lambda, T_2).$$

From (1) and (3) it is clear that the unit element in the subalgebra $\Phi[H(V)]$ of $B(\mathfrak{X})$ is

$$\Phi(\chi_V) = \{\chi_V^{(\infty, \infty)} + \chi_{V_2(\infty)}[1 - \chi_{V_1(\infty)}] + \chi_{V_1(\infty)}[1 - \chi_{V_2(\infty)}]$$

$$+ [1 - \chi_{V_1(\infty)}]\, [1 - \chi_{V_2(\infty)}]\}\, I$$

$$= I,$$

therefore Φ is a *proper* homomorphism.

Now we take a pair $\{\mu_1, \mu_2\}$ of finite points of $\mathbf{C}^2 - V$. For $g \in H(V)$ we have by (1)

$$\Phi(g) = c^2 \int_{\Gamma_1} \int_{\Gamma_2} g(\lambda_1, \lambda_2) [(\lambda_1 - \mu_1)^{-1} (\lambda_2 - \mu_2)^{-1} I - (\lambda_1 - \mu_1)^{-1} R(\lambda_2, T_2) - - (\lambda_2 - \mu_2)^{-1} R(\lambda_1, T_1) + R(\lambda_1, T_1) R(\lambda_2, T_2)] \, d\lambda_1 \, d\lambda_2 .$$

Let $f_n \in H(V)$ converge uniformly to $f \in H(V)$ on $\Gamma_1 \times \Gamma_2$. Then $\lim_n \|\Phi(f_n) - \Phi(f)\| = 0$, since $\Gamma_1 \times \Gamma_2$ is compact and the expression in the square brackets is bounded on $\Gamma_1 \times \Gamma_2$. This proves the continuity property (ii).

In order to prove the uniqueness of (1) we take a homomorphism of $H(V)$ into $B(\mathfrak{X})$ which satisfies properties (i) and (ii). Then we show that Φ has the desired representation (1). Due to (ii) we have in the uniform operator topology $\Phi(f) = \lim_n \Phi(f_n)$, if there exists a sequence f_n in $H(V)$ converging pointwise and uniformly to $f \in H(V)$ on each compact subset X of V. But by Cauchy's formula we have on $X = X_1 \times X_2$:

$$f(\lambda_1, \lambda_2) = f(\infty, \infty) + c \int_{\Gamma_1'} \frac{f(\mu_1, \infty)}{\mu_1 - \lambda_1} d\mu_1 + c \int_{\Gamma_2'} \frac{f(\infty, \mu_2)}{\mu_2 - \lambda_2} d\mu_2 + c^2 \int_{\Gamma_1'} \int_{\Gamma_2'} \frac{f(\mu_1, \mu_2)}{(\mu_1 - \lambda_1)(\mu_2 - \lambda_2)} d\mu_1 \, d\mu_2 ,$$

where Γ_j' are oriented envelopes of $X_j \cup \sigma_e(T_j)$ with respect to V_j containing $\lambda = \infty$ in its interior if V_j is unbounded. By V_j' we denote the Cauchy domain enclosed by Γ_j'. Since $X_j \cup \sigma_e(T_j)$ is closed and the boundary Γ_j of V_j is compact, the distance δ_j between $X_j \cup \sigma_e(T_j)$ and Γ_j is positive. Γ_j' may be chosen such that the greatest distance between Γ_j and the points of Γ_j' is less than $\frac{1}{2}\min\{\delta_j, 1/M_j\}$, where $M_j = \sup\{\|R(\lambda_j, T_j)\| \mid \lambda_j \in \bar{V}_j - V_j\}$. Hence there exist suitably chosen sequences of points $\mu_{j1}, \ldots, \mu_{jn} = \mu_{jo}$ on Γ_j' and $\nu_{j1}, \ldots, \nu_{jn} = \nu_{jo}$ on Γ_j, such that $|\mu_{jk} - \mu_{j,k-1}| \to 0$ uniformly with respect to k for $n \to \infty$ and $|\nu_{jk} - \mu_{jk}| < \frac{1}{2}\min\{\delta_j, 1/M_j\}$ for all k. To simplify notation the dependence on n of the points μ_{jk} and ν_{jk} is not indicated. Since the integrals are limits of Riemann sums we have for each $\varepsilon > 0$ an m such that on X for $n \geqslant m$

$$\Bigg| f(\lambda_1, \lambda_2) - f(\infty, \infty) - c \sum_{k=1}^{n} \frac{\mu_{1k} - \mu_{1,k-1}}{\mu_{1k} - \lambda_1} f(\mu_{1k}, \infty) - c \sum_{l=1}^{n} \frac{\mu_{2l} - \mu_{2,l-1}}{\mu_{2l} - \lambda_2} f(\infty, \mu_{2l}) - c^2 \sum_{k,l=1}^{n} \frac{(\mu_{1k} - \mu_{1,k-1})(\mu_{2l} - \mu_{2,l-1})}{(\mu_{1k} - \lambda_1)(\mu_{2l} - \lambda_2)} f(\mu_{1k}, \mu_{2l}) \Bigg| < \varepsilon .$$

