Some Congruence Theorems for Closed Hypersurfaces in Riemann Spaces (Part I: Method based on Stokes' Theorem)

Dedicated to the memory of Mrs. Anja Hopf

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Introduction

1. Usually two point sets $F, \bar{F}$ of a metric space R are called "congruent" if there exists an isometric mapping of R onto itself carrying F into $\bar{F}$. However we shall use a more general notion of congruence: a group G of 1-1-mappings T of R onto itself shall be distinguished; then $F, \bar{F}$ are called congruent modulo G if there exists a mapping $T \in G$ with $T(F) = \bar{F}$. A "congruence theorem" is a statement saying "F and $\bar{F}$ are congruent modulo G".

2. Our spaces R shall be $(m+1)$-dimensional Riemann spaces and $F = W$, $\bar{F} = \bar{W}$ shall be m-dimensional closed (several times differentiable) hypersurfaces. Two known special congruence theorems, namely theorems V and A as stated below, form our starting point: W, $\bar{W}$ are closed surfaces in the euclidean space $R = E^3$; a differentiable mapping ϕ of W onto $\bar{W}$ is given; we consider the straight lines $(p\bar{p})$ connecting the points $p \in W$ with their images $\bar{p} = \phi(p)$; in order to exclude certain exceptions we further assume that the set of points in which $(p\bar{p})$ is tangent to W or $\bar{W}$ does not have inner points. H and $\bar{H}$ are the mean curvatures of W and $\bar{W}$ respectively. Then the mentioned theorems read as follows:

THEOREM V: If all straight lines $(p\bar{p})$ are parallel to one another and if $H(p) = \bar{H}(\bar{p})$ for all $p \in W$, the surface $\bar{W}$ is produced from W by simple translation in the direction of $(p\bar{p})$. W and $\bar{W}$ are therefore congruent (in the elementary sense).

THEOREM A: If all straight lines $(p\bar{p})$ go through a fixed point o (which does not lie on W or $\bar{W}$) and if, assuming that $|x|$ means the distance of a point x from o,

$$|p| \cdot H(p) = |\bar{p}| \cdot \bar{H}(\bar{p}),$$

then $\bar{W}$ is produced from W through a "homothety" (="similarity") with the center o. Therefore W and $\bar{W}$ are similar (in the elementary sense).

Both of these theorems are "congruence theorems" in the sense explained under *1.*: in Theorem V, G is the group of translations in the direction of the straight lines $(p\bar{p})$

since the translations are isometric, Theorem V is an ordinary theorem of elementary geometry. In Theorem A, G is the group of homotheties with the center o. – These theorems are, by the way, known not only for $m=2$ but for arbitrary m of m-dimensional hypersurfaces in the euclidean E^{m+1} (cf. [1], [2], [3]).

3. Now, the aim of H. HOPF and the present author was to formulate as general as possible a theorem valid in arbitrary spaces and containing theorems V and A as special cases. We consider a Riemann space $R=R^{m+1}$ and a continuous differentiable 1-parameter group G of 1-1-mappings T_τ of R onto itself (the group parameter τ, $-\infty<\tau<+\infty$, is assumed to be always canonic, i.e. $T_{\tau_1}\cdot T_{\tau_2}=T_{\tau_1+\tau_2}$). The orbits (or streamlines) of the points of R produced by T_τ are regular curves covering R simply; in particular T_τ has no fixed point for $\tau\neq 0$. (In the case of Theorem V is $R=E^3$ and the orbits are parallel straight lines; in the case of Theorem A is $R=E^3-o$ and the orbits are the straight lines through o, not including o.) In R a closed hypersurface $W=W^m$ shall be given; to each point $p\in W$ we attribute a parameter value $\tau=\tau(p)$ in such a manner that $\bar{p}=T_{\tau(p)}p$ form a regular hypersurface $\bar{W}=\bar{W}^m$ and that the mapping $\phi: p\to\bar{p}$ and $W\to\bar{W}$ is regular. Furthermore we assume, as under *2.* that the set of points in which orbits are tangent to W or $\bar{W}$, has no inner points.

Now we consider an additional hypersurface $\tilde{W}_p=T_{\tau(p)}\,W$ for each $p\in W$: the point $\bar{p}$ lies therefore on $\bar{W}$ and $\tilde{W}_p$. The mean curvatures of W, $\bar{W}$, $\tilde{W}_p$ are H, $\bar{H}$, $\tilde{H}_p$ and we assume $\tilde{W}_p$ such that

$$\bar{H}(\bar{p})=\tilde{H}_p(\bar{p}) \qquad \text{for each} \qquad p\in W\,. \tag{1}$$

(If G consists of isometries of R, we always have $\tilde{H}_p(\bar{p})=H(p)$, so that the condition (1) simply means $\bar{H}(\bar{p})=H(p)$ as in Theorem V. In the case of Theorem A, $T_{\tau(p)}$ is the homothety with the similarity factor $|\bar{p}|\cdot|p|^{-1}$; this leads to $\tilde{H}_p(\bar{p})=|p|\,|\bar{p}|^{-1}H(p)$ so that (1) is also the relation between $\bar{H}$ and H assumed in Theorem A.)

4. The assumptions formulated under *3.* shall be valid. Our aim is to prove with as few other restrictions as possible the following theorem: *The function $\tau(p)$ is constant, i.e. independent of p; with other words: $\bar{W}$ is one of the hypersurfaces $\bar{W}=T_\tau W$, or: $\bar{W}$ is congruent W modulo G.*

In the following Part I of this paper we shall prove this for the case that the transformations T_τ are *homothetic*, and somewhat more general, that they are *conformal* (either all properly conformal or all improperly conformal). In all these paragraphs we are treating integral formulas which are deduced using Stokes' theorem.

In Part II (which H. HOPF and the present author hope to publish soon) the limitation to homothetic or conformal transformations will be dropped but on the other hand some more stringent restrictions concerning the relative position of the hypersurfaces W to the transformations T_τ will have to be adopted. The method of proof will be based upon the maximum principle for the solutions of elliptic differential equations.[1])

§ 1. Some integral forms for closed hypersurfaces

We suppose an $m+1$-dimensional Riemann space R^{m+1} of class C^ν ($\nu \geqq 3$) which admits an infinitesimal point transformation

$$\bar{x}^i = x^i + \xi^i(x)\delta\tau \tag{1.1}$$

(where x^i are local coordinates in R^{m+1} and ξ^i are the components of a contravariant vector ξ). We assume that the orbits of the transformations generated by ξ cover R^{m+1} simply and that ξ is everywhere continuous and $\neq 0$. Let us choose a coordinate system such that the orbits of the transformations are new x^1-coordinate curves, that is, a coordinate system in which the vector ξ^i has components $\xi^i = \delta^i_1$, where the symbol δ^i_j denotes Kronecker's delta; then (1.1) becomes as follows

$$\bar{x}^i = x^i + \delta^i_1\,\delta\tau \tag{1.1'}$$

and R^{m+1} admits a one-parameter continuous group G of transformations which are 1-1-mappings of R^{m+1} onto itself and are given by the expression

$$\bar{x}^i = x^i + \delta^i_1\,\tau \tag{1.2}$$

in the new special coordinate system ([4]).

If the infinitesimal transformation (1.1) is isometric, homothetic, conformal, affine motion etc., we shall call the one-parametric group G itself isometric, homothetic, conformal, affine motion, etc.

Now we consider two hypersurfaces W^m and $\bar{W}^m$ of class C^ν imbedded in R^{m+1} which do not pass through a singular point of the tangent vector field of the orbits, and are given as follows

$$\begin{aligned} &W^m: x^i = x^i(u^\alpha) \qquad\qquad i = 1, \dots, m+1 \qquad \alpha = 1, \dots, m \\ &\bar{W}^m: \bar{x}^i = x^i(u^\alpha) + \delta^i_1\,\tau(u^\alpha) \end{aligned} \tag{1.3}$$

where u^α are local coordinates of W^m and τ is a continuous function attached to each point of the hypersurface W^m. We shall henceforth confine ourselves to Latin indices running from 1 to $m+1$ and Greek indices from 1 to m.

