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Introduction

Let $\varphi \not\equiv 0$ be a regular single valued analytic function defined on the unit disc $\Delta = \{|z| < 1\}$. For each $0 < k < 1$ it is well known that a quasiconformal[1]) mapping f of Δ onto itself exists which solves the Beltrami equation

$$\mu_f = k \frac{\bar{\varphi}}{|\varphi|} \tag{1}$$

except at the zeros of φ. Here μ_f denotes the complex dilatation $f_{\bar{z}}/f_z$ where f_z and $f_{\bar{z}}$ are the complex derivatives

$$f_z = \tfrac{1}{2}(f_x - i f_y), \quad f_{\bar{z}} = \tfrac{1}{2}(f_x + i f_y).$$

Such an f will be called a *Teichmüller mapping corresponding to* φ. It is immediate that f has the constant dilatation $K = (1+k)/(1-k)$. A quasiconformal mapping of the unit disc onto itself is known to have continuous boundary values, hence we may consider the class $C(f)$ of all topological mappings of Δ onto itself that agree with f on the boundary $\partial\Delta$. In $C(f)$ there is at least one function whose maximal dilatation is a minimum, for the set of all quasiconformal mappings of Δ onto itself with maximal dilatation bounded above by K is a normal family whose limit functions either belong to the same family or reduce to constants (AHLFORS [1]). Such a mapping is called extremal quasiconformal in the class $C(f)$ or more simply *extremal.* An extremal mapping will be called *unique extremal* if it is the only extremal mapping in $C(f)$. The purpose of this paper is to determine conditions on the regular function φ which guarantee that a corresponding Teichmüller mapping is extremal or unique extremal.

Much is already known concerning this problem. Indeed, the "Grötzsch extremal problem" (GRÖTZSCH [2]), which eventually led to the development of the theory of quasiconformal mappings, can be formulated in such a way as to show that if a schlicht and single valued branch of $\int \sqrt{\varphi(z)}\,dz$ can be chosen which maps Δ onto a rectangle then any Teichmüller mapping corresponding to φ is unique extremal. That a Teichmüller mapping need not be extremal is seen by the following simple example.

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1) Quasiconformal is also to mean topological and sense preserving. Of course if f solves the equation (1) it must necessarily be sense preserving since $k < 1$ implies that the Jacobian $J(f) = |f_z|^2 - |f_{\bar{z}}|^2$ is positive almost everywhere.

Let Φ map Δ conformally onto the upper half plane and let $F(z)=Kx+iy$, $K>1$. It follows that $f=\Phi^{-1}\circ F\circ\Phi$ is a Teichmüller mapping corresponding to $\varphi=\Phi'^2$ which is not extremal, for if $G(z)=Kz$ then $g=\Phi^{-1}\circ G\circ\Phi$ is conformal and agrees with f on $\partial\Delta$. Finally, let us show that a Teichmüller mapping may be extremal and yet not unique extremal. K. STREBEL [3] has shown that the mapping $F(z)=Kx+iy$ minimizes the maximal dilatation in the class of all mappings of the region

$$R=\{\operatorname{Im} z<0\}\cup\{|\operatorname{Re} z|<1\}$$

onto $S=F(R)$ that agree with F on the boundary ∂R. If Φ and Ψ are conformal mappings of Δ onto R and S respectively, then $f=\Psi^{-1}\circ F\circ\Phi$ is a Teichmüller mapping corresponding to $\varphi=\Phi'^2$ which is extremal. However, if

$$G(z)=\begin{cases}F(z) & \text{for} \quad \operatorname{Im} z\geqslant 0\\ Kz & \text{for} \quad \operatorname{Im} z<0,\end{cases}$$

the mapping $g=\Psi^{-1}\circ G\circ\Phi$ also has the maximal dilatation K and moreover g agrees with f on $\partial\Delta$.

The original reason for studying extremal problems of this nature was to determine the "most nearly conformal" mapping in a given class C which, according to the most common criterion for "most nearly conformal", is a mapping in C whose maximal dilatation is a minimum. Subsequent developments have given a new significance to such extremal problems, the most pronounced being the role they play in the study of the "space" of closed Riemann surfaces. The first indication of this application is to be found in the deep and surprising result of TEICHMÜLLER [4] which can be stated as follows. In a given homotopy class of topological mappings $R\to S$, of two closed Riemann surfaces of genus $g>1$, there is a unique quasiconformal mapping f whose maximal dilatation K is a minimum. Moreover, there exists an analytic quadratic differential φ on R, unique up to a positive factor, such that if z is the local uniformizing variable the equation (1) is satisfied with $k=(K-1)/(K+1)$. Making systematic use of this theorem AHLFORS and RAUCH [5, 6] have succeeded in showing that the Teichmüller space of closed Riemann surfaces of genus $g>1$ permits a complex analytic structure which, under certain simply stated analyticity requirements is unique.

In his earlier work [7] TEICHMÜLLER presents a formulation of the general extremal problem. In addition to several general statements which together give a loose formulation of the problem, he includes specific conjectures such as the aforementioned theorem and also sets forth several possible theorems as suggestive of what might be expected in the form of specific results. The problem of determining conditions on φ in order that a corresponding Teichmüller mapping f be extremal is thus seen to be one possible interpretation of the following general problem stated by Teichmüller.

"Let a sufficiently regular topological mapping of the circumference $|z|=1$ onto itself be given. It is desired to continue this to a quasiconformal topological mapping of the circle $|z|\leqslant 1$ onto itself in such a way that the maximum of the dilatation quotient becomes as small as possible."

Teichmüller proceeds to suggest the problem of determining how regular the boundary correspondence should be in order that it be continuable at all to a quasiconformal mapping and furthermore that it be continuable to an extremal mapping which is, in the above sense, a Teichmüller mapping. Of course explicit necessary and sufficient conditions that the boundary correspondence be continuable to a quasiconformal mapping have since been given by BEURLING and AHLFORS [8], but the second part of the problem is still an open question. Indeed, it is not even known whether an arbitrary boundary correspondence of a quasiconformal mapping is continuable to one with constant dilatation.

Concerning the problem of the present paper Teichmüller asks whether the condition $\iint_\Delta |\varphi|\, dx\, dy < \infty$ is sufficient in order that a corresponding Teichmüller mapping be extremal. In 1962 K. STREBEL [3] succeeded in giving a partial answer to this question as well as to the case where the integral is infinite. The specific problem considered by Strebel is that of determining geometric properties of a surface R lying "above" the z-plane such that the horizontal stretching F of R, defined by $z \to Kx + iy$, is extremal in the class of all topological mappings $R \to F(R)$ which agree with F on the boundary ∂R. In a later work [9] STREBEL succeeded in giving an unconditional affirmative answer to Teichmüller's question. In all that follows it will therefore be assumed that

$$\iint_\Delta |\varphi|\, dx\, dy = \infty .$$

Many of the techniques employed by Strebel in both of the aforementioned works are used here. Especially, the entire technique of using horizontal paths, as set forth in section 1.3, is due to Strebel. Certain essential preliminary considerations will be presented in Section 1. In Section 2 a basic theorem which is central to all subsequent considerations is stated and proved. Section 3 is concerned with the problem of determining explicit conditions on φ which imply that a corresponding Teichmüller mapping is extremal. These take the form of growth conditions on φ and each such result is shown to have a formulation in which the growth condition is required only in a relative neighborhood of a single point on the boundary $\partial\Delta$. The main result of Section 4 is a uniqueness theorem for extremal Teichmüller mappings. This theorem is used to deduce a simple characterization of regular functions φ, possessing at worst poles on $\partial\Delta$, with respect to the property that corresponding Teichmüller mappings are unique extremal.

The author wishes to take this opportunity to express his sincere gratitude to Professor L. V. AHLFORS who, during the course of this research, not only gave generously of his time in discussing the work, but also made many important suggestions that to a great extent made possible the results contained herein.