Unfortunately, this approximation for f does not belong to $H(V)$, so that we have to move the singularities out of V. But $|(\nu_{jk}-\mu_{jk})/(\nu_{jk}-\lambda_j)| < \frac{1}{2}$ on X_j for each k. Hence there is an m_j such that on X_j for $n_j \geq m_j$

$$\left| (\mu_{jk} - \lambda_j)^{-1} - \sum_{i=0}^{n_j} \frac{(\nu_{jk} - \mu_{jk})^i}{(\nu_{jk} - \lambda_j)^{i+1}} \right| < \frac{2\pi\varepsilon}{L_j N},$$

where L_j is the length of Γ'_j and $N = \sup\{|f(\lambda_1, \lambda_2)|\ \lambda_1 \in \Gamma'_j, \lambda_2 \in \Gamma'_2\}$. Thus

$$|f(\lambda_1, \lambda_2) - f_n(\lambda_1, \lambda_2)| < 3\varepsilon + \varepsilon^2/N$$

on X, where $n \geq \max\{m, m_1, m_2\}$ and

$$f_n(\lambda_1, \lambda_2) = \chi_V(\lambda_1, \lambda_2) \Bigg\{ f(\infty, \infty) + c \sum_{k=1}^{n} (\mu_{1k} - \mu_{1,k-1}) f(\mu_{1k}, \infty)$$

$$\times \sum_{i=0}^{n} \frac{(\nu_{1k} - \mu_{1k})^i}{(\nu_{1k} - \lambda_1)^{i+1}} + c \sum_{l=1}^{n} (\mu_{2l} - \mu_{2,l-1}) f(\infty, \mu_{2l}) \sum_{i=0}^{n} \frac{(\nu_{2l} - \mu_{2l})^i}{(\nu_{2l} - \lambda_2)^{i+1}}$$

$$+ c^2 \sum_{k,l=1}^{n} (\mu_{1k} - \mu_{1,k-1})(\mu_{2l} - \mu_{2,l-1}) f(\mu_{1k}, \mu_{2l})$$

$$\times \sum_{i,j=0}^{n} \frac{(\nu_{1k} - \mu_{1k})^i (\nu_{2l} - \mu_{2l})^j}{(\nu_{1k} - \lambda_1)^{i+1} (\nu_{2l} - \lambda_2)^{j+1}} \Bigg\}.$$

Clearly f_n is a sequence with the required property and by (i) we have in the uniform operator topology

$$\Phi(f) = f(\infty, \infty) I + \lim_n \Bigg\{ c \sum_{k=1}^{n} (\mu_{1k} - \mu_{1,k-1}) f(\mu_{1k}, \infty)$$

$$\times \sum_{i=0}^{n} (\nu_{1k} - \mu_{1k})^i R(\nu_{1k}, T_1)^{i+1} + c \sum_{l=1}^{n} (\mu_{2l} - \mu_{2,l-1}) f(\infty, \mu_{2l})$$

$$\times \sum_{i=0}^{n} (\nu_{2l} - \mu_{2l})^i R(\nu_{2l}, T_2)^{i+1} + c^2 \sum_{k,l=1}^{n} (\mu_{1k} - \mu_{1,k-1})(\mu_{2l} - \mu_{2,l-1})$$

$$\times f(\mu_{1k}, \mu_{2,l}) \sum_{i,j=0}^{n} (\nu_{1k} - \mu_{1k})^i (\nu_{2l} - \mu_{2l})^j \times R(\nu_{1k}, T_1)^{i+1} R(\mu_{2l}, T_2)^{j+1} \Bigg\}.$$

From the resolvent equation we easily get

$$\sum_{i=0}^{n} (\nu_{jk} - \mu_{jk})^i R(\nu_{jk}, T_j)^{i+1} = R(\mu_{jk}, T_j) \{I - [(\nu_{jk} - \mu_{jk}) R(\nu_{jk}, T_j)]^{n+1}\}$$

so that, since $\|(\nu_{jk}-\mu_{jk}) R(\nu_{jk}, T_j)\| < \frac{1}{2}$,

$$\Phi(f) = f(\infty, \infty) I + c \int_{\Gamma_1'} f(\lambda_1, \infty) R(\lambda_1, T_1) d\lambda_1 + c \int_{\Gamma_2'} f(\infty, \lambda_2) R(\lambda_2, T_2) d\lambda_2$$

$$+ c^2 \int_{\Gamma_1'} \int_{\Gamma_2'} f(\lambda_1, \lambda_2) R(\lambda_1, T_1) R(\lambda_2, T_2) d\lambda_1 d\lambda_2$$

and the proof of Theorem 2 is complete.