As is known from (1.3) we can attach a transformation $T_{\tau(p)} \in G$ to each point on the hypersurface W^m, such that it depends regularly on p; that is, we have

$$\bar{x}^i = x^i + \delta^i_1\,\tau(u^\alpha_0)$$

[1]) Our papers (Part I and Part II which are common works by H. Hopf and the present author) have been presented in 1958 at the International Congress of Mathematicians in Edinburgh; a summary has appeared in the "Abstracts of short communications and scientific programme", p. 114.

as the transformation $T_{\tau(p_0)}$ attached to p_0 whose coordinates are u_0^α, then the corresponding hypersurface $\bar{W}^m$ indicated by (1.3) is regarded as a set of the point $\bar{p}=T_{\tau(p)}(p)$ and we also consider the hypersurface $T\cdot W^m$ for any $T\in G$.

In this section, the purpose of our study is to introduce some integral forms which we need in order to prove our congruence theorems.

Let us denote the mean curvature ([5], p. 250), the normal unit vector, the area element, etc. of a hypersurface S in R^{m+1} at a point $q\in S$, by $H(S, q)$, $n^i(S, q)$, $dA(S, q)$, etc. respectively.

For a while, we shall confine ourselves to the case $m+1=3$ to let an explanation be easy. We consider the following invariant attached to each point $\bar{p}$ on the surface $\bar{W}^m$

$$\bar{\varepsilon}_{ijk}(\bar{n}^i-\tilde{n}^i)\,\delta_1^j\,\tau(u^\alpha)\,d\bar{x}^k \tag{1.4}$$

where $\tilde{n}^i$ means $n^i(T_{\tau(p)}W, \bar{p})$ and $d\bar{x}^k$ a displacement along $\bar{W}^m$, i.e., $d\bar{x}^k=\dfrac{\partial\bar{x}^k}{\partial u^\alpha}\,du^\alpha$, and

$$\varepsilon_{ijk}\overset{\text{def.}}{=}\sqrt{g}\;e_{ijk}$$

g being the determinant of the metric tensor g_{ij} of R^{m+1}, and the symbol e_{ijk} meaning plus one or minus one, depending on whether the indices $i\,j\,k$ denote an even permutation of 1 2 3 or an odd permutation, and zero when two or three indices have the same values ([6], p. 7). Throughout this paper repeated lower case Latin indices call for summation 1 to $m+1$ and repeated lower case Greek indices for summation 1 to m. Every quantity at the corresponding point $\bar{p}$ on $\bar{W}^m$ is denoted by a symbol bar. Then we obtain locally the differential expression

$$\begin{aligned}&\delta_{[1}\,\bar{\varepsilon}_{|ijk}(\bar{n}^i-\tilde{n}^i)\delta_{1|}^j\,\tau(u^\alpha)\,d_{2]}\bar{x}^k\\&\qquad=\delta_{[1}\,\bar{\varepsilon}_{|ijk}\bar{n}^i\,\delta_{1|}^j\,\tau(u^\alpha)\,d_{2]}\bar{x}^k-\delta_{[1}\,\bar{\varepsilon}_{|ijk}\tilde{n}^i\,\delta_{1|}^j\,\tau\,d_{2]}\bar{x}^k\end{aligned} \tag{1.5}$$

where the symbol δ means the covariant differential along the surface $\bar{W}^m$, and the symbol [ ] means alternating in 2 (see [5], p. 14), e.g.

$$\delta_{[1}\,\bar{\varepsilon}_{|ijk}(\bar{n}^i-\tilde{n}^i)\delta_{1|}^j\,\tau\,d_{2]}\bar{x}^k=\tfrac{1}{2}\{\delta_1\,\bar{\varepsilon}_{ijk}(\bar{n}^i-\tilde{n}^i)\delta_1^j\,\tau\,d_2\,\bar{x}^k-\delta_2\,\bar{\varepsilon}_{ijk}(\bar{n}^i-\tilde{n}^i)\delta_1^j\,\tau\,d_1\,\bar{x}^k\}.$$

The first term of the right-hand member of (1.5) is written in the form

$$\delta_{[1}\,\bar{\varepsilon}_{|ijk}\bar{n}^i\,\delta_{1|}^j\,\tau\,d_{2]}\bar{x}^k=\bar{\varepsilon}_{ijk}\delta_{[1}\,\bar{n}^i\,\delta_{|1|}^j\,\tau\,d_{2]}\bar{x}^k+\bar{\varepsilon}_{ijk}\bar{n}^i\,\delta_{[1}\,(\delta_{|1|}^j\,\tau)\,d_{2]}\bar{x}^k. \tag{1.6}$$

The first term of the right-hand member of (1.6) becomes as follows

$$\bar{\varepsilon}_{ijk}\delta_{[1}\,\bar{n}^i\delta_{|1|}^j\,\tau\,d_{2]}\bar{x}^k=\bar{H}\,\bar{n}_i\,\delta_1^i\,\tau\,d\bar{A},$$

the second term of the right-hand member of (1.6) is rewritten in the form

$$\bar{\varepsilon}_{ijk}\bar{n}^i\delta_{[1}\,(\delta_{|1|}^j\,\tau)\,d_{2]}\bar{x}^k=\bar{\varepsilon}_{ijk}\bar{n}^i\,\delta_{[1}\,(\delta_{|1|}^j)\tau\,d_{2]}\bar{x}^k+\bar{\varepsilon}_{ijk}\bar{n}^i\,\delta_1^j\,d_{[1}\,\tau\,d_{2]}\bar{x}^k. \tag{1.7}$$

Furthermore applying the relations

$$\bar{n}_i = \frac{\bar{\varepsilon}_{jki}}{\sqrt{\bar{g}^*}} \frac{\partial \bar{x}^j}{\partial u^{[1}} \frac{\partial \bar{x}^k}{\partial u^{2]}},$$

$$\bar{\varepsilon}_{ijk} \bar{\varepsilon}^{lmk} = 2 \bar{\delta}^l_{[i} \bar{\delta}^m_{j]} \qquad ([6], \text{p. } 8, \text{ex. } 9), \tag{1.8}$$

where $\bar{g}^*$ means the determinant of the metric tensor $\bar{g}_{\alpha\beta}$ on the surface $\bar{W}^m$ and ε^{lmk} means $(1/\sqrt{g})e^{lmk}$, e^{lmk} having the same meaning as the symbol e_{lmk}, we can see that the first term of the right-hand member of (1.7) is developed as follows

$$\begin{aligned}\bar{\varepsilon}_{ijk} \bar{n}^i \delta_{[1} (\delta^j_{|1|}) \tau d_{2]} \bar{x}^k &= \bar{\varepsilon}_{ijk} \bar{n}^i \bar{\Gamma}^j_{1l} d_{[1} \bar{x}^l d_{2]} \bar{x}^k \tau \\ &= \tfrac{1}{2} \tau \bar{\varepsilon}_{ijk} \bar{n}^i \bar{\Gamma}^j_{1l} \bar{n}_p \bar{\varepsilon}^{plk} d\bar{A} = \tfrac{1}{2} \tau (\bar{\Gamma}^j_{1j} - \bar{\Gamma}_{(jl)1} \bar{n}^j \bar{n}^l) d\bar{A}\end{aligned}$$

where the symbol ( ) denotes mixing ([5], p. 14). Consequently the first term of the right-hand member of (1.5) becomes

$$\begin{aligned}\delta_{[1} \bar{\varepsilon}_{|ijk} \bar{n}^i \delta^j_{1|} \tau d_{2]} \bar{x}^k = {} & \bar{H} \bar{n}_i \delta^i_1 \tau d\bar{A} \\ & + \tfrac{1}{2} (\bar{\Gamma}^l_{l1} - \bar{\Gamma}_{(jl)1} \bar{n}^l \bar{n}^j) \tau d\bar{A} + \bar{\varepsilon}_{ijk} \bar{n}^i \delta^j_1 d_{[1} \tau d_{2]} \bar{x}^k .\end{aligned} \tag{1.9}$$

Next we shall calculate the second term of the right-hand member of (1.5):

$$\begin{aligned}\delta_{[1} \bar{\varepsilon}_{|ijk} \bar{n}^i \delta^j_{1|} \tau d_{2]} \bar{x}^k = {} & \bar{\varepsilon}_{ijk} \delta_{[1} \bar{n}^i \delta^j_{|1|} \tau d_{2]} \bar{x}^k \\ & + \bar{\varepsilon}_{ijk} \bar{n}^i \delta_{[1} \delta^j_{|1|} \tau d_{2]} \bar{x}^k + \bar{\varepsilon}_{ijk} \bar{n}^i \delta^j_1 d_{[1} \tau d_{2]} \bar{x}^k .\end{aligned} \tag{1.10}$$