1. Preliminary Considerations

1.1. *The local mapping*

Let z_0 be a point of the unit disc Δ for which the regular function φ does not vanish. Then choose a neighborhood U of z_0 so small that a single valued and schlicht branch $z'=\Phi(z)$ of $\int\sqrt{\varphi(z)}\,dz$ can be chosen in U. Defining $F(z')=Kx'+iy'$, where $z'=x'+iy'$ and $K>1$, the composition $F\circ\Phi$ satisfies $\mu_{F\circ\Phi}=k\bar{\varphi}/|\varphi|$ in U with $k=(K-1)/(K+1)$. Now two quasiconformal mappings of U, having the same complex dilatation almost everywhere in U, differ by a conformal mapping of the image domains (see e.g. LEHTO and VIRTANEN [10]). Hence, if $f:\Delta\to\Delta'=\{|w|<1\}$ is a Teichmüller mapping corresponding to φ, with the same k, the mapping

$$\Psi = F\circ\Phi\circ f^{-1} \tag{2}$$

is a conformal mapping of $f(U)$ onto $F\circ\Phi(U)$. If one continues Φ along a path γ in Δ that avoids the zeros of φ, (2) yields a corresponding continuation of Ψ along $f(\gamma)$. It follows that the relation $\Psi'\cdot f_z=F_z\cdot\Phi'$ is valid for corresponding branches Φ and Ψ. But then Ψ' is determined except for its sign since this is the case for $\Phi'=\sqrt{\varphi}$. Hence, $\psi=\Psi'^2$ in single valued and not zero in the disc Δ' punctured at the images, under f, of the zeros of φ. Furthermore, since f^{-1} is sense preserving and since F is simply a horizontal stretching it follows easily from (2) that f maps the zeros of φ into zeros of ψ of the same order. Direct calculation from (2) gives

$$\mu_{f^{-1}} = -k\frac{\bar{\psi}}{|\psi|} \tag{3}$$

except at the zeros of ψ. We have shown that the inverse of a Teichmüller mapping is again a Teichmüller mapping.

At points of Δ that are not zeros of φ we obtain the formulas

$$f_z = \tfrac{1}{2}(K+1)\frac{\Phi'}{\Psi'}, \quad f_{\bar{z}} = \tfrac{1}{2}(K-1)\frac{\bar{\Phi}'}{\Psi'}. \tag{4}$$

The Jacobian of f is then $J(f)=|f_z|^2-|f_{\bar{z}}|^2=K|\varphi|/|\psi|$ except at the zeros of φ, hence for a measurable subset G of Δ we obtain the relation

$$\iint_{f(G)} |\psi|\,du\,dv = K\iint_G |\varphi|\,dx\,dy. \tag{5}$$

1.2. *The metric* $\sqrt{|\varphi(z)|}\,|dz|$

On Δ let us consider the metric defined by $ds=\sqrt{|\varphi(z)|}\,|dz|$. It is well known (AHLFORS [1]) that an arc on which $\sqrt{\varphi(z)}dz$ has a constant argument and which does not turn back on itself should it contain a zero of φ, represents the unique shortest line between any two of its points. Such an arc on which $\varphi(z)dz^2>0$ is called a horizontal arc and one on which $\varphi(z)dz^2<0$ is called a vertical arc because any branch of $\int\sqrt{\varphi(z)}dz$ maps horizontal arcs onto horizontal segments and vertical arcs onto vertical segments. The distance between two points z_0 and z_1 of Δ will be denoted by $d_\varphi(z_0, z_1)$, the area of a subset G of Δ by $|G|_\varphi$ and the length of a path γ by $|\gamma|_\varphi$. Analogous statements and notations hold also for the metric on Δ' defined by $\sqrt{|\psi|}\,|dw|$.

From the equations (4) it is seen that, avoiding the zeros of φ, the Teichmüller mapping f has the infinitesimal representation

$$\sqrt{\psi}\,dw=\frac{1}{1-k}(\sqrt{\varphi}\,dz+k\sqrt{\bar{\varphi}}\,d\bar{z}) \tag{6}$$

where the signs of the square roots are chosen appropriately. This representation exhibits the geometrically obvious fact that f maps geodesic arcs, that is, arcs on which $\sqrt{\varphi(z)}dz$ has a constant argument, into geodesic arcs and in particular horizontal and vertical arcs into horizontal and vertical arcs respectively. Moreover, it follows from (6) that for any path γ in Δ the inequalities

$$|\gamma|_\varphi \leqslant |f(\gamma)|_\psi \leqslant K|\gamma|_\varphi \tag{7}$$

are valid with the lower bound for $|f(\gamma)|_\psi$ being attained when γ is a vertical arc and the upper bound being attained when γ is a horizontal arc.

1.3. *Area integrals*

In that which follows it will frequently be necessary to express a double integral as an iterated integral whose paths of integration are subsets of horizontal and vertical arcs. To this end let G be an open subset of Δ of finite φ-area. For each $z\in G$ for which φ does not vanish denote by γ_z (respectively β_z) the longest horizontal arc (respectively vertical arc) in G, containing z and free of the zeros of φ. From G choose a sequence $\{z_n\}$ of non-critical points of φ which are everywhere dense in G and for each $n=1,2,\ldots$ set

$$G_n=\bigcup_{z\in\beta_{z_n}}\gamma_z.$$

Now let C be a closed Jordan curve lying in G_n and let z_0 be interior to C. A horizontal arc γ passing through z_0 and extended in both directions cannot intersect itself since this would contradict the uniqueness of a shortest line between any two of its points. It follows that γ must meet C in at least two distinct points z' and z''. If z' and z'' belong to the same horizontal γ_z in G_n then $z_0 \in G_n$, for otherwise $\gamma + \gamma_z$ would be a self intersecting horizontal arc. But, if z' and z'' belong to different horizontals in G_n, then γ and β_{z_n} would yield two shortest lines between a pair of points on β_{z_n} which is contradictory. It follows that G_n is simply connected.

Since G_n is simply connected and free of the zeros of φ a single valued branch $z' = \Phi(z)$ of $\int \sqrt{\varphi(z)}\,dz$ can be chosen in G_n. If $a \neq b$ are two points of G_n lying on the same horizontal in G_n we have $\operatorname{Re}[\Phi(b) - \Phi(a)] \neq 0$ since $\int_a^b \sqrt{\varphi(z)}\,dz$ is real and not zero. If a and b lie on different horizontals in G_n, say $a \in \gamma_z$ and $b \in \gamma_{\hat{z}}$, we obtain $\operatorname{Im}[\Phi(b) - \Phi(a)] = \operatorname{Im} \int_z^{\hat{z}} \sqrt{\varphi(z)}\,dz \neq 0$. Hence, Φ is also schlicht in G_n. It follows that

$$|G_n|_\varphi = \iint_{\Phi(G_n)} dx'\,dy' = \int_{\Phi(\beta_{z_n})} \int_{\Phi(\gamma_z)} dx'\,dy'.$$

But then, if dx_1 denotes the differential of arc length $\sqrt{|\varphi(z)|}\,|dz|$ along γ_z and if dy_1 has the same meaning along β_{z_n} we obtain

$$|G_n|_\varphi = \int_{\beta_{z_n}} \int_{\gamma_z} dx_1\,dy_1. \tag{8}$$

From the way in which the G_n were constructed it is not difficult to see that $G - \{\text{zeros of } \varphi\} = \bigcup G_n$. Indeed, let z_0 be a point in G for which $\varphi(z_0) \neq 0$ and choose a neighborhood U of z_0 on which a single valued and schlicht branch Φ of $\int \sqrt{\varphi(z)}\,dz$ can be defined. Next, let C be a circle lying in $\Phi(U)$ with its center at $\Phi(z_0)$ and choose w interior to C such that $\Phi^{-1}(w) = z_j$ belongs to the dense sequence $\{z_n\}$. It is now immediate that $z_0 \in G_j$. The opposite inclusion is trivial. If for each n we set $E_n = G_n - \bigcup_1^{n-1} G_j$ we obtain $G - \{\text{zeros of } \varphi\}$ as the disjoint union

$$G - \{\text{zeros of } \varphi\} = \bigcup_1^\infty E_j.$$

Now $z \in G_i \cap G_j$ clearly implies that $\gamma_z \subset G_i \cap G_j$. This fact together with the relation (8) allows us to write

$$|E_n|_\varphi = \int_{\beta_n} |\gamma_z|_\varphi\,dy_1$$

where we have set $\beta_n = \beta_{z_n} \cap E_n$. Finally, setting $\beta = \sum_1^\infty \beta_n$, we have

$$\iint_G |\varphi|\,dx\,dy = \int_\beta \int_{\gamma_z} dx_1\,dy_1 \tag{9}$$

as was desired. Since a Teichmüller mapping f maps horizontals and verticals into arcs of the same type we also have

$$\iint_{f(G)} |\psi|\, du\, dv = \int_{f(\beta)} \int_{f(\gamma_z)} du_1\, dv_1 \tag{10}$$

where du_1 denotes the differential of arc length along $f(\gamma_z)$ and dv_1 has the same meaning along $f(\beta)$.