3. Polynomials, resolvents and spectra of sums and products of operators

In view of an application to polynomials in bounded linear operators we have the following

COROLLARY 3. *Let V_j be bounded and let $\{T_1, T_2\} \in \mathfrak{G}(\mathfrak{X}, V)$. Then for $j = 1, 2$ we have* $\Phi[\lambda_j \chi_V(\lambda_1, \lambda_2)] = T_j$.

Proof. Since T_j is bounded it immediately follows from (1) and (3) that

$$\Phi[\lambda_j \chi_V(\lambda_1, \lambda_2)] = \frac{1}{2\pi i} \int_{\Gamma_j} \lambda_j R(\lambda_j, T_j) d\lambda_j$$

$$= \frac{1}{2\pi i} \int_{\Gamma_j} [I + T_j R(\lambda_j, T_j)] d\lambda_j = T_j. \quad \text{q.e.d.}$$

For the sum of T_1 and T_2 we obtain

THEOREM 4. *Let $\{T_1, T_2\} \in \mathfrak{G}(\mathfrak{X}, V)$. Then for each $\lambda \notin V_1 + V_2$ we have*

$$\Phi[(\lambda - \lambda_1 - \lambda_2)^{-1} \chi_V(\lambda_1, \lambda_2)] = R(\lambda, T_1 + T_2).$$

Proof. Clearly $(\lambda - \lambda_1 - \lambda_2)^{-1} \chi_V(\lambda_1, \lambda_2) \in H(V)$ if there is a $\lambda \notin V_1 + V_2$. Applying the method of residues we get by (1)

$$\Phi[(\lambda - \lambda_1 - \lambda_2)^{-1} \chi_V(\lambda_1, \lambda_2)] = \frac{1}{2\pi i} \int_{\Gamma_1} R(\lambda_1, T_1) R(\lambda - \lambda_1, T_2) d\lambda_1. \qquad (5)$$

If V_1 is unbounded, then, according to our assumptions, V_2 must be bounded and vice versa. On $\mathfrak{X}$ we then have

$$(\lambda I - T_1 - T_2) \int\limits_{\Gamma_1} R(\lambda_1, T_1)\, R(\lambda - \lambda_1, T_2)\, x\, d\lambda_1$$

$$= [\lambda_1 I - T_1 + (\lambda - \lambda_1) I - T_2] \int\limits_{\Gamma_1} R(\lambda_1, T_1)\, R(\lambda - \lambda_1, T_2)\, x\, d\lambda_1$$

$$= \int\limits_{\Gamma_1} R(\lambda - \lambda_1, T_2)\, x\, d\lambda_1 + \int\limits_{\Gamma_1} R(\lambda_1, T_1)\, x\, d\lambda_1$$

$$= -\chi_{V_1}(\infty) \int\limits_{C_1} R(\lambda - \lambda_1, T_2)\, x\, d\lambda_1 + 2\pi i [1 - \chi_{V_1}(\infty)]\, x$$

$$= \chi_{V_1}(\infty) \int\limits_{C_1} R(\mu, T_2)\, x\, d\mu + 2\pi i [1 - \chi_{V_1}(\infty)]\, x$$

$$= 2\pi i\, x,$$

where we have used the fact that T_1 and T_2 commute with the integral, C_1 is a positively oriented circle of sufficiently large radius around the origin, and μ has been substituted for $\lambda - \lambda_1$. Hence the convolution (5) equals $R(\lambda, T_1 + T_2)$, q.e.d.

COROLLARY 5. *Let T_1 or T_2 be bounded and $\{T_1, T_2\} \in \mathfrak{C}(\mathfrak{X})$. Then*

$$\sigma_e(T_1 + T_2) \subset \sigma_e(T_1) + \sigma_e(T_2).$$

Proof. If $\sigma_e(T_1) + \sigma_e(T_2) = \mathbf{C}$ the statement is trivial. Otherwise we have for each complex $\lambda \notin \sigma_e(T_1) + \sigma_e(T_2)$ a neighborhood V (in the sense of Section 2) of $\sigma_e(T_1) \times \sigma_e(T_2)$ such that $(\lambda - \lambda_1 - \lambda_2)^{-1} \chi_V(\lambda_1, \lambda_2) \in H(V)$ and the resolvent $R(\lambda, T_1 + T_2)$ is given by (5). But since $\lambda - \lambda_1 \notin \sigma_e(T_2)$, the integrand in (5) is bounded on Γ_1 and, since Γ_1 is compact, $R(\lambda, T_1 + T_2)$ is also bounded. Hence $\lambda \notin \sigma_e(T_1 + T_2)$. This implies $\sigma_e(T_1 + T_2) \subset \sigma_e(T_1) + \sigma_e(T_2)$, q.e.d.