First we must discuss in detail the following quantity

$$\delta \bar{n}^i = d\bar{n}^i + \bar{\Gamma}^i_{jk} \bar{n}^j d\bar{x}^k \tag{1.11}$$

where $d\bar{n}^i$ means the infinitesimal term of the order one of $n^i(T_{\tau(p+dp)} W^m, \bar{p} + d\bar{p}) - n^i(T_{\tau(p)} W^m, \bar{p})$; therefore we can express it as follows

$$\begin{aligned}d\bar{n}^i &= n^i(T_{\tau(p+dp)} W^m, \bar{p} + d\bar{p}) - n^i(T_{\tau(p)} W^m, \bar{p}) \\ &= n^i(T_{\tau(p+dp)} W^m, \bar{p} + d\bar{p}) - n^i(T_{\tau(p)} W^m, \bar{p} + d\tilde{p}) \\ &\quad + n^i(T_{\tau(p)} W^m, \bar{p} + d\tilde{p}) - n^i(T_{\tau(p)} W^m, \bar{p})\end{aligned}$$

excepting the infinitesimal terms of the higher order than one, where $p + d\tilde{p} \equiv T_{\tau(p)}(p + dp)$, that is, the point on the surface $T_{\tau(p)} W^m$ corresponding to $p + dp \in W^m$; its coordinates are

$$\tilde{x}^i = x^i(u + du) + \delta^i_1 \tau(u).$$

Since the difference between the two normal vectors $n^i(T_{\tau(p+dp)} W^m, \bar{p} + d\tilde{p})$ and $n^i(T_{\tau(p)} W^m, \bar{p} + d\tilde{p})$ which correspond to the same point $p + dp \in W^m$ depends only on the x^1-coordinate value, the following expression is obtained

$$n^i(T_{\tau(p+dp)}W^m, \bar{p}+d\bar{p}) - n^i(T_{\tau(p)}W^m, \bar{p}+d\bar{p}) = \left(\frac{\partial \tilde{n}^i}{\partial \bar{x}^1}\right)_{u+du} d\tau$$

excepting the infinitesimal terms of the higher order than one and the infinitesimal terms of the order one of the difference between the two normals at the points $\bar{p}$ and $\bar{p}+d\bar{p}$ on the surface $T_{\tau(p)}W^m$ is the differential of the normal on the surface $T_{\tau(p)}W^m$: $\tilde{d}\tilde{n}^i$, that is,

$$\tilde{d}\tilde{n}^i \equiv n^i(T_{\tau(p)}W^m, \bar{p}+d\bar{p}) - n^i(T_{\tau(p)}W^m, \bar{p})$$

excepting the infinitesimal terms of the higher order than one. Accordingly (1.11) may be rewritten by symbols $\overset{1}{\delta}$ and $\tilde{\delta}$:

$$\delta\tilde{n}^i = \overset{1}{\delta}\tilde{n}^i + \tilde{\delta}\tilde{n}^i$$

with

$$\overset{1}{\delta}\tilde{n}^i \equiv \left(\frac{\partial \tilde{n}^i}{\partial \bar{x}^1} + \bar{\Gamma}^i_{j\,1}\tilde{n}^j\right)d\tau,$$

$$\tilde{\delta}\tilde{n}^i \equiv \tilde{d}\tilde{n}^i + \bar{\Gamma}^i_{j\,k}\tilde{n}^j\,d\tilde{x}^k. \tag{1.12}$$

$\overset{1}{\delta}\tilde{n}^i$ and $\tilde{\delta}\tilde{n}^i$ show the covariant differentials of $\tilde{n}^i$ along the x^1-curve and the surface $T_{\tau(p)}W^m$ respectively.

Therefore the first term of the right-hand member of (1.10) is reformed as follows

$$\begin{aligned}\bar{\varepsilon}_{i\,j\,k}\delta_{[1}(\tilde{n}^i)\delta^j_{|1|}\tau\,d_{2]}\bar{x}^k = {} & \bar{\varepsilon}_{i\,j\,k}\tilde{\delta}_{[1}(\tilde{n}^i)\delta^j_{|1|}\tau\,d_{2]}\bar{x}^k \\ & + \bar{\varepsilon}_{i\,j\,k}\overset{1}{\delta}_{[1}(\tilde{n}^i)\delta^j_{|1|}\tau\,d_{2]}\bar{x}^k.\end{aligned} \tag{1.13}$$

For the first term of the right-hand member of (1.13) we have

$$\bar{\varepsilon}_{i\,j\,k}\tilde{\delta}_{[1}(\tilde{n}^i)\delta^j_{|1|}\tau\,d_{2]}\bar{x}^k = \tilde{H}\tilde{n}_i\delta^i_1\tau\,d\tilde{A} \tag{1.14}$$

where the quantities with a wave mean the following

$$\tilde{H} \equiv H(T_{\tau(p)}W^m, \bar{p}), \qquad \tilde{n}_i \equiv n_i(T_{\tau(p)}W^m, \bar{p}),$$
$$d\tilde{A} \equiv dA(T_{\tau(p)}W^m, \bar{p}), \qquad d\tilde{x} \equiv dx(T_{\tau(p)}W^m, \bar{p}).$$

We can see easily that $d\tilde{x} = dx$ from the above definition.

On the other hand, making use of the expression indicated by (1.3):

$$d\bar{x}^i = dx^i + \delta^i_1\,d\tau$$

and $d\tilde{x}^i = dx^i$, we have

$$\bar{\varepsilon}_{i\,j\,k}d_{[1}\bar{x}^j d_{2]}\bar{x}^k = \bar{\varepsilon}_{i\,j\,k}d_{[1}\tilde{x}^j d_{2]}\tilde{x}^k + 2\bar{\varepsilon}_{i\,j\,k}\delta^j_1\,d_{[1}\tau\,d_{2]}\tilde{x}^k$$

and we get

$$dA\bar{n}_i = d\tilde{A}\tilde{n}_i + 2\bar{\varepsilon}_{i\,j\,k}\delta^j_1\,d_{[1}\tau\,d_{2]}\tilde{x}^k,$$

that is,

$$\bar{\varepsilon}_{i\,j\,k}\delta^j_1\, d_{[1}\,\tau\, d_{2]}\bar{x}^k = \tfrac{1}{2}(d\bar{A}\,\bar{n}_i - d\tilde{A}\,\tilde{n}_i), \tag{1.15}$$

and the relation

$$d\bar{A}\,\bar{n}_i\delta^i_1 = d\tilde{A}\,\tilde{n}_i\delta^i_1\,. \tag{1.16}$$

For the second term of the right-hand member of (1.13), the following calculations are developed:

$$\begin{aligned}\bar{\varepsilon}_{i\,j\,k}\overset{1}{\delta}_{[1}(\tilde{n}^i)\delta^j_{|1|}\,\tau\, d_{2]}\bar{x}^k &= \bar{\varepsilon}_{i\,j\,k}\left(\frac{\partial\tilde{n}^i}{\partial\bar{x}^1} + \bar{\Gamma}^i_{l\,1}\tilde{n}^l\right)\delta^j_1\, d_{[1}\,\tau\, d_{2]}\bar{x}^k\,\tau \\ &= \frac{1}{2}\left(\frac{\partial\tilde{n}^i}{\partial\bar{x}^1} + \bar{\Gamma}^i_{l\,1}\tilde{n}^l\right)(d\bar{A}\,\bar{n}_i - d\tilde{A}\,\tilde{n}_i)\,\tau \\ &= \frac{1}{2}\left(\frac{\partial\tilde{n}^i}{\partial\bar{x}^1} + \bar{\Gamma}^i_{l\,1}\tilde{n}^l\right)\bar{n}_i\,\tau\, d\bar{A}\end{aligned}$$

because of the relation $\overset{1}{\delta}(\tilde{n}^i)\tilde{n}_i=0$ at which we arrive by differentiating covariantly $\tilde{n}^i\tilde{n}_i=1$ along the x^1-curve. Now calculating the partial derivative of $\tilde{n}^i$ indicated in the form

$$\tilde{n}^i \equiv \bar{g}^{i\,l}\frac{\bar{\varepsilon}_{p\,q\,l}}{\sqrt{\tilde{g}^*}}\frac{\partial\bar{x}^p}{\partial u^{[1}}\frac{\partial\bar{x}^q}{\partial u^{2]}} \quad \left(\equiv \bar{g}^{i\,l}\frac{\bar{\varepsilon}_{p\,q\,l}}{\sqrt{\tilde{g}^*}}\frac{\partial x^p}{\partial u^{[1}}\frac{\partial x^q}{\partial u^{2]}}\right)$$

with respect to the $\bar{x}^1$, $\tilde{g}^*$ being the determinant of the metric tensor $\tilde{g}^*_{\alpha\,\beta}$ on the surface $T_{\tau(p)}W^m$ at $\bar{p}$, that is,

$$\tilde{g}^* = g^*(T_{\tau(p)}W^m, \bar{p}) = |\tilde{g}^*_{\alpha\,\beta}|\,,$$

$$\tilde{g}^*_{\alpha\,\beta} = g^*_{\alpha\,\beta}(T_{\tau(p)}W^m, \bar{p}) = \bar{g}_{i\,j}\frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^j}{\partial u^\beta},$$

and multiplying its result by $\bar{n}_i$ and contracting with respect to the index i, we get

$$\frac{\partial\tilde{n}^i}{\partial\bar{x}^1}\bar{n}_i = -\frac{\partial\bar{g}_{i\,j}}{\partial\bar{x}^1}\tilde{n}^i\bar{n}^j + \frac{1}{2}\left(\frac{\partial\bar{g}_{i\,j}}{\partial\bar{x}^1}\tilde{n}^i\tilde{n}^j\right)\tilde{n}^l\bar{n}_l\,.$$