From the preceding considerations we may write

$$|\gamma_z| \equiv \int_{\gamma_z} |dz| = \int_{\Phi(\gamma_z)} \left|\frac{d\Phi^{-1}}{dz'}\right| dx' .$$

Integrating over $\Phi(\beta_j)$ and applying the Schwarz inequality we obtain

$$\left(\int_{\Phi(\beta_j)} |\gamma_z|\, dy'\right)^2 \leqslant \iint_{\Phi(E_j)} \left|\frac{d\Phi^{-1}}{dz'}\right|^2 dx'\, dy' \iint_{\Phi(E_j)} dx'\, dy' \leqslant \pi\, |E_j|_\varphi .$$

It follows that $|\gamma_z| < \infty$ except possibly for a subset of β_j of zero measure and hence also except for a subset of β of zero measure. Thus, for almost all $z \in \beta$ one can speak of the ends of γ_z. From the relation

$$|G|_\varphi = \int_\beta |\gamma_z|_\varphi\, dy_1 < \infty$$

it also follows that $|\gamma_z|_\varphi < \infty$ except for a subset of β of zero measure. Deleting from β all z for which either $|\gamma_z| = \infty$, $|\gamma_z|_\varphi = \infty$, or γ_z has an endpoint at a zero of φ, and again calling the remaining set β, the relations (9) and (10) continue to hold. In the sequel this will always be assumed to have been done.

2. A Basic Theorem

The central theorem of this thesis, which exhibits sufficient conditions for a Teichmüller mapping to be extremal or unique extremal, is formulated and proved in the present section. All results of the subsequent sections will be seen to depend on it. We begin by fixing some notations that will allow for a compact and easily applied statement of this theorem. First we remark that $\varphi \not\equiv 0$ again denotes a regular single valued analytic function on Δ and $\varphi, f, \psi, k, \ldots$ are related as in the preceding section.

Let $\{S_n\}$ be a sequence of open subsets of Δ such that, for each n, the boundary ∂S_n is a union of countably many smooth arcs. For each n define Γ_n to be that part of $\Delta \cap \partial S_n$ remaining after all horizontal arcs are deleted. We further assume that S_n satisfies the following three conditions.

(i) $|S_n|_\varphi < \infty, \quad n = 1, 2, \ldots$.
(ii) $|S_n|_\varphi \to \infty \quad \text{as} \quad n \to \infty$.
(iii) $l_n \equiv |\Gamma_n|_\varphi < \infty, \quad n = 1, 2, \ldots$.

Next, let h be a $\tilde{K}$-quasiconformal mapping of Δ onto itself that agrees with f on $\partial \Delta$. For each n let there exist numbers $B_{n_1}, \ldots, B_{n_s}$ and a subdivision $\Gamma_n = \Gamma_{n_1} + \cdots + \Gamma_{n_s}$ (each Γ_{n_i} is again a union of countably many arcs) such that

$$d_\psi(f(z), h(z)) \leqslant B_{n_i}, \quad z \in \Gamma_{n_i}, \quad i = 1, \ldots, s. \tag{11}$$

Now fix n and let β and $\{\gamma_z\}$ be as in the equation (9) where S_n plays the role of the open set G. As a generic notation we take $\gamma_w = f(\gamma_z)$ and $\gamma'_w = h(\gamma_z)$. The following lemma will be essential for the proof of the main theorem.

LEMMA 1. *For almost all $z \in \beta$, if γ_z has $j_i(z)$ of its endpoints in Γ_{n_i}, $i = 1, \ldots, s$,* then

$$|\gamma_w|_\psi \leqslant |\gamma'_w|_\psi + \sum_{i=1}^{s} j_i(z) B_{n_i}. \tag{12}$$

Proof. If $\sum_{i=1}^{s} j_i(z) = 2$ the inequality follows from the relation (11) and the fact that γ_w represents the shortest line between its two endpoints. Now let z_0 be an endpoint of γ_z that does not belong to Γ_n. Since we consider only horizontals whose endpoints are not zeros of φ it follows from the way in which Γ_n was defined that $z_0 \in \partial \Delta$. Furthermore, since the lemma requires the inequality (12) only for almost all $z \in \beta$, it may be assumed that z_0 is an interior point of $\partial S_n \cap \partial \Delta$. If for each $r > 0$ we define

$$C_r = \{|w - f(z_0)| = r\} \cap f(S_n)$$

it follows that C_r connects γ_w and γ'_w for all sufficiently small r. An application of the Schwarz inequality yields

$$|C_r|_\psi^2 = \left(\int_{C_r} \sqrt{|\psi|}\, r\, d\theta \right)^2 \leqslant r\pi \int_{C_r} |\psi|\, r\, d\theta.$$

Then, for $r_0 > 0$ we have

$$\int_0^{r_0} \frac{|C_r|_\psi^2}{r}\, dr \leqslant \pi \int_0^{r_0} \int_{C_r} |\psi|\, r\, d\theta\, dr \leqslant \pi |f(S_n)|_\psi < \infty.$$

This shows that there exists a sequence $r_m \to 0$ for which $|C_{r_m}|_\psi \to 0$. Hence, for arbitrarily small r there exist arbitrarily short arcs connecting γ_w and γ'_w. This together with the fact that γ_w represents the shortest line between any two of its points implies that the inequality (12) also holds for the cases $\sum_{i=1}^{s} j_i(z) = 0, 1$.

THEOREM 1. *For each* $n=1,2,\dots$ *and* $i=1,2,\dots,s$ *let* $T_n=h(S_n)-f(S_n)$ *and* $l_{n_i}=|\Gamma_{n_i}|$.[2])

(1.a) *If both* $\sum_i l_{n_i}B_{n_i}$ *and* $|T_n|$ *are* $o(|S_n|)$ *as* $n\to\infty$ *then* $K\leqslant\tilde{K}$.

(1.b) *If both* $\sum_i l_{n_i}B_{n_i}$ *and* $|T_n|$ *are* $o(1)$ *as* $n\to\infty$ *and if in addition* $K=\tilde{K}$ *and* $S_n\to\Delta$ *then* $f\equiv h$.

Proof.[3]) Successively applying (5), (10), and the remark immediately following (7), we obtain

$$K^2|S_n|^2=|f(S_n)|^2=\Big(\int_{f(\beta)}|\gamma_w|\,dv_1\Big)^2=\Big(\int_{\beta}|\gamma_w|\,dy_1\Big)^2.$$

For each $i=1,\dots,s$ let $j_i(z)$ again denote the number of endpoints of γ_z lying in Γ_{n_i}. Lemma 1 and the fact that β is a union of vertical arcs yield

$$\int_\beta|\gamma_w|\,dy_1\leqslant\int_\beta|\gamma'_w|\,dy_1+\int_\beta\sum_{i=1}^{s}j_i(z)\,B_{n_i}\,dy_1$$
$$\leqslant\int_\beta|\gamma'_w|\,dy_1+2\sum_{i=1}^{s}l_{n_i}B_{n_i}.$$