In a manner similar to that used in Theorem 4 we obtain for the product of T_1 and T_2:

THEOREM 6. *Let $\{T_1, T_2\} \in \mathfrak{G}(\mathfrak{X}, V)$. Then for each $\lambda \notin V_1 \cdot V_2$ we have*

$$\Phi[(\lambda - \lambda_1 \lambda_2)^{-1} \chi_V(\lambda_1, \lambda_2)] = = \frac{1}{2\pi i} \int\limits_{\Gamma_1} R(\lambda_1, T_1)\, R(\lambda/\lambda_1, T_2) \frac{d\lambda_1}{\lambda_1} = R(\lambda, T_1 T_2). \tag{6}$$

The proof of this theorem parallels that of the preceding theorem. Here we have to show that on $\mathfrak{X}$

$$(\lambda I - T_1 T_2) \int_{\Gamma_1} R(\lambda_1, T_1) R(\lambda/\lambda_1, T_2) x \frac{d\lambda_1}{\lambda_1}$$

$$= \int_{\Gamma_1} \left[\lambda I - \lambda_1 T_2 - \left(\frac{\lambda}{\lambda_1} I - T_2\right)(\lambda_1 I - T_1) + \frac{\lambda}{\lambda_1}(\lambda_1 I - T_1)\right]$$

$$\times R(\lambda_1, T_1) R(\lambda/\lambda_1, T_2) x \frac{d\lambda_1}{\lambda_1}$$

$$= \int_{\Gamma_1} R(\lambda_1, T_1) x \, d\lambda_1 - \int_{\Gamma_1} \frac{d\lambda_1}{\lambda_1} x + \lambda \int_{\Gamma_1} R(\lambda/\lambda_1, T_2) x \frac{d\lambda_1}{\lambda_1^2}$$

$$= 2\pi i [1 - \chi_{V_1}(\infty)] x - 2\pi i [\chi_{V_1'}(0) - \chi_{V_1}(\infty)] x$$

$$+ \chi_{V_1'}(0) \lambda \int_{C_2} R(\lambda/\lambda_1, T_2) x \frac{d\lambda_1}{\lambda_1^2} - \chi_{V_1}(\infty) \lambda \int_{C_1} R(\lambda/\lambda_1, T_2) x \frac{d\lambda_1}{\lambda_1^2}$$

$$= 2\pi i [1 - \chi_{V_1'}(0)] x + \chi_{V_1'}(0) \int_{C_1} R(\mu, T_2) x \, d\mu - \chi_{V_1}(\infty) \int_{C_2} R(\mu, T_2) x \, d\mu,$$

where Γ_1 is chosen such that it does not contain the point $\lambda=0$, V_1' is the Cauchy domain enclosed by Γ_1, C_2 is a sufficiently small positively oriented circle around the origin, and μ has been substituted for λ/λ_1. If $\chi_{V1}(0)=1$, then $\lambda=0$ is in V_1, V_2 is bounded and, according to (3), the first integral in the last expression is $2\pi i x$. If $\chi_{V_1}(\infty)\neq 0$, then $\lambda=0$ is not in V_2 and the last integral vanishes. Hence the whole expression equals $2\pi i x$ and this shows that $R(\lambda, T_1 T_2)$ is given by (6).

COROLLARY 7. *Let* $\{T_1, T_2\} \in \mathfrak{C}(\mathfrak{X})$ *and let the point* $\lambda=0$ *not be contained in the spectrum of one operator if the other is unbounded. Then* $\sigma_e(T_1 T_2) \subset \sigma_e(T_1) \cdot \sigma_e(T_2)$.

Again the proof is similar to that for Corollary 5 with a few changes in signs.

Remark. Corollaries 5 and 7 have been proved by FOGUEL [5, Corollary 1] for the special case of two commuting scalar operators whose spectra are finite point sets in **C**. More generally, using the fact that there exists a complex commutative Banach algebra containing I, T_1 and T_2, Corollaries 5 and 7 for two commuting bounded linear operators T_1 and T_2 on $\mathfrak{X}$ follow from the GELFAND theory [6, Satz 6 and § 5]. A convolution integral similar to (5) for the resolvent of the sum of two bounded commuting linear operators has been established by BIANCHI and FAVELLA [1] in connection with problems in scattering theory.

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