Thus it follows that

$$\begin{aligned}&\bar{\varepsilon}_{i\,j\,k}\overset{1}{\delta}_{[1}(\tilde{n}^i)\delta^j_{|1|}\,\tau\, d_{2]}\bar{x}^k \\ &\qquad = \frac{1}{2}\left\{-\frac{\partial\bar{g}_{i\,j}}{\partial\bar{x}^1}\tilde{n}^i\bar{n}^j + \frac{1}{2}\left(\frac{\partial\bar{g}_{i\,j}}{\partial\bar{x}^1}\tilde{n}^i\tilde{n}^j\right)\tilde{n}^l\bar{n}_l + \bar{\Gamma}^i_{j\,1}\tilde{n}^j\bar{n}_i\right\}\tau\, d\bar{A}\,.\end{aligned} \tag{1.17}$$

Moreover the second term of the right-hand member of (1.10) is expressed as follows

$$\bar{\varepsilon}_{i\,j\,k}\tilde{n}^i\delta_{[1}(\delta^j_{|1|})\,\tau\, d_{2]}\bar{x}^k = \tfrac{1}{2}(\tilde{n}^i\bar{n}_i\bar{\Gamma}^l_{l\,1} - \bar{\Gamma}^j_{l\,1}\tilde{n}^l\bar{n}_j)\,\tau\, d\bar{A}\,. \tag{1.18}$$

Substituting the result of (1.13) which is replaced by the right-hand member of (1.14), (1.17) and (1.18) in the right-hand member of (1.10), we have

$$\delta_{[1}\,\varepsilon_{|ijk}\bar{n}^i\delta^j_{1|}\tau\, d_{2]}\bar{x}^k = \tilde{H}\tilde{n}_i\delta^i_1\tau\, d\tilde{A} + \frac{1}{2}\left\{\tilde{n}^i\bar{n}_i\bar{\Gamma}^l_{l\,1}\right.$$
$$\left. - \frac{\partial\bar{g}_{ij}}{\partial\bar{x}^1}\tilde{n}^i\tilde{n}^j + \frac{1}{2}\left(\frac{\partial\bar{g}_{ij}}{\partial\bar{x}^1}\tilde{n}^i\tilde{n}^j\right)\tilde{n}^l\bar{n}_l\right\}\tau\, d\bar{A} + \bar{\varepsilon}_{ijk}\tilde{n}^i\delta^j_1\, d_{[1}\tau\, d_{2]}\bar{x}^k. \tag{1.19}$$

Now substituting (1.9) and (1.19) in the right-hand member of (1.5) and making use of (1.15) and (1.16), we have

$$\begin{aligned}
\delta_{[1}\,\varepsilon_{|ijk}(\bar{n}^i - \tilde{n}^i)\delta^j_{1|}\tau\, d_{2]}\bar{x}^k &= (\bar{H} - \tilde{H})\bar{n}_i\delta^i_1\tau\, d\bar{A} + \\
&+ \frac{1}{2}\left\{\bar{\Gamma}^l_{l\,1}(1 - \tilde{n}^i\bar{n}_i) - \bar{\Gamma}_{(l\,j)1}\tilde{n}^l\tilde{n}^j + \frac{\partial\bar{g}_{ij}}{\partial\bar{x}^1}\tilde{n}^i\tilde{n}^j\right.\\
&\left. - \frac{1}{2}\left(\frac{\partial\bar{g}_{ij}}{\partial\bar{x}^1}\tilde{n}^i\tilde{n}^j\right)\tilde{n}^l\bar{n}_l\right\}\tau\, d\bar{A}\\
&+ \tfrac{1}{4}(\bar{n}_i - \tilde{n}_i)(\bar{n}^i - \tilde{n}^i)(d\bar{A} + d\tilde{A}),\\
&= (\bar{H} - \tilde{H})\bar{n}_i\delta^i_1\tau\, d\bar{A} + \frac{1}{4}\left\{\left(\bar{\Gamma}^i_{i\,1} + \frac{1}{2}\frac{\partial\bar{g}_{ij}}{\partial\bar{x}^1}\tilde{n}^i\tilde{n}^j\right)\bar{g}_{lm}\right.\\
&\left. - \frac{\partial\bar{g}_{lm}}{\partial\bar{x}^1}\right\}(\bar{n}^l - \tilde{n}^l)(\bar{n}^m - \tilde{n}^m)\tau\, d\bar{A}\\
&+ \tfrac{1}{4}(\bar{n}_i - \tilde{n}_i)(\bar{n}^i - \tilde{n}^i)(d\bar{A} + d\tilde{A}).
\end{aligned}$$

From the property of a Riemann connection that the covariant differential of a scalar is equal to an ordinary differential, we arrive at

$$\begin{aligned}
&d_{[1}\,\bar{\varepsilon}_{|ijk}(\bar{n}^i - \tilde{n}^i)\,\delta^j_{1|}\tau\, d_{2]}\bar{x}^k\\
&\quad = (\bar{H} - \tilde{H})\,\bar{n}_i\delta^i_1\tau\, d\bar{A} + \tfrac{1}{4}\bar{G}_{l\,j}(\bar{n}^l - \tilde{n}^l)(\bar{n}^j - \tilde{n}^j)\tau\, d\bar{A}\\
&\quad + \tfrac{1}{4}(\bar{n}^l - \tilde{n}^l)(\bar{n}_l - \tilde{n}_l)(d\bar{A} + d\tilde{A}),
\end{aligned}\tag{1.20}$$

putting

$$\bar{G}_{l\,j} \equiv \left(\bar{\Gamma}^i_{i\,1} + \frac{1}{2}\frac{\partial\bar{g}_{ik}}{\partial\bar{x}^1}\tilde{n}^i\tilde{n}^k\right)\bar{g}_{l\,j} - \frac{\partial\bar{g}_{l\,j}}{\partial\bar{x}^1}, \tag{1.21}$$

where the quantity $\bar{G}_{l\,j}$ is expressed as follows

$$\bar{G}_{l\,j} = \frac{1}{2}\frac{\partial\bar{g}_{ik}}{\partial\bar{x}^1}(\bar{g}^{ik} + \tilde{n}^i\tilde{n}^k)\,\bar{g}_{l\,j} - \frac{\partial\bar{g}_{l\,j}}{\partial\bar{x}^1},$$

and $\partial g_{ik}/\partial x^1$ are the components of the Lie derivative of g_{ik} with respect to this special coordinate system ([7], p. 4). Hence the quantity $\bar{G}_{l\,j}$ is a tensor.