Hence,

$$K^2|S_n|^2\leqslant\Big(\int_\beta|\gamma'_w|\,dy_1\Big)^2+L_n \tag{13}$$

where the following order relations for L_n will follow as soon as the first term on the right side of (13) is shown to be $O(|S_n|^2)$ as $n\to\infty$:

$$\left.\begin{aligned}L_n&=o(|S_n|^2)\ \text{ if }\ \sum_{i=1}^{s}l_{n_i}B_{n_i}=o(|S_n|)\\ L_n&=o(|S_n|)\ \text{ if }\ \sum_{i=1}^{s}l_{n_i}B_{n_i}=o(1).\end{aligned}\right\} \tag{14}$$

Avoiding the zeros φ we may consider the local mapping $w=h\circ\Phi^{-1}(z')$. Now

$$|dw|=\left|\frac{h_z}{\Phi'}dz'+\frac{h_{\bar z}}{\bar\Phi'}d\bar z'\right|$$

is well defined almost everywhere. But then, since $h^{-1}(\gamma'_w)=\gamma_z$ is a horizontal arc, and since $dz'=dx'$ along $\Phi(\gamma_z)$, we obtain

$$|\gamma'_w|=\int_{\Phi(\gamma_z)}\sqrt{|\psi|}\left|\frac{h_z}{\Phi'}+\frac{h_{\bar z}}{\bar\Phi'}\right|dx'.$$

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[2]) That is, $|\Gamma_{n_i}|_\varphi$. In the sequel subscripts indicating distance, length, and area in the various metrics will be dropped whenever the metric under consideration is clear from the context.

[3]) This proof is a direct generalization of Strebel's proof for the case $|\Delta|_\varphi<\infty$. Indeed, taking $S_n\equiv\Delta$ for every n, the proof reduces to that given by Strebel.

It follows that

$$\int_\beta |\gamma'_w|\, dy_1 = \int_\beta \int_{\gamma_z} \sqrt{|\psi|}\left|\frac{h_z}{\Phi'} + \frac{h_{\bar z}}{\bar\Phi'}\right| dx_1\, dy_1 = \iint_{S_n} \sqrt{|\varphi\psi|}\left|h_z + \frac{|\varphi|}{\bar\varphi}\, h_{\bar z}\right| dx\, dy.$$

Squaring and applying the Schwarz inequality we obtain

$$\left(\int_\beta |\gamma'_w|\, dy_1\right)^2 \leqslant \iint_{S_n} (|h_z|^2 - |h_{\bar z}|^2)\,|\psi|\, dx\, dy \iint_{S_n} \frac{\left|h_z + \dfrac{|\varphi|}{\bar\varphi}\, h_{\bar z}\right|^2}{|h_z|^2 - |h_{\bar z}|^2}|\varphi|\, dx\, dy. \tag{15}$$

With $\tilde k = (\tilde K - 1)/(\tilde K + 1)$ and noting that $|\mu_h| \leqslant \tilde k$ we may write

$$\frac{\left|1 + \dfrac{|\varphi|}{\bar\varphi}\mu_h\right|^2}{1 - |\mu_h|^2} = \frac{1 + |\mu_h|^2 + 2\operatorname{Re}\left[\dfrac{|\varphi|}{\bar\varphi}\mu_h\right]}{1 - |\mu_h|^2} \leqslant \tilde K - \frac{2}{1 - \tilde k^2}\left(\tilde k - \operatorname{Re}\left[\frac{|\varphi|}{\bar\varphi}\,\mu_h\right]\right).$$

Hence, the second integral on the right side of (15) has the upper estimate

$$\tilde K |S_n| - \iint_{S_n} \left(\tilde k - \operatorname{Re}\left(\frac{|\varphi|}{\bar\varphi}\mu_h\right)\right)|\varphi|\, dx\, dy. \tag{16}$$

The first integral on the right side of (15) is just $|h(S_n)|$ which obviously satisfies $|h(S_n)| \leqslant K|S_n| + |T_n|$. This remark, together with the bound (16), and the fact that $\tilde k \geqslant \operatorname{Re}((|\varphi|/\bar\varphi)\mu_h)$, shows that

$$\left(\int_\beta |\gamma'_w|\, dy_1\right)^2 = O(|S_n|^2), \quad n \to \infty.$$

It follows that the relations (14) are valid. Combining results we obtain

$$\begin{aligned} K^2 |S_n|^2 &\leqslant (K|S_n| + |T_n|)\left(\tilde K |S_n| - \iint_{S_n}\left(\tilde k - \operatorname{Re}\left(\frac{|\varphi|}{\bar\varphi}\mu_h\right)\right)|\varphi|\, dx\, dy\right) + L_n \\ &\leqslant K\tilde K |S_n|^2 + \tilde K |S_n|\,|T_n| + L_n. \end{aligned} \tag{17}$$

The first part of the theorem follows by dividing (17) by $|S_n|^2$ and letting $n \to \infty$. Now assume that the hypotheses of (1.b) are satisfied. An obvious manipulation of the first inequality in (17) gives

$$0 \leqslant \iint_{S_n}\left(k - \operatorname{Re}\left(\frac{|\varphi|}{\bar\varphi}\mu_h\right)\right)|\varphi|\, dx\, dy \leqslant |T_n| + \frac{L_n}{K|S_n|}.$$

The right side of this relation is $o(1)$ as $n \to \infty$. Since $S_n \to \Delta$ it follows that $k - \operatorname{Re}((|\varphi|/\bar\varphi)\mu_h)$ vanishes almost everywhere in Δ. From this we obtain

$$\mu_h = k \frac{\bar{\varphi}}{|\bar{\varphi}|}, \quad \text{almost everywhere.}$$

We have shown that f and h satisfy the same Beltrami equation. It follows that they differ at most by a conformal mapping of Δ' onto itself. This must be the identity mapping because of the agreement of f and h on $\partial\Delta$.[4])

3. Extremal Teichmüller Mappings

In this chapter the extremal property of Teichmüller mappings f for which the corresponding regular functions φ satisfy certain growth conditions will be exhibited. In each case it will be seen that the growth condition is required only in a relative neighborhood of a point of $\partial\Delta$. For convenience the boundary point will always be taken as $z=1$. To facilitate the discussion we define, for each $\varrho>0$, the set

$$N_\varrho = \{z \in \Delta : |z-1| < \varrho\}.$$

Again, let h denote a $\tilde{K}$-quasiconformal mapping of Δ onto itself that agrees with f on $\partial\Delta$. In order that Theorem 1 be applicable it will be necessary to estimate $d(f(z), h(z))$. This can frequently be accomplished with the use of a distortion theorem due to TEICHMÜLLER [11], which for our purposes we state as follows. Given $K_1>1$ there exists a number $0<\varrho(K_1)<1$ with the property that every K_1-quasiconformal mapping of Δ onto itself that fixes each point of $\partial\Delta$ moves the origin by a distance at most equal to $\varrho(K_1)$. First, consider a regular function φ which, for some $\varrho>0$, satisfies the inequality

$$|\varphi(z)| \leq 1/(1-|z|^2)^2, \quad z \in N_\varrho. \tag{18}$$

Next, choose $0<\varrho_1<\varrho$ so small that for all $z \in N_{\varrho_1}$, N_ϱ contains the non-euclidean line segment joining z and $f^{-1} \circ h(z)$. Now, the non-euclidean distance D between two points z_1 and z_2 in Δ is defined by $D(z_1, z_2) = \inf \int_\gamma (|dz|/1-|z|^2)$, where the infimum is taken over all paths γ, lying in Δ, and joining z_1 and z_2. Hence, it follows from (18)

$$d(z, f^{-1} \circ h(z)) \leq D(z, f^{-1} \circ h(z)), \quad \text{for} \quad z \in N_{\varrho_1}.$$

Since non-euclidean distance is invariant under Möbius transformations of Δ onto itself, and since $f^{-1} \circ h$ is $K\tilde{K}$-quasiconformal, the Teichmüller distortion theorem stated above shows that $D(z, f^{-1} \circ h(z))$ is bounded above by $B = \int_0^{\varrho(K\tilde{K})} dx/1-x^2$, for all $z \in \Delta$. It folllows that

$$d(f(z), h(z)) \leq KB, \quad \text{for} \quad z \in N_{\varrho_1}.$$

4) This is the only use made of the condition that h agrees with f on the entire boundary $\partial\Delta$. The proof of (1.a) requires only that f and h agree on $\bigcup_n \partial\Delta \cap \partial S_n$.

Even if φ does not satisfy (18) one can always write

$$|\varphi(z)| \leqslant M^2(z)/(1-|z|^2)^2, \quad z \in N_\varrho.$$

Then if $\gamma(z)$ denotes the non-euclidean line segment joining z and $f^{-1} \circ h(z)$ it follows immediately from the preceding considerations that

$$d(f(z), h(z)) \leqslant KB \max_{\zeta \in \gamma(z)} M(\zeta), \quad z \in N_{\varrho_1}.$$

It is in this form that the above result will be applied in what follows.

THEOREM 2. *With $r=|z|$ let $\varphi(z)=O(1/(1-r))$ as $r \to 1$. Then f is extremal.*

Proof. As remarked in the introduction, only the case $|\Delta|_\varphi=\infty$ needs to be considered. For each $0<\varrho<1$ set $S_\varrho=\{|z|<\varrho\}$, $\Gamma_\varrho=\partial S_\varrho$, and $l_\varrho=|\Gamma_\varrho|$. Assuming, as we may, that $|\varphi(z)| \leqslant 1/(1-r)$ we obtain

$$l_\varrho \leqslant \int_{\Gamma_\varrho} \frac{|dz|}{\sqrt{1-\varrho}} = \frac{2\pi\varrho}{\sqrt{1-\varrho}}.$$

Next, for $z_0 \in \Gamma_\varrho$ we seek an upper bound for $d(f(z_0), h(z_0))$. Let $C=\{D(z, z_0) \leqslant B\}$ where B is the bound on $D(z, f^{-1} \circ h(z))$ given by the Teichmüller distortion theorem. Setting $M(z_0)=\max z \in C \sqrt{1-|z|}$ it follows that

$$d(f(z_0), h(z_0)) \leqslant B_\varrho$$

where we have set $B_\varrho=2KBM(z_0)$. In order to determine $M(z_0)$ choose ϱ sufficiently close to one so that C does not contain the origin. Then there exists a unique $\zeta \in C$ such that $M(z_0)=\sqrt{1-|\zeta|}$. We may write

$$B = \int_{|\zeta|}^{\varrho} \frac{|dz|}{1-r^2} = \tfrac{1}{2}\log\frac{(1+\varrho)(1-|\zeta|)}{(1-\varrho)(1+|\zeta|)}.$$

Solving for $1-|\zeta|$ we obtain

$$M(z_0) = \left(\frac{2(1-\varrho)}{(1+\varrho)e^{-2B}+(1-\varrho)}\right)^{\frac{1}{2}}.$$

It follows that $B_\varrho l_\varrho$ is bounded. Similar considerations show that

$$1-\varrho_1 \equiv \min_{z \in C}(1-|z|) = \frac{2(1-\varrho)}{(1+\varrho)e^{2B}+(1-\varrho)}.$$

Setting $T_\varrho=h(S_\varrho)-f(S_\varrho)$ we may then write

$$|f^{-1}(T_\varrho)| \leqslant \int_0^{2\pi} \int_\varrho^{\varrho_1} \frac{r\,dr\,d\theta}{1-r}.$$

Direct calculations show that the integral increases to $4B\pi$ as $\varrho \to 1$. Since both $B_\varrho l_\varrho$ and $|T_\varrho|$ are bounded the conclusion follows from Theorem 1.a.

In order to give a local formulation of Theorem 2 we first observe that if Δ has infinite φ-area then at least one boundary point, say $z=1$, has the property that $|N_\varrho| = \infty$ for every $\varrho > 0$. Let φ be a function with this property and assume that $|\varphi(z)| \leqslant 1/(1-r)$ in N_{ϱ_1} for some $\varrho_1 > 0$. Now choose a sequence r_n increasing to one with $1 - r_1 \leqslant \varrho_1$. Then set $S_n = \{|z| < r_n\} \cap \{\mathrm{Re}\, z > r_1\}$, $\Gamma_{n_1} = \partial S_n \cap \{\mathrm{Re}\, z = r_1\}$, and $\Gamma_{n_2} = \partial S_n - \Gamma_{n_1}$. We observe that the chord $\mathrm{Re}\, z = r_1$ has finite φ-length. It then follows by reasonings similar to those in the proof of Theorem 2 that the conditions of Theorem 1.a are satisfied. We state this result as a theorem.

THEOREM 3. *Let* $|N_\varrho| = \infty$ *for every* $\varrho > 0$ *and for some* $\varrho_1 > 0$ *let* $|\varphi(z)| \leqslant 1/(1-r)$ *in* N_{ϱ_1}. *Then f is extremal.*

During the early stages of this research it was shown that a Teichmüller mapping f is extremal if the corresponding regular function φ possesses a second order pole at a point of $\partial\Delta$. The proof depends entirely upon the existence of positive numbers m, M, and ϱ_1 such that if we take the double pole to be at $z=1$, then $m \leqslant |(1-z)^2 \varphi(z)| \leqslant M$ for all z belonging to N_{ϱ_1}. This suggests the following more general problem. Let $\lambda(\varrho)$ be defined for all $\varrho > 0$. Next, set $\varrho = |1-z|$ and assume that φ satisfies the relations

$$m \leqslant |\varphi(z)|/\lambda(\varrho) \leqslant M, \quad z \in N_{\varrho_1} \tag{19}$$

where m, M, and ϱ_1 are all positive. What further conditions should $\lambda(\varrho)$ satisfy in order that the single requirement (19) implies that a Teichmüller mapping f corresponding to φ is extremal? The remainder of this section will be concerned with this problem. We begin by proving the following theorem which exhibits one such set of conditions on $\lambda(\varrho)$. First, for every $0 < \varrho \leqslant \varrho_1$, set

$$F(\varrho) = \int_{\varrho}^{\varrho_1} \lambda(\varrho)\, \varrho\, d\varrho\,.$$

THEOREM 4. *Let* $\varphi(z)$ *satisfy* (19) *where* $\lambda(\varrho)$ *is a monotonc function such that* $F(\varrho) \to \infty$ *as* $\varrho \to 0$. *Further assume that*

$$\lim_{\varrho \to 0} \frac{F(s\varrho) - F(\varrho)}{F(\varrho)} = 0 \tag{20}$$

for every $0 < s < 1$. *Then f is extremal.*

Proof. For every $0 < \varrho \leqslant \varrho_1$ set $S_\varrho = N_{\varrho_1} - N_\varrho$ and $\gamma_\varrho = \Delta \cap \partial N_\varrho$. Then let θ_ϱ denote the angle between the chords joining $z=1$ to the endpoints of γ_ϱ. In view of (19) we have

$$\left.\begin{aligned} |S_\varrho| &\geqslant m\,\theta_{\varrho_1} \int\limits_{\varrho}^{\varrho_1} \lambda(\varrho)\,\varrho\,d\varrho = m\,\theta_{\varrho_1} F(\varrho). \\ l_\varrho &\equiv |\gamma_\varrho| \leqslant \pi\varrho\sqrt{M}\sqrt{\lambda(\varrho)}. \end{aligned}\right\} \tag{21}$$

Next, let B be the bound on $D(z, f^{-1} \circ h(z))$ given by the Teichmüller distortion theorem and let $0<\tilde{\varrho}<\varrho$ be defined by $D(1-\tilde{\varrho}, 1-\varrho)=B$. Setting $T_\varrho=h(S_\varrho)-f(S_\varrho)$ and choosing ϱ_1 sufficiently small, it follows that the part of $f^{-1}(T_\varrho)$ lying outside of N_{ϱ_1} has finite φ-area A. But then, in view of (19), we obtain

$$|T_\varrho| = K|f^{-1}(T_\varrho)| \leqslant A\,K + M\,K\,\theta_\varrho[F(\tilde{\varrho}) - F(\varrho)]. \tag{22}$$