In the case of $m+1 \geqq 3$, we take the following invariant attached to each point on the hypersurface $\bar{W}^m$

$$\bar{\varepsilon}_{i_1 i_2 \dots i_{m+1}}(\bar{n}^{i_1} - \tilde{n}^{i_1})\, \delta_1^{i_2} \tau\, d_2 \bar{x}^{i_3} \dots d_m \bar{x}^{i_{m+1}}$$

where $d_2 \bar{x}^i, \dots, d_m \bar{x}^i$ indicate $m-1$ displacements along $\bar{W}^m$ and

$$\varepsilon_{i_1 \dots i_{m+1}} = \sqrt{g}\, e_{i_1 \dots i_{m+1}}$$

where the symbol $e_{i_1 \dots i_{m+1}}$ means plus one or minus one, depending on whether the indices $i_1 \dots i_{m+1}$ denote an even permutation of $1\ 2\ 3 \dots m+1$ or an odd permutation, and zero when at least two indices have the same value ([5], p. 25). Then, following a similar method and using similar relations,

$$n_{i_{m+1}} = \frac{\varepsilon_{i_1 \dots i_m i_{m+1}}}{\sqrt{g^*}} \frac{\partial x^{i_1}}{\partial u^{[1}} \cdots \frac{\partial x^{i_m}}{\partial u^{m]}}$$

and

$$\bar{\varepsilon}_{i_1 \dots i_{m+1}} \delta_1^{i_2} d_{[1} \tau\, d_2 \bar{x}^{i_3} \dots d_{m]} \bar{x}^{i_{m+1}} = \frac{(-1)^m}{m} (d\bar{A}\, \bar{n}_{i_1} - d\tilde{A}\, \tilde{n}_{i_1}), \tag{1.22}$$

we get easily the result

$$\begin{aligned} & d_{[1} \bar{\varepsilon}_{|i_1 \dots i_{m+1}}(\bar{n}^{i_1} - \tilde{n}^{i_1})\, \delta_{1|}^{i_2} \tau\, d_2 \bar{x}^{i_3} \dots d_{m]} \bar{x}^{i_{m+1}} \\ & \quad = (-1)^m \Big\{ (\bar{H} - \tilde{H})\, \bar{n}_i \delta_1^i \tau\, d\bar{A} + \frac{1}{2m} \bar{G}_{l\,j} (\bar{n}^l - \tilde{n}^l)(\bar{n}^j - \tilde{n}^j) \tau\, d\bar{A} \\ & \quad + \frac{1}{2m} (\bar{n}_i - \tilde{n}_i)(\bar{n}^i - \tilde{n}^i)(d\bar{A} + d\tilde{A}) \Big\} \end{aligned} \tag{1.23}$$

where the symbol [ ] means alternating in m ([5], p. 14) and $d\bar{A}$ and $d\tilde{A}$ are the area elements of the hypersurfaces $\bar{W}^m$ and $\tilde{W}^m$ at $\bar{p}$ respectively.

In the case of $m+1=2$, let us take the following

$$d\{\bar{\varepsilon}_{i\,j}(\bar{n}^i - \tilde{n}^i)\, \delta_1^j\},$$

then, by using the analogous method, we can see the chain of the following equalities

$$\begin{aligned} d\{\bar{\varepsilon}_{i\,j}(\bar{n}^i - \tilde{n}^i)\, \delta_1^j \tau\} &= \delta\{\bar{\varepsilon}_{i\,j}(\bar{n}^i - \tilde{n}^i)\, \delta_1^j \tau\} \\ &= -\{\delta(\bar{g}_{i\,j} \bar{t}^i \delta_1^j) - \delta(\bar{g}_{i\,j} \tilde{t}^i \delta_1^j)\} \tau - \bar{g}_{i\,j}(\bar{t}^i - \tilde{t}^i)\, \delta_1^j\, d\tau \\ &= -\Big[(\bar{\kappa} \bar{g}_{i\,j} \delta_1^i \bar{n}^j\, d\bar{s} + \bar{\Gamma}_{(j\,k)1} \bar{t}^j \bar{t}^k d\bar{s}) - \Big\{ \tilde{\kappa} \bar{g}_{i\,j} \delta_1^i \tilde{n}^j\, d\tilde{s} \\ &\quad + \Big(-\frac{1}{2} \frac{\partial \bar{g}_{p\,q}}{\partial \bar{x}^1} \tilde{n}^p \tilde{n}^q \tilde{t}^j \bar{t}_j + 2\bar{\Gamma}_{(j\,l)1} \tilde{t}^l \bar{t}^j \Big) d\tilde{s} \Big\} \Big] \tau \\ &\quad - \tfrac{1}{2} (\bar{n}_i - \tilde{n}_i)(\bar{n}^i - \tilde{n}^i)(d\bar{s} + d\tilde{s}) \end{aligned}$$

(because of $\delta^j_1\, d\tau = d\bar{x}^j - d\tilde{x}^j = \bar{t}^j\, d\bar{s} - \tilde{t}^j\, d\tilde{s}$)

$$= -\Big[(\bar{\kappa} - \tilde{\kappa})\, \bar{g}_{ij}\, \delta^i_1\, \bar{n}^j\, \tau\, d\bar{s}$$

$$+ \frac{1}{2}\left\{\frac{\partial \bar{g}_{jl}}{\partial \bar{x}^1} - \frac{1}{2}\left(\frac{\partial \bar{g}_{pq}}{\partial \bar{x}^1}\tilde{t}^p\, \tilde{t}^q\right)\bar{g}_{jl}\right\}(\bar{t}^j - \tilde{t}^j)\,(\bar{t}^l - \tilde{t}^l)\,\tau\, d\bar{s}$$

$$+ \tfrac{1}{2}(\bar{n}_i - \tilde{n}_i)\,(\bar{n}^i - \tilde{n}^i)\,(d\tilde{s} + d\bar{s})\Big]$$

(because of $\bar{g}_{ij}\,\delta^i_1\, \bar{n}^j\, d\bar{s} = \bar{g}_{ij}\,\delta^i_1\, \tilde{n}^j\, d\tilde{s}$)

$$= -\Big[(\bar{\kappa} - \tilde{\kappa})\, \bar{g}_{ij}\, \delta^i_1\, \bar{n}^j\, \tau\, d\bar{s} + \frac{1}{2}\left\{\frac{\partial \bar{g}_{jl}}{\partial \bar{x}^1} - \frac{1}{2}\frac{\partial \bar{g}_{rs}}{\partial \bar{x}^1}(\bar{g}^{rs} - \tilde{n}^r\,\tilde{n}^s)\,\bar{g}_{jl}\right\}$$

$$\times\, \bar{\varepsilon}^{jp}\,\bar{\varepsilon}^{lq}(\bar{n}_p - \tilde{n}_p)\,(\bar{n}_q - \tilde{n}_q)\,\tau\, d\bar{s}$$

$$+ \tfrac{1}{2}(\bar{n}_i - \tilde{n}_i)\,(\bar{n}^i - \tilde{n}^i)\,(d\tilde{s} + d\bar{s})$$

(because of $\bar{g}^{pq} = \tilde{t}^p\,\tilde{t}^q + \tilde{n}^p\,\tilde{n}^q$)

$$= -\Big[(\bar{\kappa} - \tilde{\kappa})\, \bar{g}_{ij}\, \delta^i_1\, \bar{n}^j\, \tau\, d\bar{s} + \frac{1}{2}\left\{\left(\frac{1}{2}\frac{\partial \bar{g}_{pq}}{\partial \bar{x}^1}\bar{g}^{pq} + \frac{1}{2}\frac{\partial \bar{g}_{pq}}{\partial \bar{x}^1}\tilde{n}^p\,\tilde{n}^q\right)\bar{g}_{lj}\right.$$

$$\left. - \frac{\partial \bar{g}_{lj}}{\partial \bar{x}^1}\right\}(\bar{n}^l - \tilde{n}^l)\,(\bar{n}^j - \tilde{n}^j)\,\tau\, d\bar{s}$$

$$+ \tfrac{1}{2}(\bar{n}_i - \tilde{n}_i)\,(\bar{n}^i - \tilde{n}^i)\,(d\tilde{s} + d\bar{s})\Big]$$

(because of $\bar{\varepsilon}^{jp}\,\bar{\varepsilon}^{lq} = \bar{g}^{jl}\,\bar{g}^{pq} - \bar{g}^{pl}\,\bar{g}^{jq}$),

that is,

$$d\bar{\varepsilon}_{ij}(\bar{n}^i - \tilde{n}^i)\,\delta^j_1\,\tau = -\,\{(\bar{\kappa} - \tilde{\kappa})\,\bar{n}_i\,\delta^i_1\,\tau\, d\bar{s}$$
$$+ \tfrac{1}{2}\bar{G}_{lj}(\bar{n}^l - \tilde{n}^l)\,(\bar{n}^j - \tilde{n}^j)\,\tau\, d\bar{s} + \tfrac{1}{2}(\bar{n}_i - \tilde{n}_i)\,(\bar{n}^i - \tilde{n}^i)\,(d\tilde{s} + d\bar{s})$$

where t^i, s, and κ are the tangent unit vector, the arc length, and the curvature of the curve W^m respectively; and

$$\tilde{t}^i \equiv t^i(T_{\tau(p)}\, W^m, \bar{p}), \qquad \tilde{s} \equiv s(T_{\tau(p)}\, W^m, \bar{p}), \qquad \tilde{\kappa} \equiv \kappa(T_{\tau(p)}\, W^m, \bar{p})$$

Therefore the relation (1.23) holds good in every case for $m+1 \geqq 2$.

If we take the m displacements in the left-hand member of (1.23) as follows

$$d_1 u^\alpha = (du^1, 0, \ldots, 0), \qquad d_2 u^\alpha = (0, du^2, 0, \ldots, 0), \ldots.$$
$$d_m u^\alpha = (0, 0, \ldots, du^m)$$

and multiply by $m!$, then it shows the exterior differential of the differential form of degree $m-1$ by means of E. Cartan ([8], p. 81; [5], p. 97). Accordingly (1.23) becomes

$$\frac{1}{m!} d((\bar{n} - \tilde{n}, \delta_1 \tau, \underbrace{d\bar{x}, \ldots, d\bar{x}}_{m-1}))$$

$$= (-1)^m \Big\{ (\bar{H} - \tilde{H})\, \bar{n}_i \delta_1^i \tau\, d\bar{A} + \frac{1}{2m} \bar{G}_{lj} (\bar{n}^l - \tilde{n}^l)(\bar{n}^j - \tilde{n}^j)\, \tau\, d\bar{A} \tag{1.24}$$

$$+ \frac{1}{2m} (\bar{n}_l - \tilde{n}_l)(\bar{n}^l - \tilde{n}^l)(d\tilde{A} + d\bar{A}) \Big\}$$

where

$$((\bar{n} - \tilde{n}, \delta_1 \tau, d\bar{x}, \ldots, d\bar{x})) \overset{\text{def.}}{\equiv} \bar{\varepsilon}_{i_1 \ldots i_{m+1}} (\bar{n}^{i_1} - \tilde{n}^{i_1}) \delta_1^{i_2} \tau\, d_2 \bar{x}^{i_3} \ldots d_m \bar{x}^{i_{m+1}}, \tag{1.25}$$

that is, the differential form of degree $m-1$ with respect to the du^α.

If the hypersurfaces W^m and $\tilde{W}^m$ are orientable, integrating both members of (1.24) over the whole hypersurface W^m and applying Stokes' theorem, we have

$$\frac{1}{m!} \int_{\partial W^m} ((\bar{n} - \tilde{n}, \delta_1 \tau, dx, \ldots, dx)) = (-1)^m \Big\{ \int_{W^m} (\bar{H} - \tilde{H})\, \bar{n}_i \delta_1^i \tau\, d\bar{A}$$

$$+ \frac{1}{2m} \int_{W^m} \bar{G}_{lj} (\bar{n}^l - \tilde{n}^l)(\bar{n}^j - \tilde{n}^j)\, \tau\, d\bar{A} + \tag{1.26}$$

$$+ \frac{1}{2m} \int_{W^m} (\bar{n}_l - \tilde{n}_l)(\bar{n}^l - \tilde{n}^l)(d\tilde{A} + d\bar{A}) \Big\},$$

and if the hypersurfaces W^m and $\tilde{W}^m$ are closed, then it follows that

$$\int_{W^m} (\bar{H} - \tilde{H})\, \bar{n}_i \delta_1^i \tau\, d\bar{A} + \frac{1}{2m} \int_{W^m} \bar{G}_{lj} (\bar{n}^l - \tilde{n}^l)(\bar{n}^j - \tilde{n}^j) \tau\, d\bar{A} +$$

$$+ \frac{1}{2m} \int_{W^m} (\bar{n}_l - \tilde{n}_l)(\bar{n}^l - \tilde{n}^l)(d\tilde{A} + d\bar{A}) = 0. \tag{1.27}$$

Especially we consider the differential form of degree $m-1$ for $\tau = 1$ in (1.25),

$$((\bar{n} - \tilde{n}, \delta_1, dx, \ldots, dx)),$$

then we can see easily

$$\int_{W^m} (\bar{H} - \tilde{H})\, \bar{n}_i \delta_1^i\, d\bar{A} + \frac{1}{2m} \int_{W^m} \bar{G}_{lj} (\bar{n}^l - \tilde{n}^l)(\bar{n}^j - \tilde{n}^j)\, d\bar{A} = 0 \tag{1.28}$$

for the closed orientable hypersurfaces W^m and $\tilde{W}^m$.

§ 2. Supplementary theorems

THEOREM 2.1. *If two hypersurfaces W^m and $\bar{W}^m$ in R^{m+1} whose points correspond along the orbits of the transformations $\in G$, are closed orientable and fulfil the relation*

$$\bar{H} = \tilde{H},$$

then it follows that

$$\int_{W^m} \bar{G}_{l\,j}(\bar{n}^l - \tilde{n}^l)(\bar{n}^j - \tilde{n}^l)\, d\bar{A} = 0.$$

Proof. This theorem is selfevident from (1.28).

COROLLARY. If the hypersurfaces W^m and $\bar{W}^m$ are closed and orientable and if the relation $\bar{H} = \tilde{H}$ holds, and if we shall assume that the invariant $\bar{G}_{l\,j}(\bar{n}^l - \tilde{n}^l)(\bar{n}^j - \tilde{n}^j)$ is non-negative (or non-positive) at every point on the hypersurface W^m, then it follows that

$$\bar{G}_{l\,j}(\bar{n}^l - \tilde{n}^l)(\bar{n}^j - \tilde{n}^j) = 0$$

at every point on the hypersurface W^m.

THEOREM 2.2. *If G is conformal, that is, ξ^i satisfies the equation* $\xi_{i;\,j} + \xi_{j;\,i} = 2\phi g_{i\,j}$ ([7], p. 32), *then it follows that*

$$\bar{G}_{l\,j} = m\bar{\phi}\,\bar{g}_{l\,j}$$

where ϕ is a scalar function and the symbol ";" denotes the covariant derivative.

Proof. In the special coordinate system, a necessary and sufficient condition that the infinitesimal transformation be conformal, is that

$$\Gamma_{i\,j\,1} + \Gamma_{j\,i\,1} = \frac{\partial g_{i\,j}}{\partial x^1} = 2\phi\, g_{i\,j};$$

accordingly we have

$$\bar{\Gamma}_{i\,j\,1} + \bar{\Gamma}_{j\,i\,1} = \frac{\partial \bar{g}_{i\,j}}{\partial \bar{x}^1} = 2\bar{\phi}\,\bar{g}_{i\,j}, \quad \frac{1}{2}\frac{\partial \bar{g}_{l\,j}}{\partial \bar{x}^1}\tilde{n}^l\tilde{n}^j = \bar{\phi}$$

and it follows that

$$\bar{G}_{l\,j} = \{(m+1)\,\bar{\phi} + \bar{\phi}\}\,\bar{g}_{l\,j} - 2\bar{\phi}\,\bar{g}_{l\,j} = m\bar{\phi}\,\bar{g}_{l\,j}.$$

COROLLARY 1. If G is homothetic ([7], p. 166), that is, ξ^i satisfies $\xi_{i;\,j} + \xi_{j;\,i} = 2Cg_{i\,j}$ (C=const.), then the relation

$$\bar{G}_{l\,j} = mC\,\bar{g}_{l\,j}$$

holds.

COROLLARY 2. If G is isometric, then it follows that $\bar{G}_{lj}=0$.

THEOREM 2.3. *A necessary and sufficient condition that* $\partial \bar{g}_{ij}/\partial \bar{x}^1=0$ *is that* $\bar{G}_{ij}=0$.

Proof. The necessity of the condition is selfevident from the definition of $\bar{G}_{ij}$; hence it will be enough only to show that the relation: $\partial \bar{g}_{lj}/\partial \bar{x}^1=0$ is induced from $\bar{G}_{lj}=0$.