In order to estimate $d(f(z), h(z))$ on ∂S_ϱ choose $z_1 \in \gamma_\varrho$ and set $z_2=f^{-1} \circ h(z_1)$. It is not difficult to see that $|z_2-z_1| \leqslant C(\varrho-\tilde{\varrho})$ for some constant C that does not depend on ϱ. Indeed, if $\varrho<\varrho'<1$ is defined by $D(1-\varrho', 1-\varrho)=B$ it follows that $|z_2-z_1| \leq \varrho'-\varrho$. Furthermore, from the equations $B=D(1-\tilde{\varrho}, 1-\varrho)=D(1-\varrho', 1-\varrho)$ we obtain $\varrho'-\varrho \leqslant C(\varrho-\tilde{\varrho})$ where $C=(e^{2B}-1)/(1-e^{-2B})$. Now set $d_\varrho=\int_{\tilde{\varrho}}^{\varrho} \sqrt{\lambda(\varrho)}\,d\varrho$. By making use of (19), the monotonicity of $\lambda(\varrho)$, and the relation $|z_2-z_1| \leqslant C(\varrho-\tilde{\varrho})$, we easily deduce the estimate

$$d(z_1, z_2) \leqslant \sqrt{M} \int\limits_{z_1}^{z_2} \sqrt{\lambda(\varrho)}\,|dz| \leqslant C\sqrt{M}\,d_\varrho.$$

Again using the monotonicity of $\lambda(\varrho)$ we may write

$$l_\varrho d_\varrho \leqslant \pi\varrho\sqrt{M}\sqrt{\lambda(\varrho)} \int\limits_{\tilde{\varrho}}^{\varrho} \sqrt{\lambda(\varrho)}\,d\varrho \leqslant C_1[F(\tilde{\varrho}) - F(\varrho)] \tag{23}$$

where C_1 is a constant which is independent of ϱ.

Combining the relations (20) through (23) it follows that there exists a sequence $\varrho_n \to 0$ such that both $d_{\varrho_n} l_{\varrho_n}$ and $|T_{\varrho_n}|$ are $o(|S_{\varrho_n}|)$ as $n \to \infty$. If, for $n=1, 2, \ldots$, we set $\Gamma_{n_1}=\gamma_{\varrho_1}$ and $\Gamma_{n_2}=\gamma_{\varrho_n}$, the conditions of Theorem 1.a are seen to be satisfied. This completes the proof.

An interesting special case of Theorem 4, in which the condition (20) is replaced by an explicit growth condition on $\lambda(\varrho)$ can now be deduced. First, let us show that the condition

$$\varliminf_{\varrho \to 0} \frac{\log F(\varrho)}{\log 1/\varrho} = 0 \tag{24}$$

implies that (20) holds. To see this we observe that (20) fails to hold only if there exist positive numbers a, ϱ_0, and t such that

$$\frac{F(t\varrho)}{F(\varrho)} > 1 + a, \quad 0 < \varrho \leqslant \varrho_0. \tag{25}$$

For $0<\varrho<t\varrho_0$ we may write $\varrho=t^n\hat{\varrho}$ where n is a positive integer and where $t\varrho_0 \leqslant \hat{\varrho} < \varrho_0$. In view of (25) we have $F(\varrho)>(1+a)^n F(\hat{\varrho})$ from which fact we obtain

$$\log\frac{F(\varrho)}{F(\varrho_0)} > \frac{\log(1+a)}{\log 1/t}\log\frac{t\varrho_0}{\varrho}.$$

The desired conclusion is now immediate.

Now let $\lambda(\varrho)$ satisfy the following growth condition.

$$\lambda(\varrho) = o\left(\frac{1}{\varrho^{2+\delta}}\right), \quad \varrho\to 0, \quad \text{for every } \delta > 0. \tag{26}$$

It is not difficult to see that $\lambda(\varrho)$ must also satisfy (24). Indeed, if (24) fails to hold there exists a positive number α such that $F(\varrho)>1/\varrho^\alpha$ for all small $\varrho>0$. Fix $0<\delta<\alpha$ and choose $\varrho_1>0$ so that $\lambda(\varrho)\leqslant 1/\varrho^{2+\delta}$ for $0<\varrho\leqslant\varrho_1$. Then for all sufficiently small ϱ we obtain

$$\frac{1}{\varrho^\alpha} < F(\varrho) \leqslant \int_{\varrho}^{\varrho_1} \frac{d\varrho}{\varrho^{1+\delta}} = \frac{1}{\delta}\left(\frac{1}{\varrho^\delta} - \frac{1}{\varrho_1^\delta}\right)$$

which is contradictory since $\delta<\alpha$. We may now state the following corollary to Theorem 4.

COROLLARY. Theorem 4 continues to hold if the requirement (20) is replaced by either (24) or (26).

We conclude this section by showing that the condition (20) of Theorem 4 cannot be arbitrarily relaxed. This can be seen as follows. First, map Δ onto the right half plane H by means of the transformation $\zeta=(1+z)/2(1-z)$. Given $\delta>0$ we can choose a single valued branch of $z'=\zeta^{\delta/2}$ that maps H conformally onto the wedge V defined by $-\delta\pi/4<\arg z'<\delta\pi/4$. (Of course if $\delta>8$ the wedge must be viewed as a surface lying above the z'-plane.) Let Φ denote the composition $z\to\zeta\to z'$. Then $\varphi=\Phi'^2$ is given by

$$\varphi(z) = \frac{\delta^2(1+z)^{\delta-2}}{2^\delta(1-z)^{\delta+2}}. \tag{27}$$

Taking $\lambda(\varrho)=(1/\varrho)^{2+\delta}$ it follows easily that all conditions of Theorem 4 are satisfied except for the condition (20). But, STREBEL [3] has shown that the horizontal stretching, $F(z') = Kx' + iy'$, of such a wedge V is not extremal in the class of mappings $V \to F(V)$ that agree with F on ∂V. It follows that $f = \Phi^{-1} \circ F \circ \Phi$, which is a Teichmüller mapping corresponding to φ, is not extremal.

4. Unique Extremal Teichmüller Mappings

The object of the present section is to present a uniqueness theorem (Theorem 5) for extremal Teichmüller mappings. From this theorem we will deduce a simple characterization of regular functions φ, possessing at worst poles on $\partial\Delta$, with respect to the property that corresponding Teichmüller mappings f are unique extremal.

4.1. *Distortion of plane regions.* Let R be a region of the z-plane whose boundary is a union of countably many arcs, let $S = F(R)$ where $F(z) = Kx + iy$, and for each y set $R_y = \{z \in R \mid \operatorname{Im} z > y\}$. Next, let H be a $\tilde{K}$-quasiconformal mapping of R_{y_0} into S that agrees with F on $\partial R_{y_0} \cap \partial R$ and which also satisfies $H(R_{y_1}) \subset F(R_{y_0})$ for some $y_1 \geqslant y_0$. Finally, for each $y \geqslant y_0$, let $\gamma_y = \{z \in R \mid \operatorname{Im} z = y\}$. We define $M(y)$ and $d(y)$ as follows:

$$\left.\begin{aligned} M(y) &= \sup_{y_0 \leqslant y' \leqslant y} |\gamma_{y'}|, \\ d(y) &= \sup_{z \in \gamma_y} |\operatorname{Im} H(z) - y|. \end{aligned}\right\} \tag{28}$$

Strebel [3] has shown that $d(y) \leqslant \sqrt{K\tilde{K}}\, M$ for $y \geqslant y_1$ if M is such that $M(y) \leqslant M < \infty$ for every $y \geqslant y_0$. A slight modification of Strebel's proof gives the following lemma for the general case.

Lemma 2. $d(y) \leqslant \sqrt{K\tilde{K}}\, M(y + d(y))$ for every $y \geqslant y_1$.