From the definition:

$$\bar{G}_{lj}=\left(\bar{\Gamma}^i_{i1}+\frac{1}{2}\frac{\partial \bar{g}_{ih}}{\partial \bar{x}^1}\tilde{n}^i\tilde{n}^h\right)\bar{g}_{lj}-\frac{\partial \bar{g}_{lj}}{\partial \bar{x}^1}$$

and the relations:

$$\bar{\Gamma}^i_{i1}=\tfrac{1}{2}\bar{g}^{ij}\frac{\partial \bar{g}_{ij}}{\partial \bar{x}^1},\quad \bar{g}^{ih}=\frac{\partial \tilde{x}^i}{\partial u^\alpha}\frac{\partial \tilde{x}^h}{\partial u^\beta}\tilde{g}^{*\alpha\beta}+\tilde{n}^i\tilde{n}^h,$$

we have

$$\bar{G}_{lj}=\left(\tfrac{1}{2}\tilde{g}^{*ih}\frac{\partial \bar{g}_{ih}}{\partial \bar{x}^1}+\frac{\partial \bar{g}_{ih}}{\partial \bar{x}^1}\tilde{n}^i\tilde{n}^h\right)\bar{g}_{lj}-\frac{\partial \bar{g}_{lj}}{\partial \bar{x}^1},\tag{2.1}$$

where $\tilde{x}^i=\tilde{x}^i(u^\alpha)$ denotes the hypersurface $T_{\tau(p)}W^m$, $\tilde{g}^{*\alpha\beta}$ is the contravariant metric tensor on $T_{\tau(p)}W^m$ and

$$\tilde{g}^{*ij}\equiv\frac{\partial \tilde{x}^i}{\partial u^\alpha}\frac{\partial \tilde{x}^j}{\partial u^\beta}g^{*\alpha\beta}.$$

Multiplying both members of (2.1) by $\tilde{n}^l\tilde{n}^j$ and contracting with respect to the indices l and j, we have

$$\bar{G}_{lj}\tilde{n}^l\tilde{n}^j=\tfrac{1}{2}\tilde{g}^{*ih}\frac{\partial \bar{g}_{ih}}{\partial \bar{x}^1}.$$

By making use of $\bar{G}_{lj}=0$, we can see that

$$\tilde{g}^{*ih}\frac{\partial \bar{g}_{ih}}{\partial \bar{x}^1}=0\tag{2.2}$$

and we get

$$\left(\frac{\partial \bar{g}_{ih}}{\partial \bar{x}^1}\tilde{n}^i\tilde{n}^h\right)\bar{g}_{lj}=\frac{\partial \bar{g}_{lj}}{\partial \bar{x}^1}.\tag{2.3}$$

Moreover multiplying by $\bar{g}^{lj}$ and contracting, we have

$$\begin{aligned}(m+1)\left(\frac{\partial \bar{g}_{ih}}{\partial \bar{x}^1}\tilde{n}^i\tilde{n}^h\right)&=\frac{\partial \bar{g}_{lj}}{\partial \bar{x}^1}(\tilde{g}^{*lj}+\tilde{n}^l\tilde{n}^j)\\&=\frac{\partial \bar{g}_{lj}}{\partial \bar{x}^1}\tilde{n}^l\tilde{n}^j\end{aligned}$$

from (2.2) and accordingly we have

$$m \frac{\partial \bar{g}_{ih}}{\partial \bar{x}^1} \tilde{n}^i \tilde{n}^h = 0 .$$

From the above result and (2.3), it follows that

$$\frac{\partial \bar{g}_{lj}}{\partial \bar{x}^1} = 0 . \tag{2.4}$$

§ 3. Main theorems

We shall prove the following congruence theorem concerning the mean curvature for closed hypersurfaces with the aid of the statements in the preceding sections. We shall henceforth confine ourselves to two hypersurfaces W^m and $\bar{W}^m$ which do not contain a piece of a hypersurface covered by the orbits of the transformations, which is expressed by $f(x^2, \ldots, x^{m+1}) = 0$.

THEOREM 3.1. *If the hypersurfaces W^m and $\bar{W}^m$ are closed and orientable, and if there exists the relation*

$$\bar{H} = \tilde{H}$$

at corresponding points along the orbits of the transformations, and if the following condition is satisfied

$$\bar{G}_{lj}(\bar{n}^l - \tilde{n}^l)(\bar{n}^j - \tilde{n}^j) \geqq 0 \qquad (\text{or} \leqq 0)$$

at every point on the hypersurface W^m, then W^m and $\bar{W}^m$ are congruent mod G to each other.

Proof. For the closed orientable hypersurfaces W^m and $\bar{W}^m$, we can see (1.27), that is

$$\int_{W^m} (\bar{H} - \tilde{H}) \, \bar{n}_i \delta_1^i \, \tau \, d\bar{A} + \frac{1}{2m} \int_{W^m} \bar{G}_{lj}(\bar{n}^l - \tilde{n}^l)(\bar{n}^j - \tilde{n}^j) \, \tau \, d\bar{A}$$
$$+ \frac{1}{2m} \int_{W^m} (\bar{n}_i - \tilde{n}_i)(\bar{n}^i - \tilde{n}^i)(d\tilde{A} + d\bar{A}) = 0 .$$

Let us use the hypotheses: $\bar{H} = \tilde{H}$ and $\bar{G}_{lj}(\bar{n}^l - \tilde{n}^l)(\bar{n}^j - \tilde{n}^j) \geqq 0$ (or $\leqq 0$). Then from the corollary of Theorem 2.1, we have

$$\bar{G}_{lj}(\bar{n}^l - \tilde{n}^l)(\bar{n}^j - \tilde{n}^j) = 0$$

at every point on W^m, accordingly we obtain

$$\int_{W^m} (\bar{n}_i - \tilde{n}_i)(\bar{n}^i - \tilde{n}^i)(d\tilde{A} + d\bar{A}) = 0 .$$

From $d\tilde{A} + d\bar{A} > 0$, we conclude

$$\bar{n}_i = \tilde{n}_i .$$

The relation: $\bar{n}_1 = \tilde{n}_1$ gives rise to

$$\frac{\sqrt{\bar{g}}}{\sqrt{\bar{g}^*}} \frac{\partial \bar{x}^2}{\partial u^{[1}} \cdots \frac{\partial \bar{x}^{m+1}}{\partial u^{m]}} = \frac{\sqrt{\tilde{g}}}{\sqrt{\tilde{g}^*}} \frac{\partial \tilde{x}^2}{\partial u^{[1}} \cdots \frac{\partial \tilde{x}^{m+1}}{\partial u^{m]}}$$

and

$$\sqrt{\bar{g}^*} = \sqrt{\tilde{g}^*} \tag{3.1}$$

because of $\bar{x}^i(u^\alpha) = \tilde{x}^i(u^\alpha)$ for $i = 2, \ldots, m+1$.

From (3.1), $\bar{n}_2 = \tilde{n}_2, \ldots, \bar{n}_{m+1} = \tilde{n}_{m+1}$ and $\dfrac{\partial x^i}{\partial u^\alpha} = \dfrac{\partial \bar{x}^i}{\partial u^\alpha}$, we have

$$\left.\begin{array}{l} \dfrac{\partial \tau}{\partial u^{[1}} \dfrac{\partial x^3}{\partial u^2} \cdots \dfrac{\partial x^{m+1}}{\partial u^{m]}} = 0 \\ \dfrac{\partial \tau}{\partial u^{[1}} \dfrac{\partial x^2}{\partial u^2} \dfrac{\partial x^4}{\partial u^3} \cdots \dfrac{\partial x^{m+1}}{\partial u^{m]}} = 0 \\ \vdots \\ \dfrac{\partial \tau}{\partial u^{[1}} \dfrac{\partial x^2}{\partial u^2} \cdots \dfrac{\partial x^m}{\partial u^{m]}} = 0 \end{array}\right\} \tag{3.2}$$

Let M be a set of points on the hypersurface W^m which satisfy

$$\frac{\partial x^2}{\partial u^{[1}} \cdots \frac{\partial x^{m+1}}{\partial u^{m]}} = 0 .$$

Then since

$$\frac{\partial x^2}{\partial u^{[1}} \cdots \frac{\partial x^{m+1}}{\partial u^{m]}} \neq 0 .$$

at a point on $W^m - M$, the following relation must hold

$$\frac{\partial \tau}{\partial u^1} = \cdots = \frac{\partial \tau}{\partial u^m} = 0 ,$$

that is,

$$\tau = \text{const.},$$

and also for the reason that τ is a continuous function with respect to u^α and for the reason that W^m and $\bar{W}^m$ do not contain a piece of a hypersurface covered by the orbits of the transformations, we conclude

$$\tau = \text{const.}$$

at every point on W^m. This fact shows that W^m and $\bar{W}^m$ are congruent mod G.

Further from Theorem 2.2 and Theorem 3.1, we can conclude the following

COROLLARY 1. If G is a conformal group such that ξ^i satisfies $\xi_{i;j}+\xi_{j;i}=2\phi g_{ij}$ with the condition $\phi \geqq 0$ (or $\leqq 0$), and if the closed orientable hypersurfaces W^m and $\bar{W}^m$ fulfil the relation $\bar{H}=\tilde{H}$, then W^m and $\bar{W}^m$ are congruent mod G.

As a special case of the last corollary, we get

COROLLARY 2. If G is homothetic, and if the closed orientable hypersurfaces W^m and $\bar{W}^m$ fulfil the relation $\bar{H}=\tilde{H}$, then W^m and $\bar{W}^m$ are congruent mod G.

Next we examine the problem: How is the relation $\bar{H}=\tilde{H}$ rewritten by the corresponding invariants of the hypersurfaces W^m and $\bar{W}^m$ in the case that G is homothetic?

Let us consider again the infinitesimal transformation (1.1). Then as well-known, the Lie derivative of an affine connection Γ^i_{jk} is as follows

$$L\Gamma^i_{jk}=\tfrac{1}{2}g^{il}\{(Lg_{lj})_{;k}+(Lg_{lk})_{;j}-(Lg_{jk})_{;l}\} \qquad ([7], \text{p. } 52) \tag{3.3}$$

where the symbol L means the Lie derivative.

Because the infinitesimal transformation is homothetic, we have the relation

$$Lg_{lj}=2Cg_{lj} \tag{3.4}$$

C being a constant. Replacing Lg_{lj} by $2Cg_{lj}$ in the right-hand member of (3.3), we get

$$L\Gamma^i_{jk}=0,$$

that is, a condition for an affine motion ([7], p. 7).

In our special coordinate system the last result is expressed in the form

$$\frac{\partial \Gamma^i_{jk}}{\partial x^1}=0 \qquad ([7], \text{p. } 34) \tag{3.5}$$

and (3.4) becomes

$$\frac{\partial g_{lj}}{\partial x^1}=2Cg_{lj},$$

that is,

$$g_{lj}=\varrho^2(x^1)\,g'_{lj}(x^2,\dots,x^{m+1}) \tag{3.6}$$

where

$$(\varrho(x^1))\equiv e^{cx^1} \tag{3.7}$$

which is a scalar function attached to each point on the orbits of the transformations.

We suppose a hypersurface $\tilde{W}^m$ which is obtained from W^m by the infinitesimal transformation and denote these hypersurfaces in the forms

$$\begin{aligned} W^m &: x^i = x^i(u^\alpha) \\ \tilde{W}^m &: \tilde{x}^i = x^i(u^\alpha) + \delta^i_1\,\delta\tau \end{aligned} \tag{3.8}$$

where $\delta\tau$ is an infinitesimal constant.

In the case of $m+1 \geqq 3$, the mean curvatures at the corresponding points on these hypersurfaces are given by

$$mH = n_i\left(\frac{\partial^2 x^i}{\partial u^\alpha\,\partial u^\beta} + \Gamma^i_{jk}\frac{\partial x^j}{\partial u^\alpha}\frac{\partial x^k}{\partial u^\beta}\right) g^{*\alpha\beta} \qquad \text{([5], p. 250)} \tag{3.9}$$

and

$$m\tilde{H} = \tilde{n}_i\left(\frac{\partial^2 \tilde{x}^i}{\partial u^\alpha\,\partial u^\beta} + \Gamma^i_{jk}(\tilde{x})\frac{\partial \tilde{x}^j}{\partial u^\alpha}\frac{\partial \tilde{x}^k}{\partial u^\beta}\right) \tilde{g}^{*\alpha\beta}. \tag{3.10}$$

Now from (3.5) we can see

$$\Gamma^i_{jk}(x) = \Gamma^i_{jk}(\tilde{x}). \tag{3.11}$$

From (3.6), for the corresponding normals and metric tensors, the following relations are induced

$$n_i = \frac{\varrho(x)}{\varrho(\tilde{x})}\tilde{n}_i, \qquad g^{*\alpha\beta} = \frac{\varrho^2(\tilde{x})}{\varrho^2(x)}\tilde{g}^{*\alpha\beta} \tag{3.12}$$

and by virtue of (3.8) we get

$$\frac{\partial x^i}{\partial u^\alpha} = \frac{\partial \tilde{x}^i}{\partial u^\alpha}, \qquad \frac{\partial^2 x^i}{\partial u^\alpha\,\partial u^\beta} = \frac{\partial^2 \tilde{x}^i}{\partial u^\alpha\,\partial u^\beta}. \tag{3.13}$$

Substituting (3.11), (3.12) and (3.13) in the right-hand member of (3.9) and making use of (3.10), we have

$$\varrho H = \varrho(\tilde{x})\,\tilde{H}. \tag{3.14}$$

In the case of $m+1=2$, the curvatures at the corresponding points on these curves are expressed by

$$\kappa = n_i\left(\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}\right)\left(\frac{dt}{ds}\right)^2$$

$$\tilde{\kappa} = \tilde{n}_i\left(\frac{d^2\tilde{x}^i}{dt^2} + \Gamma^i_{jk}(\tilde{x})\frac{d\tilde{x}^j}{dt}\frac{d\tilde{x}^k}{dt}\right)\left(\frac{dt}{d\tilde{s}}\right)^2,$$

from (3.11) and the following relations

$$\frac{dt}{ds} = \frac{\varrho(\tilde{x})}{\varrho(x)}\frac{dt}{d\tilde{s}}, \qquad n_i = \frac{\varrho(x)}{\varrho(\tilde{x})}\tilde{n}_i,$$

$$\frac{dx^i}{dt} = \frac{d\tilde{x}^i}{dt}, \quad \frac{d^2x^i}{dt^2} = \frac{d^2\tilde{x}^i}{dt^2};$$

we obtain similarly

$$\varrho\,\kappa = \varrho(\tilde{x})\,\tilde{\kappa}.$$

Thus the relation (3.14) holds in every case $m+1 \geqq 2$. And also the relation (3.14) means that the quantity ϱH does not change under a homothetic infinitesimal transformation. Therefore repeating continuously this infinitesimal homothetic transformation, we can see that the relation (3.14) is given for the finite transformation as

$$W^m : x^i = x^i(u^\alpha)$$
$$\tilde{W}^m : \tilde{x}^i = x^i(u^\alpha) + \delta^i_1\,\tau$$

where τ is a finite constant. Thus we can arrive at

$$\varrho\,H = \varrho(\tilde{x})\,H(T_{\tau(p)}\,W^m, \tilde{p}) \quad (\equiv \tilde{\varrho}\,\tilde{H}).$$

Consequently the assumption $\bar{H} = \tilde{H}$ is rewritten as follows

$$\varrho\,H = \bar{\varrho}\,\bar{H}.$$

Corollary 2 of Theorem 3.1 becomes the following

THEOREM 3.2. *If G is homothetic, and if the hypersurfaces W^m and $\bar{W}^m$ are closed orientable and fulfil the relation*

$$\varrho\,H = \bar{\varrho}\,\bar{H}$$

at the corresponding points on the hypersurfaces W^m and $\bar{W}^m$ along the orbits of the transformations $\in G$, then W^m and $\bar{W}^m$ are congruent mod G, where $\varrho = e^{cx^1}$.

Especially in the case of $C=0$, the homothetic transformation becomes isometric, and in this case we have $\varrho = 1$, and Theorem 3.2 becomes the following

COROLLARY. If G is isometric, and if the hypersurfaces W^m and $\bar{W}^m$ are closed orientable and fulfil the relation

$$H = \bar{H}$$

at the corresponding points on the hypersurfaces W^m and $\bar{W}^m$ along the orbits of the transformations $\in G$, then W^m and $\bar{W}^m$ are congruent mod G.

Remark 1. In an euclidean space, if G is a translation group, that is, a special isometric transformation group, the Corollary of Theorem 3.2 just coincides with the congruence theorems according to the mean curvature of H. HOPF and K. VOSS ([1], p. 187; [2], p. 203, p. 207).

Remark 2. In an euclidean space, if we take a central transformation (homothetic transformation) then, using a polar coordinate system whose origin is the center of the transformation, we can arrive at the following case without difficulty

$$C = 1, \quad \varrho = r$$

where r denotes the distance between the origin o and a point p; in this case Theorem 3.2 is nothing but the theorem of A. AEPPLI ([3], p. 178).

Here we emphasize that Theorem 3.2 includes all the congruence theorems of closed orientable hypersurfaces in an euclidean space concerning the mean curvature which have already been dealt with by H. HOPF and K. VOSS ([1], p. 187; [2], p. 203, p. 207), and A. AEPPLI ([3], p. 178).

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