Proof. Fix $y \geqslant y_1$ and assume that $H(\gamma_y)$ meets the horizontal $\operatorname{Im} w = y_2$ for some $y_2 > y$. It must be shown that $y_2 - y \leqslant \sqrt{K\tilde{K}}\, M(y_2)$. (Noting that $H(R_{y_1}) \subset F(R_{y_0})$ it will easily be seen that the case $y_2 < y$ can be treated in the same manner.) For $0 < \Delta y < y_2 - y$, define $\Gamma = \{\gamma_\eta \mid y < \eta < y + \Delta y\}$ and $\Gamma' = \{H(\gamma_\eta) \mid \gamma_\eta \in \Gamma\}$. For measurable $\varrho \geqslant 0$,

$$L(\varrho) \equiv \inf_{\gamma_\eta \in \Gamma} \int_{\gamma_\eta} \varrho\, dx \leqslant \int_{\gamma_\eta} \varrho\, dx\,;$$

hence, integrating from y to $y + \Delta y$, squaring, and then applying the Schwarz inequality, we obtain

$$\lambda(\Gamma) \leqslant \frac{M(y_2)}{\Delta y} \tag{29}$$

where $\lambda(\Gamma)$ is the extremal length of the family of curves Γ. In order to estimate $(\lambda\Gamma')$ we define ϱ as follows:

$$\varrho(w) = \begin{cases} 1 & \text{if } y + \Delta y < \operatorname{Im} w < y_2 \\ 0 & \text{otherwise.} \end{cases}$$

Since $H=F$ on $\partial R_{y_0} \cap \partial R$ it follows that $H(\gamma_\eta)$ must meet the horizontal $\operatorname{Im} w = y_2$ for every $\gamma_\eta \in \Gamma$. Hence, the ϱ-length of $H(\gamma_\eta)$ is at least $2(y_2 - y - \Delta y)$ and the ϱ-area of the region swept out by the curves in Γ' is at most $KM(y_2)(y_2 - y - \Delta y)$. It follows that

$$\lambda(\Gamma') \geqslant \frac{4(y_2 - y - \Delta y)}{KM(y_2)}. \tag{30}$$

Set $\Delta y = (y_2 - y)/2$. By making use of (29), (30), and the fact that $\lambda(\Gamma') \leqslant \tilde{K}\lambda(\Gamma)$ we obtain $y_2 - y \leqslant \sqrt{K\tilde{K}}M(y_2)$ as was desired. This completes the proof.

If, for every $y \geqslant y_1$, we set $\hat{R}_y = R_{y_1} - R_y$ it follows from Lemma 2 that

$$|H(\hat{R}_y)| \leqslant |F(\hat{R}_y)| + 2\sqrt{K\tilde{K}}M^2(y + d(y)). \tag{31}$$

Remark. By making use of the inequality (31) it can be shown that the condition

$$M^2(y + d(y)) = o(|\hat{R}_y|) \quad \text{as} \quad y \to \infty \tag{32}$$

is sufficient to conclude that $K \leqslant \tilde{K}$. The proof would simply be a rewording of the proof of Theorem 1.a and will not be repeated here. Using this fact, many regions R can be exhibited for which $K \leqslant \tilde{K}$. Suppose, for example, that R is bounded by curves $x = \pm t^n + o(t^n)$, $y = t^m + o(t^m)$, with $t \geqslant 0$ and $0 < n < m$. It is not difficult to show, for sufficiently large y, that $|\hat{R}_y| \equiv |R_{y_1} - R_y| \geqslant C_1 y^{(n+m)/m}$ and $M(y) \leqslant C_2 y^{n/m}$ where C_1, $C_2 > 0$ depend only on m and n. Setting $\tilde{y} = y + d(y)$ and using Lemma 2 we may write: $\tilde{y} - y \leqslant \sqrt{K\tilde{K}}M(\tilde{y}) \leqslant \sqrt{K\tilde{K}}C_2\tilde{y}^{n/m}$. Noting that $0 < n/m < 1$ we obtain $\tilde{y} \leqslant 2y$, for all sufficiently large y, hence $M(\tilde{y}) \leqslant C_2(2y)^{n/m}$. It follows that the condition (32) is satisfied.

Returning to the main theme of the present section, let it be assumed that $K = \tilde{K}$ and also that $M(y) \leqslant M < \infty$, for every $y \geqslant y_0$. With $\gamma'_y = H(\gamma_y)$, we may write

$$K|\gamma_y| \leqslant |\gamma'_y| = \int_{\gamma_y} |H_z + H_{\bar{z}}|\, dx. \tag{33}$$

Since H is K-quasiconformal we have $|H_z + H_{\bar{z}}|^2 \leqslant KJ(H)$ almost everywhere. Hence, squaring (33) and applying the Schwarz inequality we obtain

$$K^2|\gamma_y|^2 \leqslant |\gamma'_y|^2 \leqslant K|\gamma_y| \int_{\gamma_y} J(H)\, dx.$$

Dividing by $|\gamma_y|$ and integrating gives, by virtue of (31), the inequalities

$$K|F(\hat{R}_y)| \leqslant \int_{y_1}^{y} \frac{|\gamma_y'|^2}{|\gamma_y|} dy \leqslant K|F(\hat{R}_y)| + 2K^2M^2. \tag{34}$$

Given $\varepsilon > 0$ there exist arbitrarily large values of y for which $|\gamma_y'| \leqslant K|\gamma_y| + \varepsilon$. For, if this were not the case, for some $\varepsilon > 0$, there would exist a number y_2 such that $|\gamma_y'| > K|\gamma_y| + \varepsilon$, for all $y \geqslant y_2$. It would then follow from the second inequality of (34) that $2K\varepsilon(y - y_2) \leqslant 2K^2M^2$ for all $y \geqslant y_2$. This is of course contradictory. In particular, we have shown the following. Given a positive sequence $\varepsilon_n \to 0$, there exists an increasing sequence $y_n \to \infty$ such that $d(y_n) < \varepsilon_n$ for every n. The following lemma is now immediate

LEMMA 3 (STREBEL). *Let $T_y = \{w \in H(\hat{R}_y) | \operatorname{Im} w > y\}$. Then given a positive sequence $\varepsilon_n \to 0$ there exists an increasing sequence $y_n \to \infty$ such that*

$$|T_{y_n}| \leqslant KM\varepsilon_n.$$

4.2 *The uniqueness theorem*

LEMMA 4. *For some real number $n > -1$ and some complex number $\alpha \neq 0$ let*

$$\left| \frac{\sqrt{\varphi(z)}}{(\log(1-z))^n} - \frac{\alpha}{1-z} \right| = O(1), \quad z \to 1. \tag{35}$$

Then there exists a $\varrho' > 0$ such that a single valued and schlicht branch of $\int\sqrt{\varphi(z)}dz$ can be chosen in $N_{\varrho'}$.

Proof. Let $\varrho_1 > 0$ be such that $\varphi(z) \neq 0$ in N_{ϱ_1}. Then choose a single valued branch Φ of $\int\sqrt{\varphi(z)}dz$ in N_{ϱ_1}. Setting

$$\eta(z) = \int \left(\sqrt{\varphi(z)} - \frac{\alpha(\log(1-z))^n}{1-z} \right) dz + C$$

we may write

$$\Phi(z) = -\frac{\alpha}{1+n}(\log(1-z))^{1+n} + \eta(z), \quad z \in N_{\varrho_1}.$$

If $\Phi(z)$ is not schlicht in $N_{\varrho'}$ for every $\varrho' > 0$, it follows that there exists a double sequence $z_i, z_i' \to 1$, $z_i \neq z_i'$, for which

$$\eta(z_i') - \eta(z_i) = \frac{\alpha}{n+1}[(\log(1-z_i'))^{n+1} - (\log(1-z_i))^{n+1}]. \tag{36}$$

Now set $a_i = \log(1 - z_i)$ and $b_i = \log(1 - z_i')$. Let the pairs (z_i, z_i') be labelled so that $|b_i| \leqslant |a_i|$. By making use of (35) we easily obtain, for each i, the inequality

$$\left|\frac{\eta(z_i') - \eta(z_i)}{z_i' - z_i}\right| \leqslant M\,|a_i|^n \tag{37}$$

where $M < \infty$ does not depend on i. It then follows from (36) and elementary properties of the mapping $\log(1-z)$ that $b_i/a_i \to 1$. With

$$\lambda_i = \left|\frac{1 - (b_i/a_i)^{n+1}}{1 - (b_i/a_i)}\right|$$

we may write

$$|a_i^{n+1} - b_i^{n+1}| \equiv |a_i - b_i|\,|a_i|^n\,\lambda_i.$$

By virtue of (36) and (37) we have

$$\left|\frac{a_i - b_i}{z_i' - z_i}\right| \lambda_i \leqslant M.$$

But $\lambda_i \to n+1 > 0$ as $i \to \infty$ whereas the first factor is unbounded. We have thus achieved the desired contradiction.

THEOREM 5. *Let $z_1, \ldots, z_s$ be points of $\partial\Delta$ such that excising an arbitrary neighborhood of each z_i from Δ results in a region of finite φ-area. Let $\alpha_1, \ldots, \alpha_s$ be non-zero complex numbers and let $t_1, \ldots, t_s$ be real numbers such that $-1 < t_i \leqslant 0$,* for $i = 1, \ldots, s$. *Then f is unique extremal if φ also satisfies, for each $i = 1, \ldots, s$, the growth condition*

$$\left|\frac{\sqrt{\varphi(z)}}{(\log(z_i - z))^{t_i}} - \frac{\alpha_i}{z_i - z}\right| = O(1), \quad z \to z_i. \tag{38}$$

Proof. Assuming, as we may, that $z_1 = 1$ the condition (38) is easily seen to imply that φ satisfies the hypothesis of Theorem 4. Hence, we need only prove the uniqueness. The method of proof will be as follows. For each positive integer n a neighborhood of each of the z_i will be excised from Δ. Letting S_n denote the remainder of Δ, it will then be shown that the conditions of Theorem 1.b are satisfied. The typical case will be exhibited for $z_1 = 1$.

By Lemma 4 there exists a $\varrho > 0$ such that a single valued and schlicht branch, $z' = \Phi(z)$, of $\int\sqrt{\varphi(z)}\,dz$ can be chosen in N_ϱ. Moreover, we may write

$$\Phi(z) = -\frac{\alpha_1}{1 + t_1}(\log(1 - z))^{1+t_1} + \eta(z), \quad z \in N_\varrho$$

where $\eta(z)$ is bounded. Let $R = \Phi(N_\varrho)$ and on R define $F(z') = Kx' + iy'$. Since $0 < 1 + t_1 \leqslant 1$, it follows that R is contained in a semi-infinite parallel strip. We may assume that α_1 is such that this strip is not horizontal.[5]) Then, for some $M<\infty$, the lengths of the horizontals of R are bounded by M.

Now let h be a K-quasiconformal comparison mapping for f. Choose $0<\varrho'<\varrho$ such that $N_{\varrho'}$ and $h(N_{\varrho'})$ are free of the zeros of φ and ψ respectively. Then h can be lifted to a K-quasiconformal mapping of $\Phi(N_{\varrho'})$ into $S=F(R)$. Call this mapping H. Assuming, as we may, that the parallel strip extends indefinitely into the upper half plane, Lemma 3 guarantees the existence of a sequence $y_n'\to\infty$ for which $|T_{y'_n}|\to 0$. The sequence $\gamma_{y_n}\equiv\Phi^{-1}(\gamma_{y'_n})$ of horizontals in Δ thus determines a sequence D_n of neighborhoods of $z_1=1$ such that the part of $h(\Delta-D_n)$ that lies in $f(D_n)$ has ψ-area tending to zero. Certainly an analogous situation holds for each z_j, $j=1,\ldots,s$. We then have the following situation. For each $j=1,\ldots,s$ there exists a sequence $\gamma_{j,n}$ of horizontals in Δ such that, for each n, $\gamma_{j,n}$ excises a neighborhood $D_{j,n}$ of z_j. Furthermore, $|h(\Delta-D_{j,n})\cap f(D_{j,n})|_\psi\to 0$ as $n\to\infty$. Setting $S_n=\Delta-\bigcup_{j=1}^{s} D_{j,n}$ it follows that $|T_n|_\psi\equiv|h(S_n)-f(S_n)|_\psi\to 0$ as $n\to\infty$. Also, since $\Delta\cap\partial S_n$ is composed entirely of horizontal arcs, we have $l_n=|\Gamma_n|=0$.[6]) The theorem now follows from Theorem 1.b.

Remark. Taking into consideration the footnote (6) it is not difficult to show that f is extremal in the class of all mappings of Δ onto itself that agree with f only on some open arc of $\partial\Delta$ that contains one of the points z_j. More generally, it can be shown that f is extremal in this larger class of mappings if φ satisfies the hypothesis of Lemma 4. Indeed, let N_ϱ be a neighborhood of $z=1$ on which a single valued and schlicht branch Φ of $\int\sqrt{\varphi(z)}dz$ can be defined and set $R=\Phi(N_\varrho)$. If the real number n in the statement of Lemma 5 is positive one readily checks that R is a region of the type described in the remark following the proof of Lemma 2. If $-1<n\leqslant 0$ it follows from the proof of Theorem 5 that the condition (32) is satisfied. In either case the conclusion is immediate.

THEOREM 6. *Let φ be regular in Δ and meromorphic in the closure $\bar{\Delta}$. Then a corresponding f is unique extremal if and only if all poles of φ are of order not exceeding two.*

Proof. If φ has only simple poles then Δ has finite φ-area. For this case the unique extremal property of f is known. On the other hand if φ possesses second order poles, but no poles of higher order, then φ satisfies the hypotheses of Theorem 5. This proves the sufficiency part of the theorem.

Now let φ possess a pole of order $n>2$ which, for convenience, will be taken at $z=1$. In N_ϱ, for some $\varrho>0$, we may write $\varphi(z)=(1-z)^{-n}g(z)$ where $g(z)$ is regular and non-vanishing in the closure $\bar{N}_\varrho$. Let

[5]) If φ is replaced by $\hat{\varphi}=e^{2\theta i}\varphi$ then α_1 is replaced by $e^{\theta i}\alpha_1$. But $\hat{f}(z)=f(e^{\theta i}z)$ is a Teichmüller mapping corresponding to $\hat{\varphi}$ and clearly $\hat{f}$ and f are extremal or unique extremal together.

[6]) We remark that to this point we have only used the agreement of h and f on s boundary arcs, each containing one of the z_j in its interior.

$$\Phi(z) = -\frac{\alpha_0}{1 - n/2}(1 - z)^{1-n/2} + \eta(z), \quad z \in N_\varrho$$

denote a single valued branch of $z' = \int\sqrt{\varphi(z)}dz$. Taking $\alpha_0 = \sqrt{g(1)}$ to be $n/2 - 1$ and noting that $\eta(z) = \int(\sqrt{g(z)} - \alpha_0)/(1-z)^{n/2}dz + C = o(|1-z|^{1-n/2})$ as $z \to 1$, we may write

$$\Phi(z) = (1 - z)^{1-n/2}(1 + o(1)), \quad z \to 1.$$

It follows that, for some $R \subset N_\varrho$, Φ maps R onto a wedge V defined by $|\arg(z' - x_0')| < \theta$ where x_0' is real and $\theta < \pi/4$. In R, f can be expressed by $f = \Psi^{-1} \circ F \circ \Phi$. Now, in the class of mappings of V onto $F(V)$ that agree with F on ∂V, let $\tilde{F}$ be extremal. As noted in the concluding paragraph of section 3, $\tilde{F} \neq F$. Hence, redefining f in R to be $\Psi^{-1} \circ \tilde{F} \circ \Phi$ exhibits a second K-quasiconformal mapping with the same boundary correspondence as f. This completes the proof.

We conclude with a remark concerning a possible characterization of regular functions φ, which are meromorphic in $\bar{\Delta}$, with respect to the property that corresponding Teichmüller mappings f are extremal but not necessarily unique extremal. If φ has a double pole on $\partial\Delta$, or if φ possesses only simple poles, we know that f is extremal. On the other hand, if φ possesses poles of order exceeding two but no second order poles, it is not known whether a corresponding f is extremal. However, there is some evidence that the answer is always in the negative. Indeed, for every integer $n > 2$, there exists such a φ, with an nth order pole at $z = 1$, for which a corresponding f is not extremal. This fact was exhibited in the concluding remarks of Section 3. In view of these remarks a reasonable conjecture is that f is extremal if and only if either (i) φ has a double pole or (ii) φ has no pole of order exceeding two.

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